Non-Smooth Multi objective Programs with Generalized $(\phi-v-\rho)$-Invexity

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Abstract

In this paper, we introduce a new class of functions called locally as Lipschitz $(\phi, v, \rho)$-invex functions. Some properties are studied. In this, various optimality and duality results were derived for such a class of non smooth multi objective programming problem.

Key words: Non smooth multi objective programme, optimality, duality, $(\phi, v, \rho)$ – invexity.

1. INTRODUCTION

In the recent past, several attempts were made by different authors to define various classes of differentiable and non-differentiable non-convex functions to study their optimality and duality results [1, 29]. First Hanson et al. [10] generalized the concept of convexity by introducing invexity for differentiable mathematical programming problem. Later, Craven [29] named it as invex. Many generalizations of invexity concept have been given in the literature [1,2,7,10] for both smooth and non-smooth optimization problems. Recently, many authors (eg,[7,8,12,13]) studied various optimality and duality results for non-smooth locally Lipschitz optimization problems using Clarke’s sub differentials. Further, Hanson’s invex function was also
generalized for non smooth problem. In [7] Caristi et al., generalized invexity notion to the non-differentiability case by defining invexity for Lipschitz function. Also, he used principal analytic tool as a generalized gradient in the Clarke’s sense.

In [24, 25], Reddy and Mukherjee generalized invexity notion to the non-differentiable case by defining invexity for Lipschitz function. Also the definition of non smooth invex function was weakened function by different authors (eg.[1,18,19,25]) to establish different optimality and duality results. Consequently other generalized convex functions introduced like \( \rho \) convexity Vail et. Al.[30] F- convexity by Hanson and Mond [31], \((F, \rho)\)-convexity by Preda [23], \((\phi - \rho)\)- invexity by Caristi et al [7], and generalized \((\phi, \rho)\)-invexity by Antczak et al [1].

The main aim of this paper is to introduce \((\phi, v, \rho)\)-invexity for locally Lipschitz \((\phi, v, \rho)\)invex function to establish optimality conditions and Mond-Weir duality results for a new class of function using non smooth programs. Further, Kuk, Lee and Kim [12] discussed \( v - \rho \)-invexity for vector valued functions of non smooth multi objective programs.

2. PRELIMINARIES

Let \( \mathbb{R}^n \) be the n-dimensional Euclidean space and \( \mathbb{R}^n_+ \) be its non-negative part of \( \mathbb{R}^n \). Throughout this chapter, the following convections for vectors in \( \mathbb{R}^n \) will be used \( x > y \) if and only if \( x_i > y_i, i = 1,2,\ldots,n \) \( x \geq y \) if and only if \( x_i \geq y_i \) for \( i = 1,2,\ldots,n \) and \( x \) not less than \( y \) is the negative of \( x > y \).

**Definition 2.1.** A real –valued function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be locally Lipschitz if for \( z \in \mathbb{R}^n \) there exists a positive constant \( K \) and a neighborhood \( N \) of \( z \), such that, for \( x, y \in N \), we have \( |f(x) - f(y)| \leq K\|x - y\| \), where \( \| \| \) denotes a norm in \( \mathbb{R}^n \).

**Definition 2.2.** The Clarke generalized directional derivative of a locally Lipschitz function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) at \( x \in \mathbb{R}^n \) in the direction \( d \in \mathbb{R}^n \) denoted by \( f^0(x;d) \)

\[
f^0(x;d) = \lim_{\substack{y \to x \\text{or} \\text{for} \ t \downarrow 0}} \frac{f(y+td) - f(y)}{t}
\]
Definition 2.3. The Clarke generalized sub gradient of \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) at \( x \in \mathbb{R}^n \) denoted \( \partial f(x) \) is defined as:

\[
\partial f(x) = \left\{ \xi \in \mathbb{R}^n : f^0(x, d) \geq \xi^T d, \forall d \in \mathbb{R}^n \right\}
\]

Definition 2.4. Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a locally Lipschitz function on \( \mathbb{R}^n \). A point \( \bar{x} \in \mathbb{R}^n \) is said to be a stationary point of \( f \), if \( 0 \in \partial f(\bar{x}) \).

In this chapter, we consider the following non smooth multi objective composite program (MCP).

(MCP) Minimize \( (f_1(F_1(x)), f_2(F_2(x)), \ldots, f_n(F_n(x))) \)

Subject to \( g_j(G_j(x)) \leq 0, j = 1, 2, \ldots, m \)

And \( x \in X = \left\{ x \in \mathbb{R}^n : g_j(G_j(x)) \leq 0, j = 1, 2, \ldots, m \right\} \)

Where \( f_i(F_i) : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, 2, \ldots, n \)

\( g_j(G_j) : \mathbb{R}^n \rightarrow \mathbb{R}, j = 1, 2, \ldots, m \) are locally Lipschitz functions.

Now, we define \((\phi - \nu - \rho)\)-invexity for locally Lipschitz functions as follows.

Definition 2.5. Let \( f_i(F_i) : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( g_j(G_j) : \mathbb{R}^n \rightarrow \mathbb{R} \) be locally Lipschitz functions for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \), respectively.

(i) \( f(F) = (f_1(F_1), f_2(F_2), \ldots, f_n(F_n)) \) is \((\phi - \nu - \rho)\)-invex with respect to function \( \eta \) and \( \theta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), if there exists a function \( \phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n+1} \rightarrow \mathbb{R} \)

Where \( \phi(x,u,\ldots) \) is convex on \( \mathbb{R}^{n+1} \phi(x,u,(0,a)) \geq 0 \) for every \( x \in \mathbb{R}^n \), and any \( a \in \mathbb{R} \), such that \( \alpha_i(x,u)((f_i((F_i)(x)) - f_i(F_i))(x)) \geq \phi(x,u)(\xi, \eta) + \rho \| \phi(x,u) \|^2 \)

(ii) \( g(G) = (g_1(G_1), g_2(G_2), \ldots, g_m(G_m)) \) is said to be \((\phi - \nu - \sigma)\)-invex with respect to function \( \eta \) and \( \theta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), there exists \( \beta_j : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \setminus \{0\} \) and \( \sigma_j \in \mathbb{R}, j = 1, 2, \ldots, m \), and \( \phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) where \( \phi(x,u) \) is convex on \( \mathbb{R}^{n+1} \), \( \phi = (x,u,(0,a)) \geq 0 \) for every \( x \in \mathbb{R}^n \), and any \( a \in \mathbb{R} \), such that \( \beta_j(x,u)[(g_j(G_j)(x) - g_j(G_j))(u)] \geq \phi(x,u)\xi \eta(x,u) + \sigma \| (x,u) \| \)^2 \)

Based on Sawaragi et.al [32], we have the following.
Definition 2.6. Let \( u \in X \subset \mathbb{R}^n \) is said to be a weak minimum of (MCP) if there exists no integers \( x \in X \subset \mathbb{R}^n \), such that \( f_i(F_i(x)) \leq f_i(F_i(u)), i = 1, 2, ..., p \).

3. RESULTS

Theorem 3.1. (Generalized Fritz - John Necessary optimality conditions)

Let \( \bar{x} \in \mathbb{R}^n \) be an optimal solution in problem (MCP) then, there exists \( \bar{\lambda} \in \mathbb{R}^m_+ \), \( \bar{\tau} \in \mathbb{R}^m_+ \) such that

\[
0 \in \bar{\tau}(f_i'(F_i(x))) + \sum_{j=1}^{m} \bar{\lambda}_j \partial g_j(G_j(\bar{x}))
\]  
(3.1)

\[
\bar{\lambda}_j(g_j(G_j(\bar{x})) = 0, j \in \{1, 2, ..., m\}
\]  
(3.2)

\[
(\bar{\tau}, \bar{\lambda}) \neq (0, 0)
\]  
(3.3)

Then, corresponding Karush-Kuhn Tucker conditions.

Theorem 3.2. Let \( \bar{x} \in \mathbb{R}^n \) be an optimal solution in problem (MCP) and some suitable constraints qualification to be satisfied at \( \bar{x} \), then, there exists \( \bar{\lambda} \in \mathbb{R}^m \) such that

\[
0 \in \partial(F(\bar{x}))) + \sum_{j=1}^{m} \bar{\lambda}_j \partial g_j(G_j(\bar{x}))
\]  
(3.4)

\[
\bar{\lambda}_j(g_j(G_j(\bar{x})) = 0, j \in \{1, 2, ..., m\}
\]  
(3.5)

\[
\bar{\lambda}_j \in \mathbb{R}_+, j \in I
\]  
(3.6)

Definition 3.1. The point \( (\bar{x}, \bar{\lambda}) \in \mathbb{R}^n \times \mathbb{R}^m_+ \) is said to be a Karush-Kuhn-Tucker Point for the considered non smooth optimization problem (MCP), if there exits conditions (3.1), (3.2), (3.3, 3.4)), (3.5), (3.6) are satisfied at \( \bar{x} \) with the Lagrange multipliers \( \bar{\lambda} \in \mathbb{R}^m_+ \).

Definition 3.2. Let \( (x, \tau, \lambda) \) be a Lagrange’s function. Then, the Lagrange function associated with the constraint minimization problem (MCP) is the function \( L : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^m_+ \rightarrow \mathbb{R} \), defined as

\[
L(x_0, \lambda_0, \tau_0) = \tau_0 f(F(x_0)) + \sum_{j=1}^{m} \lambda_0 g_j(G_j)(x_0).
\]

3.1. Generalized Slater constraint qualification

Let \( \bar{x} \in \mathbb{R}^n \) is said to satisfy the Slater constraint qualification for the
Non smooth multiobjective programs with generalized \(-\text{invexity}\)

considered non smooth multi objective composite program, if there exists a feasible solution \(\tilde{x}\) such that \((g_j(G_j(\tilde{x}))) < 0, j \in I(\tilde{x})\). And more over \((g_j(G_j)), j \in I(\tilde{x})\) are locally Lipschitz \((\phi - \nu - \rho g_j(G_j))\) invex at \(\tilde{x}\) on \(R^n\).

Now, we state and prove optimality conditions for the considered non smooth multi objective composite program (MCP) under generalized \((\phi - \nu - p)\)-invexity.

**Theorem 3.3.** Let \((\tilde{x}, \tilde{\lambda}, \tilde{\tau}) \in R^n \times R_+ \times R_+^m\) be a feasible solution in the consider problem (MCP) at which Generalized Fritz John necessary optimality conditions are satisfied. Further, assume that the constraint functions \((g_j(G_j)), j \in I(\tilde{x})\) Lipschitz \((\phi - \nu - \rho g_j(G_j))\) invex at \(\tilde{x}\) on \(R^n\) and \(\sum_{j \in I(x)} \tilde{\lambda}_j \rho g_j(G_j) \geq 0\). If the generalized Slater’s constraint qualification is satisfied at \(\tilde{x}\) than the Generalized Karush-Khun Tucker optimality conditions are satisfied.

**Proof:** Suppose \(\tilde{x}\) be a feasible solution to the problem (MCP), at which the above stated Generalized Fritz-John optimality conditions \((3.1), (3.2), (3.3)\) are satisfied with the corresponding Lagrange multipliers \(\tilde{\lambda} \in R^m, \tilde{\tau} \in R^n\).

Again since the generalized Slater’s constraint qualification is satisfied at \(\tilde{x}\) then we need to prove that the Lagrange multiplies \(\tilde{x}\) associated with the objective function is not equal to 0.

We prove this by contradiction.

Let us suppose that \(\tilde{x} = 0\)

Then, by condition \((3.3)\) we have \(\sum_{j=1}^m \tilde{\tau}_j > 0\)

Let us denote \(\alpha_j = \frac{\tilde{\lambda}_j}{\sum_{j=1}^m \tilde{\tau}_j}, j \in I\)

Hence \(0 \leq \alpha_j \leq 1, j \in I\) and \(\sum_{j=1}^m \alpha_j = 1\)

Also, \(\alpha_j(g_j(G_j)(\tilde{x})), j \in I\) \((3.7)\)

But, the constraints functions \((g_j(G_j(\tilde{x}))), j \in I, (\tilde{x})\) are locally Lipschitz \((\phi - \nu - \rho_j)\) invex at \(\tilde{x}\) and \(R^n\) then, by det, we have

\[(g_j(G_j)) (x) - (g_j(G_j))(\tilde{x}) \geq \phi \eta(x, \tilde{x})(\xi_j, \rho(g_j(G_j))) + \rho \|\theta(x, \tilde{x})\|^p, j \in I(x)\]
Holds for any \( \zeta_j \in \partial g_j(G_j(\bar{x})), j \in I(\bar{x}) \) and for all \( \left( \frac{\partial \eta}{\partial t} \right)^T h_i(t,x,x^*,u^*). \)

Since \( 0 \leq \nu_j \leq 1, j \in I \) then, the above in equations yields,

\[
\alpha_i(x,\bar{x})(g_j(G_j)(x) - g_j(G_j)(\bar{x})) \geq \alpha_i \phi \eta(x,\bar{x}(\xi_j,\rho g_j(G_j))) + \rho \| \theta(x,\bar{x}) \|^2, \quad j \in I(\bar{x})
\]

Adding both sides of the above, we get

\[
\sum_{j \in I(\bar{x})} \alpha_j(g_j(G_j))(x) - \sum_{j \in I(\bar{x})} \alpha_j(\rho g_j(G_j))(x) \geq \sum_{j \in I(\bar{x})} \alpha_j \phi \eta(x,\bar{x}(\xi_j,\rho g_j(G_j)))
\]

But, by definition, \( \phi(x,\bar{x}) \) is a convex function on \( \mathbb{R}^{n+1} \).

Also, since \( 0 \leq \alpha_j \leq 1, j \in I \) and \( \sum_{j=1}^m \alpha_j = 1 \) then, definition of generalize invex,

we have

\[
\sum_{j \in I(\bar{x})} \alpha_j \phi \eta(x,\bar{x})(\xi_j,\rho g_j(G_j)) \geq \phi \eta(x,\bar{x}) \left( \sum_{j \in I(\bar{x})} \alpha_j \xi_j, \sum_{j \in I(\bar{x})} \alpha_j \rho g_j(G_j) \right)
\]

Now, combining (3.7) and (3.8), we obtain

\[
\sum_{j \in I(\bar{x})} \alpha_j(g_j(G_j))(x) - \sum_{j \in I(\bar{x})} \alpha_j(g_j(G_j))(\bar{x}) \geq \phi \eta(x,\bar{x}) \left[ \sum_{j \in I(\bar{x})} \alpha_j \xi_j, \sum_{j \in I(\bar{x})} \alpha_j \rho g_j(G_j) \right]
\]

But our assumption is \( \bar{x} = 0 \). Therefore, by Generalized Fritz- John necessary optimality condition (3.3), we have \( \sum_{j \in I(\bar{x})} \alpha_j \xi_j = 0 \) then by definition of \( \alpha_j \)

we have

\[
\sum_{j \in I(\bar{x})} \alpha_j \xi_j = 0
\]

From (3.10) and (3.11), one can obtain

\[
\sum_{j \in I(\bar{x})} \alpha_j(g_j(G_j))(x) - \sum_{j \in I(\bar{x})} \alpha_j(g_j(G_j))(\bar{x}) \geq \phi \eta(x,\bar{x},(0, \sum_{j \in I(\bar{x})} \alpha_j \rho g_j(G_j)))
\]

But from defamation of convexity we have \( \phi(x,\bar{x},(0,u)) \geq 0 \) for every \( x \in \mathbb{R}^n \) any \( a \in \mathbb{R}^r \).

By assumption we have

\[
\sum_{j \in I(\bar{x})} \alpha_j(g_j(G_j))(x) - \sum_{j \in I(\bar{x})} \alpha_j(g_j(G_j))(\bar{x}) \geq 0 \quad ...
\]

which holds for all \( x \in \mathbb{R}^n \).
According to assumption, generalized Slater’s constraints qualification is satisfied then, there exists a feasible point \( \tilde{x} \) such that \( (g_j(G_j))(\tilde{x}) < 0, \ j \in I(\tilde{x}). \) Since the above inequality (5.3.6.7) is satisfied for all \( x \in \mathbb{R}^n \) then it is also satisfied for \( x = \tilde{x} \)

\[
i.e. \quad \sum_{j \in I(\tilde{x})} \alpha_j (g_j(G_j))(\tilde{x}) = 0
\]

\[
0 \leq \alpha_j \leq 1, \ j \in I(\tilde{x}).
\]

\[
\sum_{j=1}^{m} \alpha_j = 1, \text{ and } (g_j(G_j))(\tilde{x}) < 0, \ j \in I(\tilde{x})
\]

We get

\[
\sum_{j \in I(\tilde{x})} \alpha_j (g_j(G_j))(\tilde{x}) < 0
\]

This is a contradiction to inequality (3.15), hence \( \tilde{x} \neq 0 \).

This completes the proof of the theorem.

**Theorem 3.4.** If \( (\tilde{x}, \tilde{\lambda}) \in \mathbb{R}^n \times \mathbb{R}_+^m \) is a Krause- Kuhn Tenses point in the problem (MCP),

Let us assume that the objective \( f_i(F_i) \) is locally Lipschitz \( (\phi - \nu - \rho f_i(F_i)) \) - invex at \( \tilde{x} \in \mathbb{R}^n \) also the constraint function \( g_j(G_j), \ j \in I(\tilde{x}) \) are locally Lipschitz

\[
(f - \nu - \rho g_j(G_j)) \text{-invex at } x \text{ on } \mathbb{R}^n \quad \text{and} \quad \rho f(F) + \sum_{j \in I(\tilde{x})} \lambda_j \rho g_j(G_j) \geq 0.
\]

Then \( \tilde{x} \) is an optimal for the problem (MCP).

**Proof:** Proof is straight forward [25].

The following theorem illustration sufficient optimality conditions

**Theorem 3.5.** If \( \tilde{x} \) is a feasible solution in problem (MCP), in which Generalized Krause- Kuhn- Tucker necessary optimality conditions are satisfied. Further, assume that the Lagrangian \( L(x, \tau, \tilde{\lambda}_0) \) locally Lipschitz \( (\phi - \nu - \rho) \)-invex for \( \tilde{x} \in \mathbb{R}^n \) with \( \rho \geq 0 \) then \( \tilde{x} \) is optimal solution to the problem (MCP).

**Proof:** Proof is similar to [25]
4. DUALITY

Here, we consider the Mond-Weir type duality for the problem (MCP) (MDP) maximize $f(F(y))$

Subject to $0 \in \partial(\tilde{r} f(F)) + \sum_{j=1}^{m} \tilde{\lambda}_j \partial(g_j(G_j))y$

$$\sum_{j=1}^{m} \tilde{\lambda}_j g_j(G_j(y)) \geq 0, \tilde{r} \in R,$$

$$\tilde{\lambda}_j \in R, (\tilde{r}, \tilde{\lambda}_j) \geq 0$$

Now, the set of feasible solutions are denoted by

$$D = \left\{ (y, \tilde{r}, \tilde{\lambda}_0) \in R^n \times R \times R^m : 0 \in \partial(\tilde{r} f(F)) + \sum_{j=1}^{m} \tilde{\lambda}_j g_j(G_j)(y) - \right.$$

$$\sum_{j=1}^{m} \tilde{\lambda}_j (g_j(G_j)(y)) \geq 0, \tilde{r} > 0, \tilde{\lambda}_0 \geq 0 \left\}$$

Also $E = \left\{ Y \in R^n : (y, \tilde{r}, \tilde{\lambda}_0) \in D \right\}$. The following is used in the sequel according Antczak [1].

**Lemma 4.1.** If $(y, \tilde{r}, \tilde{\lambda}_0)$ is a certain feasible solution for dual (MDP) Further, assume that $g_j(G_j) \quad j \in I(y)$ is locally Lipschitz $(\phi, v, \rho) -$ invex not necessary with respect to the same $\rho$ and $\eta$.

**Theorem 4.1. (Weak Duality).** If $\tilde{x}$ and $(\tilde{y}, \tilde{r}, \tilde{\lambda})$ are feasible solutions for primal problem (MCP) and corresponding dual problem (MDP), respectively further assume that $(f(F))$ is locally Lipschitz $(\phi - v - \rho f(F))$ invex at $\tilde{y}$ on $R^n UE$.

$$\tilde{r} \rho f(F) + \sum_{j \in I(y)} \tilde{\lambda}_j \rho g_j(G_j) \geq 0 \text{ then } (f(F(x)) \geq (f(F(y)))$$

**Proof:** proof is similar to [25].

**Theorem 4.2. (Strong Duality).** If $\tilde{x}$ is an optimal solution for problem (MCP) and certain Slater’s Constraint qualification is satisfied at $\tilde{x}$ then there exists $\tilde{r} \in R$ such that $\tilde{\lambda}_0 \in R^m$ such that $(\tilde{x}, \tilde{r}, \tilde{\lambda}_0)$ is feasible for (MDP) and the objective functions of primal (MCP) and dual (MDP) are equal.
Proof: proof is similar to [25].

Theorem: 4.3. (Converse Duality). Assume that \( \bar{x} \) and \((\bar{y}, \bar{z}, \bar{\lambda})\) are efficient solution of (MCP) and (MCD) respectively, then there exists vectors \( \bar{\lambda}, \mu \) such that the optimal values are equal.

Proof: proof is similar to [25].

5. CONCLUSION

In this paper, the authors discussed optimality conditions. As an application Mond-Weir type duality results were proved under generalized invexity. These are the dynamic generalizations of the earlier results existing in the literature as a special case.

REFERENCES


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Non smooth multiobjective programs with generalized \(-\)-invexity


