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The New Triangle in Normed Space

Muhammad Zakir[1]
Department of Mathematics,
Hasanuddin University, Indonesia.

Eridani and Fatmawati
Department of Mathematics,
Airlangga University, Indonesia.

Abstract

This paper discusses the concept of a new triangle in normed space. This triangle is the development of triangle in Euclidean space and inner product space. Discussion about triangle in a normed space is defined using the Wilson angle. Next will be attested to some fundamental trait of a triangle in the normed space.

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1. Introduction

Bottema O, 2008, has discussed the angle between two lines in Euclid space $\mathbb{R}^2$ by using dot product [2]. Anton H, 2010, with the idea of inequality Cauchy - Schwarz has provided a sense of the angle between two vectors in the inner product space [1]. Furthermore, Gunawan H, Lindiarni J, and Neswan O, 2008, in his writing has also, discussed some angles between the two vectors in the normed space, i.e angle P, angle I, angle g [3, 4, 5, 6]. As well as Milicic PM, 2011, has covered the angle Thy [4] Valentine and Waymant has covered the Wilson angle [8]. Milicic PM, 2007, also already discuss about the angle B and angle g. This paper will define a new triangle in a normed space with using an angle Wilson. The Wilson angle is introduced by Valentine and Wayment [8]. The study of the Wilson angle is discussed as follows.

[1]Corresponding author E-mail: mzakirab9@gmail.com

Let $(V, \|\cdot\|)$ be is a normed space, defined non-linear function,

$$2\langle a,b\rangle := \|a\|^2 + \|b\|^2 - \|a-b\|^2, \qquad \forall \;\; a,b \in V. \tag{1}$$

From the norms it is known that:

$$\begin{aligned} |\|a\| - \|b\||^2 &\leq \|a-b\|^2 \\ &\Leftrightarrow \|a\|^2 - 2\|a\|\cdot\|b\| + \|b\|^2 \leq \|a-b\|^2 \\ &\Leftrightarrow \langle a,b\rangle \leq \|a\|\cdot\|b\| \end{aligned} \tag{2}$$

On the other side is obtained:

$$\begin{aligned} \|a-b\|^2 &\leq (\|a\| + \|b\|)^2 \\ &\Leftrightarrow \|a-b\|^2 - \|a\|^2 - |b\|^2 \leq 2\|a\|\cdot\|b\| \\ &\Leftrightarrow -\langle a,b\rangle \leq \|a\|\cdot\|b\| \end{aligned} \tag{3}$$

From the equation (2) and (3)are obtained :

$$|\langle a,b\rangle| \leq \|a\|\cdot\|b\|, \qquad \forall a,b \in V. \tag{4}$$

which fulfills the inequality Caucy-Schwarz [1]. From the equation (4) defined angle Wilson as an angle between two vectors a and b has properties:

$$\angle(a,b) := \arccos\left(\frac{\|a\|^2 + \|b\|^2 - \|a-b\|^2}{2\|a\|\cdot\|b\|}\right) \tag{5}$$

Let $c = a - b$ be, then of the equation (5) obtained by cosine rules [9]:

$$\|c\|^2 = \|a\|^2 + \|b\|^2 - 2\|a\|\cdot\|b\|\cos\angle(a,b). \tag{6}$$

From the equation (6) can be obtained the following sine rules:

$$\|a\|\cdot\|b\|\sin\angle(a,b) = K \tag{7}$$

with $K = 2\sqrt{(s-\|a\|)(s-\|b\|)(s-\|c\|)}$, and $2s = \|a\| + |b\| + \|c\|$

2. Result

Definition 2.1. Let $(V, \|\cdot\|)$ be, is a normed space, for $a,b,c \in V\backslash\{0\}$, defined a triangle that is symbolized $\triangle[a,b,c]$ is $\{a,b,c\}$ have the properties $a + c = b$ completed with Wilson angles $\angle(a,b)$, $\angle(b,c)$ and $\angle(-a,c)$.

Theorem 2.2. Let $(V, \|\cdot\|)$ be, is a normed space, In the triangle $\triangle[a,b,c]$ that, cosine rules $\Leftrightarrow$ sine rules.

Proof. ($\Rightarrow$) Let $\cos\angle(a,b) = \left(\dfrac{\|a\|^2+\|b\|^2-\|a-b\|^2}{2\|a\|\cdot\|b\|}\right)$ be, then :

$$\begin{aligned}
\sin^2\angle(a,b) &= 1-\cos^2\angle(a,b)\\
&= 1-\left(\frac{\|a\|^2+\|b\|^2-\|c\|^2}{2\|a\|\|b\|}\right)^2\\
&= \frac{(2\|a\|\|b\|)^2-(\|a\|^2+\|b\|^2-\|c\|^2)^2}{4\|a\|^2\|b\|^2}\\
&= \frac{\begin{array}{c}(2\|a\|\|b\|)-(\|a\|^2+\|b\|^2-\|c\|^2)\\ ((2\|a\|\|b\|)+(\|a\|^2+\|b\|^2-\|c\|^2))\end{array}}{4\|a\|^2\|b\|^2}\\
&= \frac{\|c\|^2-(\|a\|-\|b\|)^2(\|a\|+\|b\|)^2-\|c\|^2}{4\|a\|^2\|b\|^2}\\
&= \frac{\begin{array}{c}(\|a\|+\|b\|+\|c\|)(\|b\|+\|c\|-\|a\|)\\ (\|a\|+\|c\|-\|b\|)(\|a\|+\|b\|-\|c\|)\end{array}}{4\|a\|^2\|b\|^2}\\
&= \frac{(2s)2(s-\|a\|)2(s-\|b\|)2(s-\|c\|)}{4\|a\|^2\|b\|^2}\\
&= \frac{16(s)(s-\|a\|)(s-\|b\|)(s-\|c\|)}{4\|a\|^2\|b\|^2}\\
\|a\|\|b\|\sin\angle(a,b) &= 2\sqrt{(s-\|a\|)(s-\|b\|)(s-\|c\|)} = K.
\end{aligned}$$

($\Leftarrow$) By eliminating the sine rule it is obtained:

$$\begin{aligned}
K^2(\|a\|^2+\|b\|^2-\|c\|^2) &= \|a\|^2\|b\|^2\|c\|^2(\sin^2\angle(b,c)+\sin^2\angle(-a,c)\\
&\quad -\sin^2\angle(a,b))\\
&= \|a\|^2\|b\|^2\|c\|^2(\sin^2\angle(b,c)+\sin^2\angle(-a,c)\\
&\quad -\sin^2(\angle(b,c)+\angle(-a,c)))\\
&= \|a\|^2\|b\|^2\|c\|^2(\sin^2\angle(b,c)+\sin^2\angle(-a,c)\\
&\quad -(\sin\angle(b,c)\cos\angle(-a,c)\\
&\quad +\cos\angle(b,c)\sin\angle(-a,c))^2)\\
&= \|a\|^2\|b\|^2\|c\|^2(\sin^2\angle(b,c)+\sin^2\angle(-a,c)\\
&\quad -\sin^2\angle(b,c)\cos^2\angle(-a,c)\\
&\quad -\cos^2\angle(b,c)\sin^2\angle(-a,c)\\
&\quad -2\sin\angle(b,c)\cos\angle(-a,c).\cos\angle(b,c)\sin\angle(-a,c)
\end{aligned}$$

$$\begin{aligned}
&= \|a\|^2\|b\|^2\|c\|^2(\sin^2\angle(b,c)(1-\cos^2\angle(-a,c))+\sin^2\angle(b,c)(1-\cos^2\angle(b,c)) \\
&\quad -2\sin\angle(b,c)\cos\angle(-a,c).\cos\angle(b,c)\sin\angle(-a,c)) \\
&= \|a\|^2\|b\|^2\|c\|^2(\sin^2\angle(b,c)(\sin^2\angle(-a,c))+\sin^2\angle(b,c)(\sin^2\angle(b,c)) \\
&\quad -2\sin\angle(b,c)\cos\angle(-a,c).\cos\angle(b,c)\sin\angle(-a,c))
\end{aligned}$$

$$\begin{aligned}
&= 2\|a\|^2\|b\|^2\|c\|^2(\sin^2\angle(b,c)(\sin^2\angle(-a,c)) \\
&\quad -\sin\angle(b,c)\cos\angle(-a,c).\cos\angle(b,c)\sin\angle(-a,c)) \\
&= 2\|a\|^2\|b\|^2\|c\|^2(\sin\angle(b,c)\sin\angle(-a,c)(\sin\angle(b,c)\sin\angle(-a,c) \\
&\quad -\cos\angle(-a,c).\cos\angle(-a,c))) \\
&= 2\|a\|^2\|b\|^2\|c\|^2\sin\angle(b,c)\sin\angle(-a,c)\cos\angle(a,b) \\
&= 2\|a\|K\|b\|K\cos\angle(a,b)
\end{aligned}$$

so, the following cosine rules are obtained:

$$\|c\|^2 = \|a\|^2 + \|b\|^2 - 2\|a\|\|b\|\cos\angle(a,b). \tag{8}$$

■

Theorem 2.3. Let $(V, \|\cdot\|)$ be, is a normed space, In the triangle $\triangle[a,b,c]$ that, cosine rules $\Leftrightarrow$ side triangle rules.

Proof. $(\Rightarrow)$

$$\|a\|\cos\angle(a,b) = \|a\|\frac{\|a\|^2+\|b\|^2-\|c\|^2}{2\|a\|.\|b\|} \tag{9}$$

$$\|c\|\cos\angle(b,c) = \|c\|\frac{\|b\|^2+\|c\|^2-\|a\|^2}{2\|b\|.\|c\|} \tag{10}$$

by eliminating the equation (9) and (10), then obtained the rules of the triangle side:

$$\|a\|\cos\angle(a,b) + \|c\|\cos\angle(b,c) = \|b\|. \tag{11}$$

$(\Leftarrow)$ From the equation (11) obtained by triangle side rules:

$$\|a\|^2 = \|a\|\|b\|\cos\angle(a,b) + \|a\|\|c\|\cos\angle(-a,c) \tag{12}$$

$$\|b\|^2 = \|a\|\|b\|\cos\angle(a,b) + \|b\|\|c\|\cos\angle(b,c) \tag{13}$$

$$\|c\|^2 = \|b\|\|c\|\cos\angle(b,c) + \|a\|\|c\|\cos\angle(-a,c) \tag{14}$$

by eliminating the equation (12), (13) and (14) obtained by cosine rules:

$$\|a\|^2 + \|b\|^2 - 2\|a\|\|b\|\cos\angle(a,b) = \|c\|^2. \tag{15}$$

and this puts an end to the proof. ■

Theorem 2.4. Let $(V, \|\cdot\|)$ be,is a normed space. In the triangle $\triangle[a, b, c]$, then $\angle(a, b) + \angle(b, c) + \angle(-a, c) = \pi$.

Proof. To prove the theorem, it will first show that:

$$\cos\angle(a, b)\cos\angle(b, c) - \sin\angle(a, b)\sin\angle(b, c) + \cos\angle(-a, c) = 0 \tag{16}$$

By using the rules of cosine (6) and sine rules (7), then equation (16) proved as follows:

$$\begin{aligned}
&\cos\angle(a, b)\cos\angle(b, c) - \sin\angle(a, b)\sin\angle(b, c) + \cos\angle(-a, c) \\
=\ & \frac{\|a\|^2 + \|b\|^2 - \|c\|^2}{2\|a\|\cdot\|b\|}\cdot\frac{\|b\|^2 + \|c\|^2 - \|a\|^2}{2\|b\|\cdot\|c\|} \\
&+\frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\|\cdot\|c\|} - \frac{K}{\|a\|\cdot\|b\|}\cdot\frac{K}{\|b\|\cdot\|c\|} \\
=\ & \frac{(\|a\|^2 + \|c\|^2 - \|b\|^2)(\|b\|^2 + \|c\|^2 - \|a\|^2)}{4\|a\|\cdot\|b\|^2\cdot\|c\|} \\
&+\frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\|\cdot\|c\|} - \frac{K^2}{\|a\|\cdot\|b\|^2\cdot\|c\|} \\
=\ & \frac{(-\|a\|^4 + \|b\|^4 - \|c\|^4)}{4\|a\|\cdot\|b\|^2\cdot\|c\|} + \frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\|\cdot\|c\|} - \frac{K^2}{\|a\|\cdot\|b\|^2\cdot\|c\|} \\
=\ & \frac{(-\|a\|^4 + \|b\|^4 - \|c\|^4 + 2\|a\|^2\cdot\|b\|^2 + 2\|b\|^2\cdot\|c\|^2) - 4K^2}{4\|a\|\cdot\|b\|^2\cdot\|c\|} \\
=\ & \frac{4K^2 - 4K^2}{4\|a\|\cdot\|b\|^2\cdot\|c\|} \\
=\ & 0.
\end{aligned}$$

then

$$\begin{aligned}
&\cos\angle(a, b)\cos\angle(b, c) - \sin\angle(a, b)\sin\angle(b, c) + \cos\angle(-a, c) = 0 \\
\Leftrightarrow\ & \cos\angle(a, b)\cos\angle(b, c) - \sin\angle(a, b)\sin\angle(b, c) = -\cos\angle(-a, c) \\
\Leftrightarrow\ & \cos(\angle(a, b) + \angle(b, c)) = -\cos\angle(-a, c) \\
\Leftrightarrow\ & \cos(\angle(a, b) + \angle(b, c)) = \cos(\pi - \angle(-a, c)) \\
\Leftrightarrow\ & \angle(a, b) + \angle(b, c) = \pi - \angle(-a, c) \\
\Leftrightarrow\ & \angle(a, b) + \angle(b, c) + \angle(-a, c) = \pi
\end{aligned}$$

and this puts an end to the proof. ■

Example 2.5. Let $(\boldsymbol{L}^3([0,1]), \|\cdot\|)$ be is a normed space, and Let $\Delta[a,b,c]$ be, $\{a,b,c\} \subseteq \boldsymbol{L}^3([0,1])$ and satisfy $a+c=b$ with $a(t) := t^3$, $b(t) := t^2$ and $c(t) := t^2 - t^3$, further obtained:

$$\begin{aligned}
\|a\| &= \left(\int_0^1 |t^3|^3 dt\right)^{\frac{1}{3}} \\
&= \left(\int_0^1 t^9 dt\right)^{\frac{1}{3}} \\
&= 0,464
\end{aligned}$$

$$\begin{aligned}
\|b\| &= \left(\int_0^1 |t^2|^3 dt\right)^{\frac{1}{3}} \\
&= \left(\int_0^1 t^6 dt\right)^{\frac{1}{3}} \\
&= 0,523
\end{aligned}$$

$$\begin{aligned}
\|c\| &= \left(\int_0^1 |t^2 - t^3|^3 dt\right)^{\frac{1}{3}} \\
&= \left(\int_0^1 (t^6 - 3t^7 + 3t^8 - t^9) dt\right)^{\frac{1}{3}} \\
&= 0,106
\end{aligned}$$

So, obtained:

$$\begin{aligned}
\angle(a,b) &= \arccos\left(\frac{\|a\|^2 + \|b\|^2 - \|c\|^2}{2\|a\|\|b\|}\right) = 10,29 \\
\angle(-a,c) &= \arccos\left(\frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\|\|c\|}\right) = 118,27 \\
\angle(b,c) &= \arccos\left(\frac{\|b\|^2 + \|c\|^2 - \|a\|^2}{2\|b\|\|c\|}\right) = 51,44
\end{aligned}$$

then obtained $\angle(a,b) + \angle(a,b) + \angle(a,b) = \pi$.

Theorem 2.6. Let $(V, \|\cdot\|)$ be,is a normed space. In the triangle $\triangle[a,b,c]$ with $\|a\| = \|c\| \Leftrightarrow \angle(a,b) = \angle(b,c)$.

Proof. $(\Rightarrow)$ from triangle side rules:

$$\begin{aligned}
\|a\| &= \|b\| \cos\angle(a,b) + \|c\| \cos\angle(-a,c) \\
\|c\| &= \|b\| \cos\angle(b,c) + \|a\| \cos\angle(-a,c)
\end{aligned}$$

$\|a\| = \|b\|$, so obtained $\cos\angle(a,b) = \cos\angle(b,c)$ and because $\angle(a,b), \angle(b,c) \in [0,\pi]$ then $\angle(a,b) = \angle(b,c)$
$(\Leftarrow)$ Because $\angle(a,b) = \angle(b,c)$ then from sine rules obtained $\|a\|\sin\angle(a,b) = \|c\|\sin\angle(b,c)$ so obtained $\|a\| = \|c\|$. ■

Corollary 2.7. Let $(V, \|\cdot\|)$ be, normed space. In the triangle $\triangle[a,b,c]$ with $\|a\| = \|b\| = \|c\| \Leftrightarrow \angle(a,b) = \angle(b,c) = \angle(-a,c)$.

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