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Uniqueness of Product of Difference Polynomials of Meromorphic Functions Sharing Fixed Point with Finite Weight

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Abstract

In this paper, we investigate the uniqueness of product of difference polynomials of meromorphic functions sharing the fixed point z with finite weight l. We generalise the results of R S Dyavanal and A M Hattikal [11].

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1. Introduction and Results

In this paper, the term "meromorphic" will always mean meromorphic in the complex plane $\mathbb{C}$. We shall use the standard notations in Nevanlinna's value distribution theory (see, e.g. [15],[21],[23]). We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$, as $r \to \infty$ possibly outside a set of finite linear measure. We denote $\rho(f)$ for order of $f(z)$ and hyper order of a function $f(z)$, defined as

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}.$$

The following definitions we use while proving our results.

Definition 1.1. Let 'a' be a finite complex number, and k be a positive integer. We denote by $N_{(k}(r, a, f)$ the counting function for zeros of $f - a$ with multiplicities at least k, and by $\underline{N}_{(k}(r, a, f)$ the one for which multiplicity is not counted. Similarly, we denote by $N_{k)}(r,a,f)$ the counting function for zeros of $f-a$ with multiplicities at most k, and by $\overline{N}_{(k}(r,a,f)$ the one for which multiplicity is not counted. Then

$$N_k(r,a,f)=\overline{N}_{(1}(r,a,f)+\overline{N}_{(2}(r,a,f)+\cdots+\overline{N}_{(k}(r,a,f).$$

Definition 1.2. Let $f(z)$ and $g(z)$ be two meromorphic functions in the complex plane $\mathbb{C}$. If $f(z)-a$ and $g(z)-a$ assume the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value a CM, and if we do not consider the multiplicity, then we say that $f(z)$ and $g(z)$ share the value a IM, where a is a complex number.

Definition 1.3. [2] Let f and g be two nonconstant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p and also a 1-point of g with multiplicity q. We denote by $N_L(r,1;f)$ the counting function of those 1-points of f and g, where $p>q$, by $N_E^{(k}(r,1;f)$ ($k\geq 2$ is an integer) the counting function of those 1-points of f and g, where $p=q\geq k$, where each point in these counting functions is counted only once. In the same manner we can define $N_L(r,1;g)$ and $N_E^{(k}(r,1;g)$.

Definition 1.4. [16, 17] Let f and g be two nonconstant meromorphic functions such that f and g share the value a IM. We denote by $N_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly $N_*(r,a;f,g)=N_*(r,a;g,f)$ and $N_*(r,a;f,g)=N_L(r,a;f)+N_L(r,a;g)$.

Recently, people have raised great interest in difference analogues of Nevanlinna's theory and many articles have focused on value distribution and uniqueness of difference polynomials of entire or meromorphic functions (see example [6]-[14]).

In 2012, Y H Cao and X B Zhang ([3]) obtained the following theorem.

Theorem 1.5. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, whose zeros are of multiplicities at least k, where k is a positive integer. Let $n>\max\{2k-1;4+4/k+4\}$ be a positive integer. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share z CM, and $f(z)$ and $g(z)$ share ∞ IM, then one of the following two conclusions holds

(1) $f^n(z)f^{(k)}(z)=g^n(z)g^{(k)}(z)$;

(2) $f(z)=c_1e^{cz^2}, g(z)=c_2e^{-cz^2}$, where c, c_1 and c_2 are constants such that $4(c_1c_2)^{n+1}c_2=-1$.

In 2014, X. B. Zhang ([26]) reduced the lower bond on n and relax the condition on multiplicity of zeros in Theorem 1.5 and proved the following result.

Theorem 1.6. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and n,k two positive integers with $n>k+6$. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share z CM, and $f(z)$ and $g(z)$ share ∞ IM, then one of the following two conclusions holds

(1) $f^n(z)f^{(k)}(z) = g^n(z)g^{(k)}(z)$;

(2) $f(z) = c_1e^{cz^2}, g(z) = c_2e^{-cz^2}$, where c, c_1 and c_2 are constants such that $4(c_1c_2)^{n+1}c_2 = -1$.

In 2016, R. S. Dyavanal and A. M. Hattikal ([11]) obtained the following result.

Theorem 1.7. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of hyper orders $\rho_2(f) < 1$ and $\rho_2(g) < 1$. Let $k, n, d, s_j (j = 1, 2, \ldots, d), \lambda = \sum_{j=1}^{d} s_j$ be positive integers, $c_j \in \mathbb{C} - \{0\} (j = 1, 2, \ldots, d)$ are distinct constants and $n > \max\{2d(k + 2) + \lambda(k + 3) + 7, \lambda_1; \lambda_2\}$, where $\lambda_1 = \sum_{j=1}^{d} \alpha_j s_j$ and $\lambda_2 = \sum_{j=1}^{d} \beta_j s_j$, $j = 1, 2, \ldots, d$ and $f(z + c_j)$ and $g(z + c_j)$ have zeros with maximum orders α_j and β_j respectively. If $f(z)^n \left[\prod_{j=1}^{d} f(z + c_j)^{s_j}\right]^{(k)}$ and $g(z)^n \left[\prod_{j=1}^{d} g(z + c_j)^{s_j}\right]^{(k)}$ share z CM and $f(z), g(z)$ share ∞ IM, then one of the following two conclusions holds.

(1) $f(z)^n \left[\prod_{j=1}^{d} f(z + c_j)^{s_j}\right]^{(k)} = g(z)^n \left[\prod_{j=1}^{d} g(z + c_j)^{s_j}\right]^{(k)}$

(2) $\prod_{j=1}^{d} f(z + c_j)^{s_j} = C_1e^{Cz^2}$, $\prod_{j=1}^{d} g(z + c_j)^{s_j} = C_2e^{-Cz^2}$, where C_1, C_2 and C are constants such that $4(C_1C_2)^{n+1}C^2 = -1$.

The following is the unicity theorem for meromorphic functions sharing the value z with weight l.

Theorem 1.8. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of hyper orders $\rho_2(f) < 1$ and $\rho_2(g) < 1$. Let $k, n, d, s_j (j = 1, 2, \ldots, d), \lambda = \sum_{j=1}^{d} s_j$ be positive integers, $c_j \in \mathbb{C} - \{0\} (j = 1, 2, \ldots, d)$ are distinct constants. If $f(z)^n \left[\prod_{j=1}^{d} f(z + c_j)^{s_j}\right]^{(k)}$ and $g(z)^n \left[\prod_{j=1}^{d} g(z + c_j)^{s_j}\right]^{(k)}$ share (z, l) and $f(z), g(z)$ share ∞ IM, where $0 \leq l < \infty$. Then the conclusions of Theorem C hold provided

1. if $2 \leq l < \infty$ then $n \geq 2(k + 2)d + (k + 4)\lambda + 8$,

2. if $l = 1$ then $n \geq \lambda\left(\frac{3}{2}k + \frac{7}{2}\right) + d\left(2(k+2) + \frac{1}{2}(k+1)\right) + 9$,
3. if $l = 0$ then $n \geq d(5k+7) + 7\lambda + 4k\lambda + 14$.

Corollary 1.9. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of hyper orders $\rho_2(f) < 1$ and $\rho_2(g) < 1$. Let $k, n, d, s_j (j = 1, 2, \ldots, d), \lambda = \sum_{j=1}^{d} s_j$ be positive integers. If $f(z)^n \left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}$ and $g(z)^n \left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$ share $(1, l)$ and $f(z), g(z)$ share ∞ IM, where $0 \leq l < \infty$. Then the conclusions of Theorem 1.7 hold provided

1. if $2 \leq l < \infty$ then $n \geq 2(k+2)d + (k+4)\lambda + 8$,
2. if $l = 1$ then $n \geq \lambda\left(\frac{3}{2}k + \frac{7}{2}\right) + d\left(2(k+2) + \frac{1}{2}(k+1)\right) + 9$,
3. if $l = 0$ then $n \geq d(5k+7) + 7\lambda + 4k\lambda + 14$.

2. Some Preliminary Results

In this section we present some Lemmas which will be needed in the sequel.

Let F and G be two nonconstant meromorphic functions defined in $\mathbb{C}$. We shall denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 2.1. [4] Let $f(z)$ be a transcendental meromorphic function of finite order. Then

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.2. [2] Let F, G be two nonconstant meromorphic functions sharing $(1, l), (\infty, 0)$, where $2 \leq l < \infty$ and $H \not\equiv 0$. Then

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G)$$
$$-m(r,1;G) - N_E^{(3}(r,1;F) - \overline{N}_L(r,1;G) + S(r,F) + S(r,G)$$
$$T(r,G) \leq N_2(r,0;F) + N_2(r,0;G) + \overline{N}(r,\infty;G) + \overline{N}(r,\infty;F) + \overline{N}_*(r,\infty;F,G)$$
$$-m(r,1;F) - N_E^{(3}(r,1;G) - \overline{N}_L(r,1;F) + S(r,F) + S(r,G)$$

Lemma 2.3. [[20], Lemma 2.3] Let F, G be two nonconstant meromorphic functions sharing $(1,1),(\infty,0)$ and $H\not\equiv 0$. Then

$$T(r,F)\le N_2(r,0;F)+N_2(r,0;G)+\frac{3}{2}\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)$$
$$+\frac{1}{2}\overline{N}(r,0;F)+S(r,F)+S(r,G)$$
$$T(r,G)\le N_2(r,0;F)+N_2(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}(r,\infty;F)+\overline{N}_*(r,\infty;F,G)$$
$$+\frac{1}{2}\overline{N}(r,0;G)+S(r,F)+S(r,G)$$

Lemma 2.4. [[20], Lemma 2.5] Let F, G be two nonconstant meromorphic functions sharing $(1,0),(\infty,0)$ and $H\not\equiv 0$. Then

$$T(r,F)\le N_2(r,0;F)+N_2(r,0;G)+3\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)$$
$$+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G)$$
$$T(r,G)\le N_2(r,0;F)+N_2(r,0;G)+3\overline{N}(r,\infty;G)+2\overline{N}(r,\infty;F)+\overline{N}_*(r,\infty;F,G)$$
$$+2\overline{N}(r,0;G)+\overline{N}(r,0;F)+S(r,F)+S(r,G)$$

Lemma 2.5. [23] Let $f(z)$ be a non-constant meromorphic function, and let $a_0(z),a_1(z),\dots,$ $a_n(z)(\not\equiv 0)$ be small functions with respect to f. Then

$$T(r,a_nf^n+a_{n-1}f^{n-1}+\cdots+a_1f+a_0)=nT(r,f)+S(r,f).$$

Lemma 2.6. [[18], Lemma 2.11] Let $f(z)$ be a non-constant meromorphic function, and p,k be positive integers. Then

$$T(r,f^{(k)})\le T(r,f)+k\overline{N}(r,f)+S(r,f),$$
$$N_p\left(r,\frac{1}{f^{(k)}}\right)\le T(r,f^{(k)})-T(r,f)+N_{p+k}\left(r,\frac{1}{f}\right)+S(r,f),$$
$$N_p\left(r,\frac{1}{f^{(k)}}\right)\le N_{p+k}\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f).$$

Lemma 2.7. [19] Let $f(z)$ be a transcendental meromorphic function of finite order. Then

$$N(r,f(z+c))=N(r,f(z))+S(r,f),$$
$$N\left(r,\frac{1}{f(z+c)}\right)=N\left(r,\frac{1}{f(z)}\right)+S(r,f).$$

Lemma 2.8. ([11]) Let $f(z)$ be a transcendental meromorphic function of hyper order $\rho_2(f)<1$ and $F_1(z)=f(z)^n\left[\prod_{j=1}^{d}f(z+c_j)^{s_j}\right]$. Then

$$(n-\lambda)T(r,f)+S(r,f)\le T(r,F_1(z))\le(n+\lambda)T(r,f)+S(r,f).$$

3. Proof of Theorem

Proof of Theorem 1.1

Let

$$F(z) = f(z)^n \left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}, \quad G(z) = g(z)^n \left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)} \tag{3.1}$$

$$F_1(z) = f(z)^n \left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right], \quad G_1(z) = g(z)^n \left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right], \tag{3.2}$$

$$F_2(z) = \frac{f(z)^n \left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}}{z}, \; G_2(z) = \frac{g(z)^n \left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}}{z}. \tag{3.3}$$

Then $F_2(z), G_2(z)$ are transcendental meromorphic functions that share $(1,l)$ and f, g share ∞ IM. Since f and g are transcendental, z is a small function with respect to both F and G.

Let us consider two cases separately.

Case 1: Assume that $H \not\equiv 0$.
Now, we consider the following three subcases.

Subcase 1.1: Suppose that $2 \le l < \infty$. Then using Lemma 2.2, we obtain

$$\begin{aligned} T(r,F) &= T(r,F_2) + S(r,F_2) \\ &\le N_2(r,0,F_2) + N_2(r,0,G_2) + \overline{N}(r,\infty,F_2) + \overline{N}(r,\infty,G_2) \\ &\quad + \overline{N}_*(r,\infty;F_2,G_2) - m(r,1,G_2) - N_E^{(3}(r,1,F_2) - \overline{N}_L(r,1,G_2) \\ &\quad + S(r,F_2) + S(r,G_2) \\ &\le N_2(r,0,F) + N_2(r,0,G) + \overline{N}_*(r,\infty;F,G) + \overline{N}(r,\infty,F) + \overline{N}(r,\infty,G) \\ &\quad + S(r,F) + S(r,G) \end{aligned} \tag{3.4}$$

Noting that

$$\begin{aligned} \overline{N}_*(r,\infty;F,G) &= \overline{N}_L(r,\infty,F) + \overline{N}_L(r,\infty,G) \\ &\le \overline{N}(r,\infty,F) = \overline{N}(r,\infty,G) \end{aligned} \tag{3.5}$$

we obtain from (3.4),(3.5), Lemma 2.8 and Lemma 2.6 that

$$\begin{aligned}
T(r,F) &\leq N_2(r,0,F)+N_2(r,0,G)+2\overline{N}(r,\infty,F)+\overline{N}(r,\infty,G)\\
&\quad +S(r,F)+S(r,G)\\
&\leq N_2(r,0,f^n)+N_2\left(r,0,\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right)+N_2(r,0,g^n)\\
&\quad +N_2\left(r,0,\left(\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)}\right)+2\overline{N}\left(r,\infty,f^n\right)\\
&\quad +2\overline{N}\left(r,\infty,\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right)+\overline{N}\left(r,\infty,g^n\right)\\
&\quad +\overline{N}\left(r,\infty,\left(\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)}\right)+S(r,f)+S(r,g)\\
&\leq 2\overline{N}_{(2}\left(r,0,f^n\right)+T\left(r,\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right)\\
&\quad -T\left(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)\\
&\quad +N_{k+2}\left(r,0,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)+2\overline{N}_{(2}\left(r,0,g^n\right)\\
&\quad +T\left(r,\left(\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)}\right)-T\left(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)\\
&\quad +N_{k+2}\left(r,0,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)+2\overline{N}\left(r,\infty,f\right)\\
&\quad +2\overline{N}\left(r,\infty,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\overline{N}(r,\infty,g)+\overline{N}\left(r,\infty,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)+S(r,f)+S(r,g)\\
\leq\ & 2T(r,f)+T\left(r,\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right)+T(r,f^n)-T(r,f^n)\\
&-T\left(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)+d(k+2)T(r,f)+2T(r,g)\\
&+T\left(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)+k\overline{N}\left(r,\infty,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)\\
&-T\left(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)+d(k+2)T(r,g)+2T(r,f)\\
&+2\lambda T(r,f)+T(r,g)+\lambda T(r,g)+S(r,f)+S(r,g)\\
\leq\ & 2T(r,f)+T(r,F)-T(r,F_1)+d(k+2)T(r,f)\\
&+2T(r,g)+k\lambda T(r,g)\\
&+d(k+2)T(r,g)+2(\lambda+1)T(r,f)+(\lambda+1)T(r,g)\\
&+S(r,f)+S(r,g)
\end{aligned}
$$

That is,

$$
\begin{aligned}
T(r,F_1) &\leq (d(k+2)+2(\lambda+1)+2)T(r,f)\\
&\quad +(d(k+2)+(k+1)\lambda+3)T(r,g)+S(r,f)+S(r,g)\\
(n-\lambda)T(r,f) &\leq (d(k+2)+\lambda+3)(T(r,f)+T(r,g))\\
&\quad +(\lambda+1)T(r,f)+k\lambda T(r,g)+S(r,f)+S(r,g)
\end{aligned}
\tag{3.6}
$$

Similarly,

$$
\begin{aligned}
(n-\lambda)T(r,g) &\leq (d(k+2)+\lambda+3)(T(r,f)+T(r,g))\\
&\quad +(\lambda+1)T(r,g)+k\lambda T(r,f)+S(r,f)+S(r,g)
\end{aligned}
\tag{3.7}
$$

From (3.6) and (3.7), we get

$$(n-4\lambda-2(k+2)d-k\lambda-7)(T(r,f)+T(r,g))\leq S(r,f)+S(r,g)$$

Which is contradiction to $n\geq 2(k+2)d+(k+4)\lambda+8$.

Subcase 1.2: When $l = 1$.
Then using (3.5) and Lemma 2.3, we deduce that

$$\begin{aligned}
T(r,F) &= T(r,F_2) + S(r,F_2) \\
&\leq N_2(r,0,F_2) + N_2(r,0,G_2) + \frac{3}{2}\overline{N}(r,\infty,F_2) + \overline{N}(r,\infty,G_2) \\
&\quad + \overline{N}_*(r,\infty,F_2,G_2) + \frac{1}{2}\overline{N}(r,0,F_2) + S(r,F_2) + S(r,G_2) \\
&\leq N_2(r,0,F) + N_2(r,0,G) + \frac{3}{2}\overline{N}(r,\infty,F) + \overline{N}(r,\infty,G) \\
&\quad + \overline{N}_*(r,\infty,F,G) + \frac{1}{2}\overline{N}(r,0,F) + S(r,F) + S(r,G) \\
&\leq N_2(r,0,F) + N_2(r,0,G) + \frac{5}{2}\overline{N}(r,\infty,F) + \overline{N}(r,\infty,G) \\
&\quad + \frac{1}{2}\overline{N}(r,0,F) + S(r,F) + S(r,G) \\
&\leq N_2\left(r,0,f^n\right) + N_2\left(r,0,\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right) + N_2\left(r,0,g^n\right) \\
&\quad + N_2\left(r,0,\left(\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)}\right) + \frac{5}{2}\overline{N}\left(r,\infty,f^n\right) \\
&\quad + \frac{5}{2}\overline{N}\left(r,\infty,\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right) + \overline{N}\left(r,\infty,g^n\right) \\
&\quad + \overline{N}\left(r,\infty,\left(\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)}\right) + \frac{1}{2}\overline{N}\left(r,0,f^n\right) \\
&\quad + \frac{1}{2}\overline{N}\left(r,0,\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right) + S(r,f) + S(r,g) \\
&\leq 2\overline{N}_{(2}\left(r,0,f^n\right) + T\left(r,\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right) - T\left(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right) \\
&\quad + N_{k+2}\left(r,0,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + 2\overline{N}_{(2}\left(r,0,g^n\right)
\end{aligned}$$

$$
\begin{aligned}
& +T\left(r,\left(\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)}\right) - T\left(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right) \\
& +N_{k+2}\left(r,0,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right) + \frac{5}{2}\overline{N}\,(r,\infty,f) \\
& +\frac{5}{2}\overline{N}\left(r,\infty,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right) \\
& +\overline{N}\,(r,\infty,g) + \overline{N}\left(r,\infty,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right) + \frac{1}{2}N\,(r,0,f) \\
& +\frac{1}{2}N_{k+1}\left(r,0,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + \frac{1}{2}k\overline{N}\left(r,\infty,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right) \\
& +S(r,f) + S(r,g) \\
\leq\ & 2T(r,f) + T\left(r,\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right) + T(r,f^n) - T(r,f^n) \\
& -T\left(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + d(k+2)T(r,f) + 2T(r,g) \\
& +T\left(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right) + k\overline{N}\left(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right) \\
& -T\left(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right) \\
& +d(k+2)T(r,g) + \frac{5}{2}T(r,f) + \frac{5}{2}\lambda T(r,f) + T(r,g) \\
& +\lambda T(r,g) + \frac{1}{2}T(r,f) \\
& +\frac{1}{2}d(k+1)T(r,f) + \frac{1}{2}k\lambda T(r,f) + S(r,f) + S(r,g) \\
\leq\ & 2T(r,f) + T(r,F) - T(r,F_1) + d(k+2)T(r,f) \\
& +2T(r,g) + k\lambda T(r,g)
\end{aligned}
$$

$$+d(k+2)T(r,g)+\frac{5}{2}T(r,f)+\frac{5}{2}\lambda T(r,f)+T(r,g)$$
$$+\lambda T(r,g)+\frac{1}{2}T(r,f)$$
$$+\frac{1}{2}d(k+1)T(r,f)+\frac{1}{2}k\lambda T(r,f)+S(r,f)+S(r,g)$$

That is,

$$\begin{aligned} T(r,F_1) &\leq \left(d(k+2)+\frac{1}{2}k\lambda+3+\lambda\right)(T(r,f)+T(r,g)) \\ &\quad +\left(\frac{1}{2}d(k+1)+\frac{3}{2}\lambda+2\right)T(r,f)+\frac{1}{2}k\lambda T(r,g)+S(r,f)+S(r,g) \end{aligned}$$

$$\begin{aligned} (n-\lambda)T(r,f) &\leq \left(d(k+2)+\frac{1}{2}k\lambda+3+\lambda\right)(T(r,f)+T(r,g)) \\ &\quad +\left(\frac{1}{2}d(k+1)+\frac{3}{2}\lambda+2\right)T(r,f)+\frac{1}{2}k\lambda T(r,g) \\ &\quad +S(r,f)+S(r,g) \end{aligned} \tag{3.8}$$

Similarly

$$\begin{aligned} (n-\lambda)T(r,g) &\leq \left(d(k+2)+\frac{1}{2}k\lambda+3+\lambda\right)(T(r,f)+T(r,g)) \\ &\quad +\left(\frac{1}{2}d(k+1)+\frac{3}{2}\lambda+2\right)T(r,g)+\frac{1}{2}k\lambda T(r,f) \\ &\quad +S(r,f)+S(r,g) \end{aligned} \tag{3.9}$$

From (3.8) and (3.9), we get

$$\left(n-\lambda\left(\frac{3}{2}k+\frac{7}{2}\right)-d\left(2(k+2)+\frac{1}{2}(k+1)\right)-8\right)(T(r,f)+T(r,g))\leq S(r,f)+S(r,g)$$

Which is contradiction to $n\geq\lambda\left(\frac{3}{2}k+\frac{7}{2}\right)+d\left(2(k+2)+\frac{1}{2}(k+1)\right)+9$.

Subcase 1.3 : When $l=0$.
As in Subcase 1.1 and using (3.5) and Lemma 2.4, we simplify

$$\begin{aligned} T(r,F) &= T(r,F_2)+S(r,F_2) \\ &\leq N_2(r,0,F_2)+N_2(r,0,G_2)+3\overline{N}(r,\infty,F_2)+2\overline{N}(r,\infty,G_2) \\ &\quad +\overline{N}_*(r,\infty,F_2,G_2)+2\overline{N}(r,0,F_2)+\overline{N}(r,0,G_2)+S(r,F_2)+S(r,G_2) \\ &\leq N_2(r,0,F)+N_2(r,0,G)+3\overline{N}(r,\infty,F)+2\overline{N}(r,\infty,G) \\ &\quad +\overline{N}_*(r,\infty,F,G)+2\overline{N}(r,0,F)+\overline{N}(r,0,G)+S(r,F)+S(r,G) \end{aligned}$$

$$\begin{aligned}
&\leq N_2(r,0,F)+N_2(r,0,G)+4\overline{N}(r,\infty,F)+2\overline{N}(r,\infty,G)+2\overline{N}(r,0,F)\\
&\quad+\overline{N}(r,0,G)+S(r,F)+S(r,G)\\
&\leq 2T(r,f)+T(r,F)-T(r,F_1)+d(k+2)T(r,f)+2T(r,g)+k\lambda T(r,g)\\
&\quad+d(k+2)T(r,g)+4\overline{N}(r,\infty,f^n)+4\overline{N}\left(r,\infty,\left(\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)^{(k)}\right)\\
&\quad+2\overline{N}(r,\infty,g^n)+2\overline{N}\left(r,\infty,\left(\prod_{j=1}^{d}g(z+c_j)^{s_j}\right)^{(k)}\right)+2\overline{N}\left(r,0,f^n\right)\\
&\qquad+2\overline{N}\left(r,0,\left(\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)^{(k)}\right)+\overline{N}\left(r,0,g^n\right)\\
&\qquad+\overline{N}\left(r,0,\left(\prod_{j=1}^{d}g(z+c_j)^{s_j}\right)^{(k)}\right)+S(r,f)+S(r,g)
\end{aligned}$$

That is,

$$\begin{aligned}
T(r,F_1) \leq\; & 2T(r,f)+d(k+2)T(r,f)+2T(r,g)+k\lambda T(r,g)+d(k+2)T(r,g)\\
&+4T(r,f)+4\overline{N}\left(r,\infty,\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)+2T(r,g)\\
&+2\overline{N}\left(r,\infty,\prod_{j=1}^{d}g(z+c_j)^{s_j}\right)+2T(r,f)+2N_{k+1}\left(r,0,\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)\\
&+2k\overline{N}\left(r,\infty,\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)+T(r,g)+N_{k+1}\left(r,0,\prod_{j=1}^{d}g(z+c_j)^{s_j}\right)\\
&+k\overline{N}\left(r,\infty,\prod_{j=1}^{d}g(z+c_j)^{s_j}\right)+S(r,f)+S(r,g)
\end{aligned}$$

$$\begin{aligned}
(n-\lambda)T(r,f) \leq\; & (d(k+2)+d(k+1)+2k\lambda+2\lambda+5)(T(r,f)+T(r,g))\\
&+(2\lambda+d(k+1)+3)T(r,f)+S(r,f)+S(r,g) \qquad (3.10)
\end{aligned}$$

Similarly,

$$\begin{aligned}
(n-\lambda)T(r,g) \leq\; & (d(k+2)+d(k+1)+2k\lambda+2\lambda+5)(T(r,f)+T(r,g))\\
&+(2\lambda+d(k+1)+3)T(r,g)+S(r,f)+S(r,g) \qquad (3.11)
\end{aligned}$$

From (3.10) and (3.11), we get

$$(n-7\lambda-d(2k+4+2k+2+k+1)-4k\lambda-13)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g)$$

Which is contradiction to $n \geq 7\lambda + d(5k+7) + 4k\lambda + 14$.

Case 2. We now assume that $H \equiv 0$. That is

$$\left(\frac{F_2''}{F_2'} - \frac{2F_2'}{F_2'-1}\right) - \left(\frac{G_2''}{G_2'} - \frac{2G_2'}{G_2'-1}\right) = 0$$

Integrating both sides twice, we get

$$\frac{1}{F_2-1} = \frac{A}{G_2-1} + B \tag{3.12}$$

where $A(\neq 0)$ and B are constants. From (3.12) , we have following subcases.

Subcase 2.1. Let $B \neq 0$ and $A = B$. Then from (3.12), we get

$$\frac{1}{F_2-1} = \frac{BG_2}{G_2-1} \tag{3.13}$$

If $B = -1$, then from (3.13), we obtain, $F_2G_2 = 1$.

That is, $f(z)^n \left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)} \cdot g(z)^n \left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)} = z^2$

We proceed as in the proof of Theorem 1.7, we obtain $\prod_{j=1}^{d} f(z+c_j)^{s_j} = c_1e^{cz^2}$ and $\prod_{j=1}^{d} g(z+c_j)^{s_j} = c_2e^{cz^2}$, where c_1, c_2 and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.

If $B \neq -1$, from (3.13), we have

$$\frac{1}{F_2} = \frac{BG_2}{(1+B)G_2-1} \quad and \quad therefore \quad \overline{N}\left(r, \frac{1}{B+1}, G_2\right) = \overline{N}(r,0,F_2). \tag{3.14}$$

Using second fundamental theorem of Nevanlinna, we get

$$\begin{aligned}
T(r,G) &= T(r,G_2) + S(r,G) \\
&\leq \overline{N}(r,0,G_2) + \overline{N}(r,\infty,G_2) + \overline{N}\left(r, \frac{1}{B+1}, G_2\right) + S(r,G) \\
&\leq \overline{N}(r,0,G_2) + \overline{N}(r,\infty,G_2) + \overline{N}(r,0,F_2) + S(r,G) \\
&\leq \overline{N}(r,0,G) + \overline{N}(r,\infty,G) + \overline{N}(r,0,F) + S(r,G)
\end{aligned}$$

Using this with Lemma 2.8, we deduce that

$$\begin{aligned}
(n-\lambda)T(r,g) &\leq \overline{N}\left(r,0,g(z)^n\right)+\overline{N}\left(r,0,\left(\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)}\right)+\overline{N}(r,\infty,g(z)^n)\\
&\quad +\overline{N}\left(r,\infty,\left(\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)}\right)+\overline{N}\left(r,0,f(z)^n\right)\\
&\quad +\overline{N}\left(r,0,\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right)+S(r,g)\\
&\leq T(r,g)+k\overline{N}\left(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)+N_{k+1}\left(r,\frac{1}{\prod_{j=1}^{d} g(z+c_j)^{s_j}}\right)\\
&\quad +T(r,g)+\overline{N}\left(r,\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)+\overline{N}\left(r,\frac{1}{f(z)^n}\right)\\
&\quad +N_{k+1}\left(r,\frac{1}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right)+k\overline{N}\left(r,\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)+S(r,g)\\
&\leq T(r,g)+k\lambda T(r,g)+d(k+1)T(r,g)+T(r,g)+\lambda T(r,g)\\
&\quad +T(r,f)+k\lambda T(r,f)+d(k+1)T(r,f)+S(r,f)+S(r,g)
\end{aligned}$$

Thus,

$$\begin{aligned}
(n-\lambda)T(r,g) &\leq (d(k+1)+k\lambda+1)(T(r,f)+T(r,g))\\
&\quad +(1+\lambda)T(r,g)+S(r,f)+S(r,g) \qquad (3.15)\\
&\textit{Similarly}\\
(n-\lambda)T(r,f) &\leq (d(k+1)+k\lambda+1)(T(r,f)+T(r,g))\\
&\quad +(1+\lambda)T(r,f)+S(r,f)+S(r,g) \qquad (3.16)
\end{aligned}$$

From (3.15) and (3.16), we get

$$(n-2\lambda-2d(k+1)-2k\lambda-3)(T(r,f)+T(r,g))\leq S(r,f)+S(r,g)$$

which contradicts to $n\geq 2\lambda+2d(k+1)+2k\lambda+4$.

Subcase 2.2. Let $B\neq 0$ and $A\neq B$.
Then by (3.12), we get

$$F_2=\frac{(B+1)G_2-(B-A+1)}{BG_2+(A-B)}$$

and so, $\overline{N}\left(r, \dfrac{B-A+1}{B+1}, G_2\right) = \overline{N}(r, 0, F_2)$.
Proceeding as in Subcase 2.1 , we obtain a contradiction.

Subcase 2.3. Let $B = 0$ and $A \neq 0$. Then by (3.12), we get

$$F_2 = \frac{G_2 + A - 1}{A} \quad and \quad G_2 = AF_2 - (A-1)$$

If $A \neq 1$, we have $\overline{N}\left(r, \dfrac{A-1}{A}, F_2\right) = \overline{N}(r, 0, G_2)$ and $\overline{N}\,(r, A-1, G_2) = \overline{N}(r, 0, F_2)$.
Using the similar arguments as in Subcase 2.1, we obtain a contradiction. Thus $A = 1$. Which implies $F = G$ and therefore

$$f(z)^n \left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)} = g(z)^n \left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}.$$

■

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