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Some properties of the weighted Berezin transform in the unit disc and bidisc

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Abstract

For $c > -1$, let ν_c denote a weighted radial measure on the unit disc D normalized so that $\nu_c(D) = 1$. If $u \in L^1(D, \nu_c)$ and $z \in D$, we define $T_c u$ the weighted Berezin transform of u by $\big(T_c u\big)(z) = \int_D u\big(\varphi_z(x)\big)\, d\nu_c(x)$. Likewise, for $f \in L^1(D^2, \nu_{c_1} \times \nu_{c_2})$, the weighted Berezin transform $B_{c_1,c_2} f$ on D^2 is defined by

$$\big(B_{c_1,c_2}\big) f(z, w) = \int_D \int_D f\big(\varphi_z(x), \varphi_w(y)\big)\, d\nu_{c_1}(x) d\nu_{c_2}(y).$$

Here, we introduce and prove some boundedness, iterations and invariant properties of the weighted Berezin transform $T_c u$ and $B_{c_1,c_2} f$ on the unit disc and bidisc.

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1. Introduction

Let D be the unit disc of $\mathbb{C}$ and let m be the Lebesgue measure on $\mathbb{C}$ normalized to $m(D) = 1$. For $c > -1$, we define a normalized measure ν_c on D by

$$d\nu_c(z) = (c + 1)(1 - |z|^2)^c\, dm(z)$$

so that $\nu_c(D) = 1$. If a function $u \in L^1(D, \nu_c)$ is harmonic, then $u \circ \psi$ is also harmonic for every $\psi \in \mathrm{Aut}(D)$. Thus u satisfies a mean value property

$$\int_D (u \circ \psi)\, d\nu_c = u(\psi(0)) \quad \text{for every } \psi \in \mathrm{Aut}(D), \tag{1.1}$$

which is equivalent to saying that

$$\int_D (u \circ \varphi_z)\, d\nu_c = u(z)$$

for every $z \in D$, where φ_z is the canonical automorphism given by

$$\varphi_z(x) = \frac{z - x}{1 - \bar{z}x}.$$

Now for $c > -1$, $u \in L^1(D, \nu_c)$ and $z \in D$, we define $T_c u$ the weighted Berezin transform of u by

$$\begin{aligned} \big(T_c u\big)(z) &= \int_D u\big(\varphi_z(x)\big)\, d\nu_c(x) \\ &= \int_D u(x) K_c(z, x)\, d\nu_c(x) \end{aligned}$$

where

$$K_c(z, x) = \frac{(1 - |z|^2)^{2+c}}{|1 - \bar{z}x|^{4+2c}}. \tag{1.2}$$

Let μ be an invariant measure on D defined by $d\mu(z) = (1 - |z|^2)^{-2}\, dm(z)$, satisfying

$$\int_D u\, d\mu = \int_D u \circ \psi\, d\mu$$

for $u \in L^1(D, \mu)$ and $\psi \in \mathrm{Aut}(D)$. The advantage of using the invariant measure μ is that even though μ is not a finite measure on D, the space $L^\infty(D, m)$ is the dual space of $L^1(D, d\mu)$ on which the operator T has a nice behavior. Indeed, by simple calculation, we can show that T_c is a self-adjoint positive contraction on $L^p(D, \mu)$ for $1 \le p \le \infty$.

For $c_1, c_2 > -1$ and $f \in L^1(D^2, \nu_{c_1} \times \nu_{c_2})$, we can also define the weighted Berezin transform $B_{c_1,c_2} f$ on the bidisc D^2 by

$$\begin{aligned} \big(B_{c_1,c_2}\big) f(z, w) &= \int_D \int_D f\big(\varphi_z(x), \varphi_w(y)\big)\, d\nu_{c_1}(x) d\nu_{c_2}(y) \\ &= \int \int_{D^2} f(x, y) K_{c_1}(z, x) K_{c_2}(w, y)\, d\nu_{c_1}(x) d\nu_{c_2}(y). \end{aligned}$$

If f is $2-$harmonic, then the mean-value property implies that $B_{c_1,c_2} f = f$ for every $c_1, c_2 > -1$. Conversely, from the theorem of Furstenberg ([1], [2]), it is known that if $f \in L^\infty(D^2)$ satisfies $B_{c_1,c_2} f = f$ for some $c_1, c_2 > -1$, then it $2-$harmonic. Indeed, Furstenberg's result says much more is true: On any dimensional symmetric domain, a bounded function which is invariant under a weighted Berezin transform is harmonic with respect to the intrinsic metric. In [3] and [5], the author gave pure analytic proofs of Furstenberg's theorem in the bidisc. However, in Furstenberg's theorem, boundedness of function is essential. Indeed, the author ([4]) proved that for every $1 \le p < \infty$ and $c_1, c_2 > -1$, a function $f \in L^p(D^2, \ \nu_{c_1} \times \nu_{c_2})$ satisfying $B_{c_1,c_2} f = f$ needs not be 2−harmonic. This paper, we introduce and prove some additional properties of the weighted Berezin transform on the unit disc and bidisc. In section 2, we prove the boundedness properties of the weighted Berezin transform. We extend one of the main results (Lemma 3.1) of [5] on the iterations of the weighted Berezin transform. And we generalize the Proposition 3.2 of [4], which states that a certain class of functions on the bidisc invariant under the weighted Berezin transform is 2-harmonic.

2. Properties of the weighted Berezin transform

Here, we state and prove various properties of of the weighted Berezin transform. Next two propositions are about the boundedness of weighted Berezin operators.

Proposition 2.1. For $c_1, c_2 > -1$ and $p > 1$, B_{c_1,c_2} is bounded on $L^p(D^2, \nu_{c_1} \times \nu_{c_2})$. And B_{c_1,c_2} is not bounded on $L^1(D^2, \nu_{c_1} \times \nu_{c_2})$.

Proof. The proof on boundedness of B_{c_1,c_2} consists of two steps.

Step (i): First we prove that for $c > -1$ and $p > 1$ the operator T_c is bounded on $L^p(D, \nu_c)$. Let $q = p/(p-1)$ so that $1/p + 1/q = 1$. By (1.2) and 1.4.10 of [6], there exist $C_1, C_2 > 0$ such that

$$\int_D K_c(z,x)\,(1-|x|^2)^{-\frac{1}{p}}\,d\nu_c(x) \;\le\; C_1\,(1-|z|^2)^{-\frac{1}{p}} \tag{2.1}$$

and

$$\int_D K_c(z,x)\,(1-|z|^2)^{-\frac{1}{q}}\,d\nu_c(z) \;\le\; C_2\,(1-|x|^2)^{-\frac{1}{q}}. \tag{2.2}$$

Now for $u \in L^p(D, \nu_c)$, we have

$$\begin{aligned}
|T_c u(z)| &\le \int_D K_c(z,x)\,|u(x)|\,d\nu_c(x) \\
&= \int_D K_c(z,x)^{\frac{1}{q}}\,(1-|x|^2)^{-\frac{1}{pq}}\,K_c(z,x)^{\frac{1}{p}}\,(1-|x|^2)^{\frac{1}{pq}}\,|u(x)|\,d\nu_c(x) \\
&\le \left\{\int_D K_c(z,x)\,(1-|x|^2)^{-\frac{1}{p}}\,d\nu_c(x)\right\}^{\frac{1}{q}} \cdot \\
&\qquad \left\{\int_D K_c(z,x)\,(1-|x|^2)^{\frac{1}{q}}\,|u(x)|^p\,d\nu_c(x)\right\}^{\frac{1}{p}} \\
&\le C_1^{\frac{1}{q}}(1-|z|^2)^{-\frac{1}{pq}}\left\{\int_D K_c(z,x)\,(1-|x|^2)^{\frac{1}{q}}\,|u(x)|^p\,d\nu_c(x)\right\}^{\frac{1}{p}} \text{ by (2.1).}
\end{aligned}$$

Thus by Fubini's theorem along with (2.2),

$$\begin{aligned}
&\int_D |T_c u(z)|^p \, d\nu_c(z) \\
\leq \;& \int_D C_1^{\frac{p}{q}} \, (1-|z|^2)^{-\frac{1}{q}} \int_D K_c(z,x) \, (1-|x|^2)^{\frac{1}{q}} \, |u(x)|^p \, d\nu_c(x) \, d\nu_c(z) \\
\leq \;& C_1^{\frac{p}{q}} \int_D (1-|x|^2)^{\frac{1}{q}} \, |u(x)|^p \, C_2 \, (1-|x|^2)^{-\frac{1}{q}} \, d\nu_c(x) \\
= \;& C_1^{\frac{p}{q}} \, C_2 \int_D |u(x)|^p \, d\nu_c(x).
\end{aligned}$$

Hence if we let $C = C_1^{\frac{1}{q}} \, C_2^{\frac{1}{p}}$, then we have

$$\|T_c u\|_p \leq C\|u\|_p. \tag{2.3}$$

Step (ii): Let $f \in L^1(D^2, \nu_{c_1} \times \nu_{c_2})$, then

$$\big(B_{c_1,c_2} f\big)(z,w) = \int\int_{D^2} f(x,y) K_{c_1}(z,x) K_{c_2}(w,y) d\nu_{c_1}(x) d\nu_{c_2}(y).$$

Thus by Fubini's theorem along with (2.3),

$$\begin{aligned}
&\int\int_{D^2} |B_{c_1,c_2} f(z,w)|^p d\nu_{c_1}(x) d\nu_{c_2}(w) \\
\leq \;& \int\int_{D^2} \left\{ \int\int_{D^2} |f(x,y)| K_{c_1}(z,x) K_{c_2}(w,y) d\nu_{c_1}(x) d\nu_{c_2}(w) \right\}^p d\nu_{c_1}(x) d\nu_{c_2}(w) \\
= \;& \int\int_{D^2} \left\{ \int_D K_{c_2}(w,y) \left(\int_D |f(x,y)| \, K_{c_1}(z,x) \, d\nu_{c_1}(x) \right) dm(y) \right\}^p d\nu_{c_1}(x) d\nu_{c_2}(w) \\
\leq \;& \int\int_{D^2} C^p \left(\int_D |f(x,y)| \, K_{c_1}(z,x) d\nu_{c_1}(x) \right)^p d\nu_{c_1}(x) d\nu_{c_2}(w) \\
\leq \;& \int\int_{D^2} C^{2p} \, |f(z,w)|^p d\nu_{c_1}(x) d\nu_{c_2}(w).
\end{aligned}$$

This proves that B_{c_1,c_2} is a bounded operator on $L^p(D^2, \nu_{c_1} \times \nu_{c_2})$ for $p > 1$.
When $p = 1$, the norm of B_{c_1,c_2} on $L^1(D^2, \nu_{c_1} \times \nu_{c_2})$ is

$$\begin{aligned}
\|B_{c_1,c_2}\| &= \sup_{(x,y)\in D\times D} \int\int_{D^2} K_{c_1}(z,x) K_{c_2}(w,y) d\nu_{c_1}(z) d\nu_{c_2}(w) \\
&= \sup_{(x,y)\in D\times D} \int_D \frac{(1-|z|^2)^{2+c_1}}{|1-x\overline{z}|^{4+2c_1}} d\nu_{c_1}(z) \int_D \frac{(1-|w|^2)^{2+c_2}}{|1-y\overline{w}|^{4+2c_2}} \, d\nu_{c_2}(w)
\end{aligned}$$

But by 1.4.10 of [6], we get

$$\int_D \frac{(1-|z|^2)^{2+c_1}}{|1-z\overline{x}|^{4+2c_1}} \, d\nu_{c_1}(z) \; \approx \; \log \frac{1}{1-|x|^2}$$

which is unbounded on D. Hence B_{c_1,c_2} is not bounded on $L^1(D^2, \nu_{c_1} \times \nu_{c_2})$ for any $c_1, c_2 > -1$ and this completes the proof of the proposition. ■

Even though T_c is not bounded on $L^1(D, \nu_c)$, next proposition tells that it is bounded on $L^1(D, m)$ when $c > 0$ and its norm tends to 1 as $c \to \infty$.

Proposition 2.2. If $c > 0$ then T_c is bounded on $L^1(D, m)$ and $\lim_{c\to\infty} \|T_c\| = 1$.

Proof. The norm of T_c on $L^1(D, m)$ is

$$\|T_c\| \;=\; \sup_{x\in D} (c+1)(1-|x|^2)^c \int_D \frac{(1-|z|^2)^{2+c}}{|1-\bar{x}z|^{4+2c}} dm(z).$$

By 1.4.10 of [6], if $c > 0$ then we get

$$\int_D \frac{(1-|z|^2)^{2+c}}{|1-z\bar{x}|^{4+2c}}\, dm(z) \;\approx\; (1-|x|^2)^{-c},$$

which implies T_c is bounded on $L^1(D, m)$. Indeed, by direct calculation, we get

$$\begin{aligned} \|T_c\| &= \sup_{x\in D} (c+1)(1-|x|^2)^c \int_D \frac{(1-|z|^2)^{2+c}}{|1-\bar{x}z|^{4+2c}} dm(z) \\ &= \sup_{x\in D} (c+1)(1-|x|^2)^c \frac{c+2}{c(c+1)} \sum_{k=0}^{\infty} \frac{\Gamma^2(k+c+2)\,|x|^{2k}}{k!\,\Gamma(k+c+4)}. \end{aligned}$$

Using the inequality $\Gamma^2(a+b) \le \Gamma(2a+b)\Gamma(b)$ for $a, b > 0$, we have

$$\sum_{k=0}^{\infty} \frac{\Gamma^2(k+c+2)\,|x|^{2k}}{k!\,\Gamma(k+c+4)} \le \sum_{k=0}^{\infty} \frac{\Gamma(k+c)\,|x|^{2k}}{k!\,\Gamma(c)} = (1-|x|^2)^{-c.}$$

Hence, if $c > 0$ then we get

$$1 \le \|T_c\| \le \frac{c+2}{c}$$

on $L^1(D, m)$. By taking $c \to \infty$ we complete the proof. ■

Lemma 3.1 of [5] is one of the the crucial ingredients of that paper stating that: Suppose $u \in L^1(D, \mu)$ is radial and $c > -1$, then $\lim_{n\to\infty} \int_D \left|T_c^n u\right| d\mu = 0$ if and only if $\int_D u\, d\mu = 0$.

Next proposition is a generalization and extension of Lemma 3.1 of [5]. We denote $L^p_R(D^n)$ as the subspace of $L^p(D^n)$ which consists of all radial functions so that

$$L^p_R(D^n) = \{f \in L^p(D^n) | f(z_1, \cdots, z_n) = f(|z_1|, \cdots, |z_n|), \text{ for every } (z_1, \cdots, z_n) \in D^n\}.$$

Proposition 2.3. If $u \in L^1(D,\mu)$ is radial, then for $c > -1$ we have

$$\lim_{n\to\infty} \int_D \left|T_c^n u\right| d\mu = \left| \int_D u \, d\mu \right|.$$

Proof. Let's denote $A = \left\{\ell \in L_R^\infty(D) \mid \|\ell\|_\infty \le 1\right\},\ E_k = T_c^k A$ and $E = \bigcap_{k=1}^{\infty} E_k$. Then for $u \in L_R^1(\mu)$,

$$\begin{aligned} \int_D | \, T_c^k u \, | \, d\mu &= \sup\left\{ \left| \int_D (T_c^k u)\cdot \ell \, d\mu \right| \,\middle|\, \ell \in A \right\} \\ &= \sup\left\{ \left| \int_D u \cdot (T_c^k \ell) \, d\mu \right| \,\middle|\, \ell \in A \right\}. \end{aligned}$$

Hence

$$\lim_{k\to\infty} \| \, T_c^k u \, \|_{L^1(\mu)} \ \ge\ \sup \left\{ \left| \int_D u \cdot h \, d\mu \right| \,\middle|\, h \in E \right\}. \tag{2.4}$$

On the other hand, for every $\epsilon > 0$ and $k \ge 1$ there exists $h_k \in A$ with

$$\begin{aligned} \| \, T_c^k u \, \|_{L^1(\mu)} &\le \left| \int_D \left(T_c^k u\right) \cdot h_k \, d\mu \right| + \epsilon \\ &= \left| \int_D u \cdot \left(T_c^k h_k\right) d\mu \right| + \epsilon. \end{aligned}$$

Since E_k is weak $*$ compact and $E_k \downarrow E$, E is also weak $*$ compact. If g is a weak $*$ limit of a subsequence $\{T_c^{k_j}(h_{k_j})\}$ of $\{T_c^k h_k\}$, then $g \in E$ and

$$\begin{aligned} \left| \int_D u \cdot g \, d\mu \right| &= \lim_{j\to\infty} \left| \int_D u \left(T_c^{k_j} h_{k_j}\right) d\mu \right| \\ &\ge \lim_{j\to\infty} \| \, T_c^{k_j} u \, \|_{L^1(\mu)} - \epsilon. \end{aligned}$$

Hence we have

$$\lim_{k\to\infty} \| \, T_c^k u \, \|_{L^1(\mu)} \ \le\ \sup \left\{ \left| \int_D u \cdot h \, d\mu \right| \,\middle|\, h \in E \right\}. \tag{2.5}$$

From (2.4), (2.5) we get

$$\lim_{k\to\infty} \| \, T_c^k u \, \|_{L^1(\mu)} = \sup \left\{ \left| \int_D u \cdot h \, d\mu \right| \,\middle|\, h \in E \right\}. \tag{2.6}$$

From (2.6) and Lemma 3.1 of [5], if $u \in L_R^1(\mu)$ then for every $h \in E$

$$\int_D u \, d\mu = 0 \quad \text{if and only if} \quad \int_D u \cdot h \, d\mu = 0.$$

Therefore, we conclude

$$E = \{ c \in \mathbb{C} \mid |c| \leq 1 \},$$

and we can rewrite (2.6) as

$$\begin{aligned} \lim_{k\to\infty} \| T_c^k u \|_{L^1(\mu)} &= \sup \left\{ \left| \int_D cu \, d\mu \right| \,\middle|\, |c| \leq 1 \right\} \\ &= \left| \int_D u \, d\mu \right|. \end{aligned}$$

■

Next corollary comes directly from Proposition 2.3.

Corollary 2.4. Let $f \in L^1_R(D^2, \mu \times \mu)$, then for every $c_1, c_2 > -1$

$$\lim_{n\to\infty} \int\int_{D^2} |B^n_{c_1,c_2} f| \, d\mu \times d\mu = \left| \int\int_{D^2} f \, d\mu \times d\mu \right|.$$

Proof. For $f \in L^1(D^2, \mu \times \mu)$ and $q \in L^\infty(D^2)$ we use the property

$$\int\int_{D^2} (B_{c_1,c_2} f) \cdot g \, d\mu \times d\mu = \int\int_{D^2} f \cdot (B_{c_1,c_2} g) \, d\mu \times d\mu.$$

by self-adjointness of B_{c_1,c_2}. The rest of the proof is identical to that of the Proposition 2.3. ■

For $f \in L^1(\nu_{c_1} \times \nu_{c_2})$, we define the radialization of f by

$$(Rf)(z, w) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f(ze^{i\theta}, we^{i\xi}) \, d\theta \, d\xi$$

so that $Rf \in L^1_R(D^2, \nu_{c_1} \times \nu_{c_2})$. Next proposition gives a generalization of Proposition 3.2 of [4], which states that: If $f \in L^1(D^2, \nu_{c_1} \times \nu_{c_2})$ satisfies $B_{c_1,c_2} f = f$ and $R(f \circ \psi) \in L^\infty(D^2)$ for every $\psi \in \mathrm{Aut}(D^2)$, then f is 2-harmonic.

We denote for $f \in L^p(D^2, \mu \times \mu)$ and $q \in L^q(D^2, \mu \times \mu)$ $(1 \leq p \leq \infty,\ 1/p + 1/q = 1)$,

$$\langle f, g \rangle = \int\int_{D^2} f \cdot g \, d\mu \times d\mu.$$

Then by self-adjointness of B_{c_1,c_2}, we get $\langle B_{c_1,c_2} f, g \rangle = \langle f, B_{c_1,c_2} g \rangle$.

Proposition 2.5. If $f \in L^1(D^2, \nu_{c_1} \times \nu_{c_2})$ and $r \in \mathbb{C}$ with $|r| = 1$ satisfy $B_{c_1,c_2} f = rf$ and $R(f \circ \psi) \in L^\infty(D^2)$ for every $\psi \in \mathrm{Aut}(D^2)$, then f is 2-harmonic.

Proof. First, we assume that f is a bounded radial function. Suppose $f \in L^\infty_R(D^{2)}$ satisfy $B_{c_1,c_2} f = rf$ for some $|r| = 1$. Pick any $g \in L^1_R(D^2, \mu \times \mu)$ satisfying

$$\int\int_{D^2} g \, d\mu \times d\mu = 0.$$

Then by Corollary 2.4, we get

$$\lim_{k\to\infty} \|(B_{c_1,c_2})^k g\|_{L^1(\mu\times\mu)} = 0.$$

Hence

$$\lim_{k\to\infty} \left|\langle\, (B_{c_1,c_2})^k g,\, f\,\rangle\right| \;\le\; \|f\|_\infty \lim_{k\to\infty} \|B^k g\|_{L^1(\mu\times\mu)} = 0.$$

But for all $k \ge 0$, by self-adjointness of B_{c_1,c_2} we have

$$\langle\, (B_{c_1,c_2})^k g,\, f\,\rangle = \langle\, g,\, (B_{c_1,c_2})^k f\,\rangle = r^k\langle g, f\rangle.$$

Hence $\langle g, f\rangle = 0$. This implies that f is a constant, which implies $r = 1$ since $f \not\equiv 0$. For a general $f \in L^\infty(D^n)$, the radialization Rf satisfies

$$B_{c_1,c_2}(Rf) = R(B_{c_1,c_2} f) = rRf.$$

Hence Rf is a constant and $r = 1$.
The remaining part of the proof is almost identical to that of Proposition 3.2 of [4].
Let $\psi(z_1, z_2) = \big(\psi_1(z_1), \psi_2(z_2)\big)$ for $\psi_1, \psi_2 \in \mathrm{Aut}(D^2)$.
Then for every $f \in L^1(D^2, \nu_{c_1} \times \nu_{c_2})$, we get $B_{c_1,c_2}(f \circ \psi) = (B_{c_1,c_2} f) \circ \psi$.
Since R, B_{c_1,c_2} are bounded on $L^\infty(D^2)$ with norm 1, by Fubini's theorem we have

$$B_{c_1,c_2}\big(R(f\circ\psi)\big) = R\big(B_{c_1,c_2}(f\circ\psi)\big) = R\big((B_{c_1,c_2} f)\circ\psi\big) = R\big(f\circ\psi\big).$$

Thus by the theorem of Furstenberg, $R(f \circ \psi)$ is 2−harmonic and radial, which means $R(f \circ \psi)$ is constant. Therefore, we have

$$(f\circ\psi)(0,0) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} (f\circ\psi)(w_1 e^{i\theta}, w_2 e^{i\xi})\, d\theta\, d\xi \quad \text{for every } \; w_1, w_2 \in D.$$

Now pick $z_1, z_2 \in D$ and let $\psi \in Aut(D^2)$ be $\psi = (\psi_1, \psi_2) = (\varphi_{z_1}, \varphi_{z_2})$. Then the above equality becomes

$$f(z_1, z_2) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f\left(\varphi_{z_1}(w_1 e^{i\theta}),\, \varphi_{z_2}(w_2 e^{i\xi})\right) d\theta\, d\xi \quad \text{for every } \; w_1, w_2 \in D.$$

Now put $z_2 = 0$, then we get $\Delta_1 f = 0$. Likewise we also get $\Delta_2 f = 0$, which completes the proof. ■

References

[1] H. Furstenberg, *A Poisson formula for semi-simple Lie groups*, Ann. of Math., (2) 77, 1963, 335–386.

[2] H. Furstenberg, *Boundaries of Riemannian symmetric spaces*, Symmetric spaces (Short Courses, Washington Univ., St. Louis, Mo., 1969–1970),

[3] J. Lee., *An invariant mean value property in the polydisc*, Illinois J. Math. 42(1998), Issue 3, 406–419.

[4] J. Lee, *Weighted Berezin transform in the polydisc*, J. Math. Anal. Appl., 338 (2008), no. 2, 1489–1493.

[5] J. Lee, *Some properties of the Berezin transform in the bidisc*, Comm. Korean Math. Soc., 32(2017), vol. 3, 779–787.

[6] W. Rudin, Function theory in the unit ball of $\mathbb{C}^n$, Springer-Verlag, New York Inc., 1980.