Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 14, Number 2 (2018), pp. 195-207

http://www.ripublication.com

Second Order Fuzzy Proximity Structures – I

A.Kalaichelvi

Professor, Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore – 641 043, Tamil Nadu, India.

Abstract

In this paper the concept of second order fuzzy proximity is introduced in twodifferent ways by extending the fuzzy proximity introduced by Artico.G., and Moresco.R. [1,4].

Keywords:AM1-fuzzy proximity, AM2-fuzzy proximity, AM1-fuzzy proximity mapping, Second order AM1-fuzzy proximity, Second order AM2-fuzzy proximity, Second order AM1-fuzzy proximity mapping.

INTRODUCTION

A fuzzy set on a set X is a map defined on X with values in I, where I is the closed unit interval [0, 1]. Equivalently fuzzy sets which are named as first order fuzzy sets in this paper deal with crisply defined membership functions or degrees of membership. It is doubtful whether, for instance, human beings have or can have a crisp image of membership functions in their minds. Zadeh [7] therefore suggested the notion of a fuzzy set whose membership function itself is a fuzzy set. This leads to the following definition of a second order fuzzy set or a fuzzy set of type 2. A **second order fuzzy set** on a nonempty set X is a map from X to I^I.

First order fuzzy sets are denoted by f, g, h, …and second order fuzzy sets are denoted by $\hat{f}, \hat{g}, \hat{h}, \ldots$

In this paper the terms 'fuzzy set' and 'first order fuzzy set' are used synonymously.

Whenever a fuzzy set is considered without mentioning the order, it always refers to a first order fuzzy set.

Similar terminology applies to all concepts related to first order fuzzy sets.

Fundamental definitions and properties of second order fuzzy sets and second order fuzzy topological spaces are introduced in [6].

In this paper the second order fuzzy proximity is defined in two different waysby extending the first order fuzzy proximity of Artico, G., and Moresco, R. [1, 4]. Given a first order fuzzy proximity ρ (Artico-Moresco [1, 4]) on a set X, a second order fuzzy proximity $\hat{\rho}$ on X is constructed. It is proved that the association $\rho \rightarrow \hat{\rho}$ is functorial. Every second order fuzzy proximity $\hat{\rho}$ induces a second order fuzzy topology $\hat{\delta}(\hat{\rho})$. The map $\hat{\rho} \rightarrow \hat{\delta}(\hat{\rho})$ is a functor.

FUNDAMENTAL DEFINITIONS

Definition: 2.1 [6]

A second order Chang fuzzy topology $\hat{\delta}$ ona nonempty set X is a collection of second order fuzzy sets on X satisfying the following conditions:

(i) $\hat{0}, \hat{1} \in \hat{\delta}$ where, for any $x \in X$.

$\hat{0}(x)$ = the zero function **0** on I and $\hat{1}(x)$ = the constant function **1** on I.

(ii) $\hat{f}_\lambda \in \hat{\delta}$ for each $\lambda \in \Lambda$ implies $\left(\bigvee_{\lambda \in \Lambda} \hat{f}_\lambda \right) \in \hat{\delta}$

(iii) $\hat{f}_\lambda \in \hat{\delta}$ for i = 1, 2, . . . , implies $\left(\bigwedge_{\lambda \in \Lambda}^{m} \hat{f}_i \right) \in \hat{\delta}$

The pair $(X, \hat{\delta})$ is called a **second order Chang fuzzy topological space.**

Definition: 2.2 [6]

For any two second order fuzzy sets $\hat{f}, \hat{g}$ on a set X,

(i) $\hat{f} \, \Lambda_1 \, \hat{g} = \hat{0}$ means given $x \in X$, either $\hat{f}(x) = \mathbf{0}$ or $\hat{g}(x) = \mathbf{0}$

(ii) $\hat{f} \, \Lambda_2 \, \hat{g} = \hat{0}$ means given $x \in X$, $\alpha \in I$, either $\hat{f}(x)(\alpha) = 0$ or $\hat{g}(x)(\alpha) = 0$.

Definition: 2.3[6]

For a second order fuzzy set $\hat{f}$ on a set X, the **complement** of $\hat{f}$ is defined as $(\hat{f})^c(x)(\alpha) = 1 - \hat{f}(x)(\alpha)$ for every $x \in X$ and for every $\alpha \in I$.

Definition: 2.4[6]

For a second order fuzzy set $\hat{f}$ on a set X, the **support** of $\hat{f}$ is defined as $S(\hat{f}) = \{x \in X / \hat{f}(x)(\alpha) > 0$ for some $\alpha \in I\}$

Definition: 2.5 [6]

$(X, \hat{\delta}_1)$, $(Y, \hat{\delta}_2)$ be two second order fuzzy topological spaces. Then a function θ : $X \to Y$ is said to be **2-f continuous** if the following condition is satisfied: $\theta^{-1}(\hat{f}) \in \hat{\delta}_1$ if $\hat{f} \in \hat{\delta}_2$.

Definition: 2.6 [1]

A function $\rho : I^X \times I^X \to \{0,1\}$ is called an **AM1 – fuzzy proximity** if the following conditions are satisfied:

For any f, g, h ∈ I^X,

(AMFP1) $\rho(0,1) = 0$

(AMFP2) $\rho(f,g) = \rho(g,f)$

(AMFP3) $\rho(f,g) \vee \rho(h,g) = \rho(f \vee h, g)$

(AMFP4) If $\rho(f,g) = 0$, then there exists a h∈ I^X such that $\rho(f,h) = 0$, $\rho(g,h^c) = 0$.

(AMFP5) $\rho(f,g) = 0 \Rightarrow f \leq g^c$

The pair (X, ρ) is called an **AM1-fuzzy proximity space**.

Definition: 2.7 [4]

A function $\rho : I^X \times I^X \to \{0,1\}$ is called an **AM2 – fuzzy proximity** if the following conditions are satisfied:

For any f, g, h ∈ I^X,

(*AMFP1) $\rho(0,1) = 0$

(*AMFP2) $\rho(f,g) = \rho(g,f)$

(*AMFP3) $\rho(f,g) \vee \rho(h,g) = \rho(f \vee h, g)$

(*AMFP4)If $\rho(f,g) = a$, for every $\varepsilon > 0$, there exists a

$C \subseteq X$ such that $\rho(f, \chi_C) < a + \varepsilon$ and $\rho(\chi_{X-C}, g) < a + \varepsilon$

(*AMFP5) $\rho(f, g) \geq (f \wedge g)(x)$, for every $x \in X$.

(*AMFP6)

If $|g_1 - g_2| \leq \varepsilon$, for $\varepsilon \in I$, then $|\rho(f, g_1) - \rho(f, g_2)| \leq \varepsilon$, for every $f \in I^X$.

(Here $|g_1 - g_2| = \vee_{x \in X} |g_1(x) - g_2(x)|$)

Result: 2.8 [1]

Let (X, ρ) be an AM1 – fuzzy proximity space.

(1) If $\rho(f, g) = 0, f \geq h, g \geq k$, then $\rho(h, k) = 0$.

(2) If $\rho(f_i, g_i) = 0$ for $i = 1, 2, ..., n$, then $\rho(\wedge_{i=1}^{n} f_i, \vee_{i=1}^{n} g_i) = 0$.

Definition: 2.9 [1]

Let (X, ρ_1) and (Y, ρ_2) be two AM1-fuzzy proximity spaces. A function $\theta: X \to Y$ is said to be a**AM1-fuzzy proximity mapping** if any one of the following equivalent conditions hold:

(1) For every $h, k \in I^Y$

$\rho_2(h, k) = 0 \Rightarrow \rho_1(\theta^{-1}(h), \theta^{-1}(k)) = 0$

(2) For every $f, g \in I^X$,

$\rho_1(f, g) = 1 \Rightarrow \rho_2(\theta(f), \theta(g)) = 1$.

Definition: 2.10 [1]

Let (X, ρ) be an AM1 – fuzzy proximity space. For any $f \in I^X$, define intf = $\vee\{g \in I^X / \rho(g, f^C) = 0\}$. The function $f \to \text{int } f$ is an interior operator on I^X.The collection $\delta(\rho) = \{f \in I^X / \text{int } f = f\}$ is a fuzzy topology on X and it is called the **fuzzy topology induced by** ρ.

Result: 2.11 [1]

Let (X, ρ_1) and (Y, ρ_2) be two AM1-fuzzy proximity spaces. If $\theta: X \to Y$ is said to be an AM1-fuzzy proximity map, then $\theta$is fuzzy continuous with respect to the corresponding fuzzy topologies $\delta(\rho_1)$ and $\delta(\rho_2)$.

SECOND ORDER AM - FUZZY PROXIMITY STRUCTURES

Definition: 3.1

A function $\hat{\rho}$: $(I^I)^X \times (I^I)^X \rightarrow \{0, 1\}$ is said to be a **second order AM1-fuzzy proximity** if the following conditions are satisfied:

For any $\hat{f}, \hat{g}, \hat{h} \in (I^I)^X$

(SAMFP 1) $\hat{\rho}(\hat{0}, \hat{1}) = 0$

(SAMFP 2) $\hat{\rho}(\hat{f}, \hat{g}) = \hat{\rho}(\hat{g}, \hat{f})$

(SAMFP 3) $\hat{\rho}(\hat{f}, \hat{g}) \vee \hat{\rho}(\hat{h}, \hat{g}) = \hat{\rho}(\hat{f} \vee \hat{h}, \hat{g})$

(SAMFP 4)
If $\hat{\rho}(\hat{f}, \hat{g}) = 0$, there exists $\hat{h} \in (I^I)^X$ such that $\hat{\rho}(\hat{f}, \hat{h}) = 0$ and $\hat{\rho}(\hat{g}, (\hat{h})^c) = 0$

(SAMFP 5) $\hat{\rho}(\hat{f}, \hat{g}) = 0 \Rightarrow \hat{f} \leq (\hat{g})^c$

The pair $(X, \hat{\rho})$ is said to be a **second order AM1-fuzzy proximity space**.

Definition: 3.2

A function $\hat{\rho}$: $(I^I)^X \times (I^I)^X \rightarrow I$ is said to be a **second order AM2-fuzzy proximity** if the following conditions are satisfied:

For any $\hat{f}, \hat{g}, \hat{h} \in (I^I)^X$

(*SAMFP 1) $\hat{\rho}(\hat{0}, \hat{1}) = 0$

(*SAMFP 2) $\hat{\rho}(\hat{f}, \hat{g}) = \hat{\rho}(\hat{g}, \hat{f})$

(*SAMFP 3) $\hat{\rho}(\hat{f}, \hat{g}) \vee \hat{\rho}(\hat{h}, \hat{g}) = \hat{\rho}(\hat{f} \vee \hat{h}, \hat{g})$

(*SAMFP4)
If $\hat{\rho}(\hat{f}, \hat{g}) = a$, for every $\varepsilon > 0$, there exists $\hat{h} \in (I^I)^X$ such that $\hat{\rho}(\hat{f}, \hat{h}) < a + \varepsilon$ and $\hat{\rho}((\hat{h})^c, \hat{g}) < a + \varepsilon$

(*SAMFP 5) $\hat{\rho}(\hat{f}, \hat{g}) \geq (\hat{f} \wedge \hat{g})(x)(\alpha)$, for every $x \in X$ and for every $\alpha \in I$.

(*SAMFP6)
If $|\hat{g}_1 - \hat{g}_2| \leq \varepsilon$ for $\varepsilon \in I$, then $|\hat{\rho}(\hat{f}, \hat{g}_1) - \hat{\rho}(\hat{f}, \hat{g}_2)| \leq \varepsilon$, for every $\hat{f} \in (I^I)^X$

$$(\text{Here } |\hat{g}_1 - \hat{g}_2| = \vee\{|\hat{g}_1(x)(\alpha) - \hat{g}_2(x)(\alpha)| \,/\, x \in X, \alpha \in I\})$$

The pair $(X, \hat{\rho})$ is said to be a **second order AM2-fuzzy proximity space**.

Note: 3.3

Every second order AM1 – fuzzy proximity (resp., second order AM2 – fuzzy proximity) $\hat{\rho}$ on a set X induces an AM1 – fuzzy proximity (resp., AM2 – fuzzy proximity) ρ on X. To prove this it is enough to observe that every fuzzy set f can be considered as a second order fuzzy set $\hat{f}$ where $\hat{f}(x)(\alpha) = f(x)$, for every $\alpha \in I$. Here $f \rho g$ iff $\hat{f}\ \hat{\rho}\ \hat{g}$.

Theorem: 3.4

Given an AM1-fuzzy proximity (respectively, AM2 – fuzzy proximity) ρ on X, define

$\hat{\rho} : (I^I)^X \times (I^I)^X \to I$ such that $\hat{\rho}(\hat{f}, \hat{g}) = \vee_{\alpha \in I} \rho(f_\alpha, g_\alpha)$ where

$f_\alpha(x) = \hat{f}(x)(\alpha)$, for every $x \in X$ and $g_\alpha(x) = \hat{g}(x)(\alpha)$, for every $x \in X$. Then $\hat{\rho}$ is a second order AM1 – fuzzy proximity (respectively, second order AM2 – fuzzy proximity) on X.

Proof

(SAMFP1) Obvious.

(SAMFP2) $\hat{\rho}(\hat{f}, \hat{g}) = \hat{\rho}(\hat{g}, \hat{f})$ since

$\rho(f_\alpha, g_\alpha) = \rho(g_\alpha, f_\alpha)$, for every $\alpha \in I$

(SAMFP3) Consider

$$\begin{aligned}\hat{\rho}(\hat{f}, \hat{g}) \vee \hat{\rho}(\hat{h}, \hat{g}) &= [\vee_{\alpha \in I} \rho(f_\alpha, g_\alpha)] \vee [\vee_{\alpha \in I} \rho(h_\alpha, g_\alpha)] \\ &= \vee_{\alpha \in I} [\rho(f_\alpha, g_\alpha) \vee \rho(h_\alpha, g_\alpha)] \\ &= \vee_{\alpha \in I} \rho(f_\alpha \vee h_\alpha, g_\alpha) \\ &= \hat{\rho}(\hat{f} \vee \hat{h}, \hat{g})\end{aligned}$$

$(\because (f_\alpha \vee h_\alpha)(x) = (\hat{f} \vee \hat{h})(x)(\alpha)$, for every $x \in X$ and for every $\alpha \in I)$

(SAMFP4) $\hat{\rho}(\hat{f}, \hat{g}) = 0 \Rightarrow \vee_{\alpha \in I} \rho(f_\alpha, g_\alpha) = 0$

$\Rightarrow \rho(f_\alpha, g_\alpha) = 0$, for every $\alpha \in I$.

$\therefore$ For $\alpha \in I$, there exists a $h_\alpha \in I^X$ such that $\rho(f_\alpha, h_\alpha) = 0$ and $\rho(g_\alpha, h_\alpha^C) = 0$

Define $\hat{h}: X \to I^I$ such that $\hat{h}(x)(\alpha) = h_\alpha(x)$, for every $\alpha \in I$.

$\therefore (\hat{h})^C(x)(\alpha) = 1 - \hat{h}(x)(\alpha)$

$= 1 - h_\alpha(x)$

$= h_\alpha^C(x)$

Consider $\hat{\rho}(\hat{f}, \hat{h}) = \vee_{\alpha \in I} \rho(f_\alpha, h_\alpha) = 0$

Similarly $\hat{\rho}(\hat{g}, (\hat{h})^C) = 0$

(SAMFP 5) $\hat{\rho}(\hat{f}, \hat{g}) = 0 \Rightarrow \vee_{\alpha \in I} \rho(f_\alpha, g_\alpha) = 0$

$\Rightarrow \rho(f_\alpha, g_\alpha) = 0$, for every $\alpha \in I$

$\Rightarrow f_\alpha \leq g_\alpha^c$, for every $\alpha \in I$

$\Rightarrow f_\alpha(x) \leq g_\alpha^c(x)$, for every $\alpha \in I$ and for every $x \in X$.

$\Rightarrow \hat{f} \leq (\hat{g})^c$

$\therefore \hat{\rho}$ is a second order AM1 fuzzy proximity on X.

Next to prove $\hat{\rho}$ is a second order AM2 – fuzzy proximity on X.

(*SAMFP1), (*SAMFP2) and (*SAMFP3) are same as (SAMFP1), (SAMFP2) and (SAMFP3).

(*SAMFP4) Let $\hat{\rho}(\hat{f}, \hat{g}) = a$

$\therefore \vee_{\alpha \in I} \rho(f_\alpha, g_\alpha) = a$

$\therefore \rho(f_\alpha, g_\alpha) \leq a$, for every $\alpha \in I$

Let $\rho(f_\alpha, g_\alpha) = a_\alpha$, for every $\alpha \in I$

$\therefore a_\alpha \leq a$, for every $\alpha \in I$

Given $\varepsilon > 0$, choose $\varepsilon' < \varepsilon$..

$\therefore \varepsilon' > 0$ and for $\alpha \in I$, there exists $H_\alpha \subseteq X$ such that $\rho(f_\alpha, \chi_{H_\alpha}) < a_\alpha + \varepsilon'$ (1)

and $\rho(\chi_{X-H_\alpha}, g_\alpha) < a_\alpha + \varepsilon'$ (2)

Define

$\hat{h} : X \to I^I$ such that $\hat{h}(x)(\alpha) = \chi_{H_\alpha}(x)$, for every $x \in X$ and for every $\alpha \in I$.

$\therefore h_\alpha(x) = \chi_{H_\alpha}(x)$, for every $x \in X$ and for every $\alpha \in I$.

$\therefore h_\alpha = \chi_{H_\alpha}$, for every $\alpha \in I$.

$\therefore h_\alpha{}^c = \chi_{X-H_\alpha}$, for every $\alpha \in I$.

$\therefore$ From (1) and (2),

$\rho(f_\alpha, h_\alpha) < a + \varepsilon'$ for every $\alpha \in I$

and $\rho(h_\alpha{}^c, g_\alpha) < a + \varepsilon'$ for every $\alpha \in I$.

$\therefore \bigvee_{\alpha \in I} \rho(f_\alpha, h_\alpha) \leq a + \varepsilon' < a + \varepsilon$

$\therefore \hat{\rho}(\hat{f}, \hat{g}) < a + \varepsilon$

Similarly $\hat{\rho}((\hat{h})^c, \hat{g}) < a + \varepsilon$

(*SAMFP5) For $\alpha \in I$,

$\rho(f_\alpha, g_\alpha) \geq (f_\alpha \wedge g_\alpha)(x)$, for every $x \in X$

$\geq (\hat{f} \wedge \hat{g})(x)(\alpha)$, for every $x \in X$

$\therefore \bigvee_{\alpha \in I} \rho(f_\alpha, g_\alpha) \geq \bigvee_{\alpha \in I} (\hat{f} \wedge \hat{g})(x)(\alpha)$, for every $x \in X$

$\therefore \hat{\rho}(\hat{f}, \hat{g}) \geq \bigvee_{\alpha \in I} (\hat{f} \wedge \hat{g})(x)(\alpha)$, for every $x \in X$

$\geq (\hat{f} \wedge \hat{g})(x)(\alpha)$, for every $x \in X$

(*SAMFP6)

$|\hat{g}_1 - \hat{g}_2| \leq \varepsilon$, for $\varepsilon \in I \Rightarrow |\hat{g}_1(x)(\alpha) - \hat{g}_2(x)(\alpha)| \leq \varepsilon$, for every $x \in X$ and for every $\alpha \in I$

$\Rightarrow |(g_1)_\alpha(x) - (g_2)_\alpha(x)| \leq \varepsilon$, for every $x \in X$ and for every $\alpha \in I$

$\Rightarrow \vee_{x \in X} |(g_1)_\alpha(x) - (g_2)_\alpha(x)| \leq \varepsilon$, for every $\alpha \in I$

$\Rightarrow |(g_1)_\alpha - (g_2)_\alpha| \leq \varepsilon$, for every $\alpha \in I$

$\Rightarrow |\rho(f, (g_1)_\alpha) - \rho(f, (g_2)_\alpha)| \leq \varepsilon$, for every $\alpha \in I$

and for every $f \in I^X$ (3)

To prove

$|\hat{\rho}(f, \hat{g}_1) - \hat{\rho}(\hat{f}, \hat{g}_2)| \leq \varepsilon$, for every $\hat{f} \in (I^I)^X$.

Given $\hat{f} : X \to I^I$, $f_\alpha(x) = \hat{f}(x)(\alpha)$ and $f_\alpha \in I^X$, for every $\alpha \in I$.

$\therefore$ (3) $\Rightarrow |\rho(f_\alpha, (g_1)_\alpha) - \rho(f_\alpha, (g_2)_\alpha)| \leq \varepsilon$, for every $\alpha \in I$

$\therefore |\hat{\rho}(\hat{f}, \hat{g}_1) - \hat{\rho}(\hat{f}, \hat{g}_2)| \leq \varepsilon$, for every $\hat{f} \in (I^I)^X$

$\therefore$ $\hat{\rho}$ is a second order AM2 – fuzzy proximity on X.

Definition: 3.5

Let $(X, \hat{\rho})$ be a second order AM1-fuzzy proximity space. For $\hat{f} \in (I^I)^X$, define

int: $(I^I)^X \to (I^I)^X$ such that $\text{int}\,\hat{f} = \vee \{\hat{g} \in (I^I)^X \,/\, \hat{\rho}(\hat{g}, (\hat{f})^c) = 0\}$.

Proposition: 3.6

The function int: $(I^I)^X \to (I^I)^X$ is an interior operator.

Proof

(i) $\text{int}\,\hat{1} = \hat{1}$ follows from the axiom (SAMFP1)

(ii) $\text{int}\,\hat{f} \leq \hat{f}$ follows from the axiom (SAMFP5)

(iii) Trivially $\text{int}(\text{int}\,\hat{f}) \leq \text{int}\,\hat{f}$

Take $\hat{g} \in (I^I)^X$ such that $\hat{\rho}(\hat{g}, (\hat{f})^c) = 0$

$\therefore$ There exists $\hat{h} \in (I^I)^X$ such that $\hat{\rho}(\hat{g}, (\hat{h})^c) = 0$ and $\hat{\rho}(\hat{h}, (\hat{f})^c) = 0$

$\therefore \hat{g} \leq \text{int}\,\hat{h}$ and $\hat{h} \leq \text{int}\,\hat{f}$

$\therefore \hat{g} \leq \text{int}\,(\text{int}\,\hat{f})$, for every $\hat{g} \in (I^I)^X$ for which $\hat{\rho}(\hat{g},(\hat{f})^c) = 0$

$\therefore \text{int}\,\hat{f} \leq \text{int}\,(\text{int}\,\hat{f})$

$\therefore \text{int}\,(\text{int}\,\hat{f}) = \text{int}\,\hat{f}$

(iv) Trivially $\text{int}\,(\hat{f} \wedge \hat{g}) \leq (\text{int}\,\hat{f}) \wedge (\text{int}\,\hat{g})$

Suppose

$\text{int}\,(\hat{f} \wedge \hat{g}) < (\text{int}\,\hat{f}) \wedge (\text{int}\,\hat{g})$, then there exists $x \in X, \alpha \in I$ such that

$t_1 = (\text{int}\,(\hat{f} \wedge \hat{g}))(x)(\alpha) < t < ((\text{int}\,\hat{f}) \wedge (\text{int}\,\hat{g}))(x)(\alpha) = t_2$

$t_2 > t \Rightarrow$ there exist $\hat{h}_1, \hat{h}_2 \in (I^I)^X$ such that $\hat{\rho}(\hat{h}_1,(\hat{f})^c) = 0, \hat{\rho}(\hat{h}_2,(\hat{g})^c) = 0$

and $\hat{h}_1(x)(\alpha) > t, \hat{h}_2(x)(\alpha) > t$

$\therefore \hat{\rho}(\hat{h}_1 \wedge \hat{h}_2, (\hat{f})^c \vee (\hat{g})^c) = 0$ and $(\hat{h}_1 \wedge \hat{h}_2)(x)(\alpha) > t$

$\therefore \hat{\rho}(\hat{h}_1 \wedge \hat{h}_2, (\hat{f} \wedge \hat{g})^c) = 0$ and $(\hat{h}_1 \wedge \hat{h}_2)(x)(\alpha) > t$

(1)

$t_1 = (\text{int}\,(\hat{f} \wedge \hat{g}))(x)(\alpha) < t$

$\Rightarrow$ For every $\hat{h} \in (I^I)^X$,

$\hat{\rho}(\hat{h},(\hat{f} \wedge \hat{g})^c) = 0$ and $\hat{h}(x)(\alpha) < t$

$\therefore$ (1) and (2) are contradictory.

$\therefore \text{int}\,(\hat{f} \wedge \hat{g}) = (\text{int}\,\hat{f}) \wedge (\text{int}\,\hat{g})$

$\therefore$ The function $\hat{f} \to \text{int}\,\hat{f}$ is an interior operator.

Definition: 3.7

The **second order fuzzy topology induced by the second order AM1-fuzzy proximity** $\hat{\rho}$ on X is denoted by $\hat{\delta}(\hat{\rho})$ and it consists of all the second order fuzzy sets $\hat{f} \in (I^I)^X$ for which $\hat{f} = \text{int}\,\hat{f}$.

Definition: 3.8

Let $(X, \hat{\rho}_1)$ and $(Y, \hat{\rho}_2)$ be two second order AM1-fuzzy proximity spaces. A function $\theta: X \to Y$ is said to be a **second order AM1-fuzzy proximity mapping** if any one of the following equivalent conditions hold:

(1) For every $\hat{h}$, $\hat{k} \in (I^I)^Y$,

$$\hat{\rho}_2(\hat{h}, \hat{k}) = 0 \Rightarrow \hat{\rho}_1(\theta^{-1}(\hat{h}), \theta^{-1}(\hat{k})) = 0$$

(2) For every $\hat{f}$, $\hat{g} \in (I^I)^X$,

$$\hat{\rho}_1(\hat{f}, \hat{g}) = 1 \Rightarrow \hat{\rho}_2(\theta(\hat{f}), \theta(\hat{g})) = 1$$

Theorem: 3.9

Let $(X, \hat{\rho}_1)$ and $(Y, \hat{\rho}_2)$ be two second order AM1-fuzzy proximity spaces. If $\theta: (X, \hat{\rho}_1) \to (Y, \hat{\rho}_2)$ is a second order AM1-fuzzy proximity map, then $\theta: (X, \hat{\delta}(\hat{\rho}_1)) \to (Y, \hat{\delta}(\hat{\rho}_2))$ is 2 – f continuous. That is the map $\hat{\rho} \to \hat{\delta}(\hat{\rho})$ is a functor.

Proof

Consider $\hat{g} \in \hat{\delta}(\hat{\rho}_2)$.

$\therefore \hat{g} = \text{int } \hat{g}$

$= \vee\{\hat{h} \in (I^I)^Y / \hat{\rho}_2(\hat{h}, (\hat{g})^c) = 0\}$

$\therefore \theta^{-1}(\hat{g}) = \vee\{\theta^{-1}(\hat{h}) / \hat{\rho}_2(\hat{h}, (\hat{g})^c) = 0\}$

$\hat{\rho}_2(\hat{h}, (\hat{g})^c) = 0 \Rightarrow \hat{\rho}_1(\theta^{-1}(\hat{h}), \theta^{-1}((\hat{g})^c)) = 0$

$\Rightarrow \hat{\rho}_1(\theta^{-1}(\hat{h}), (\theta^{-1}(\hat{g}))^C = 0$

$\therefore \theta^{-1}(\hat{g}) \leq \vee\{\theta^{-1}(\hat{h}) / \hat{\rho}_1(\theta^{-1}(\hat{h}), (\theta^{-1}(\hat{g}))^c) = 0\}$

$\leq \vee\{\hat{k} \in (I^I)^X / \hat{\rho}_1(\hat{k}, (\theta^{-1}(\hat{g}))^c) = 0\}$

$= \text{int}\,\theta^{-1}(\hat{g})$

$\therefore\ \theta^{-1}(\hat{g}) \leq \text{int}\,\theta^{-1}(\hat{g})$

Hence $\theta^{-1}(\hat{g}) = \text{int}\,\theta^{-1}(\hat{g})$

$\therefore\ \theta^{-1}(\hat{g}) \in \hat{\delta}(\hat{\rho}_1)$

$\therefore$ θ is 2-f continuous.

In theorem 3.4, it is proved that every first order AM1-fuzzy proximity ρ induces a second order AM1-fuzzy proximity $\hat{\rho}$.The following theorem shows that the map $\rho \to \hat{\rho}$ is a functor.

Theorem: 3.10

Let θ: $(X, \rho_1) \to (Y, \rho_2)$ be an AM1-fuzzy proximity map. Then $\theta : (X,\hat{\rho}_1) \to (Y,\hat{\rho}_2)$ is a second order AM1-fuzzy proximity map.

Proof

Assume $\hat{\rho}_2(\hat{h},\hat{k}) = 0$

$\Rightarrow \rho_2(h_\alpha, k_\alpha) = 0$, for every $\alpha \in I$

$\Rightarrow \rho_1(\theta^{-1}(h_\alpha), \theta^{-1}(k_\alpha)) = 0$, for every $\alpha \in I$

$\Rightarrow \rho_1(p_\alpha, q_\alpha) = 0$, for every $\alpha \in I$ Where $p_\alpha = \theta^{-1}(h_\alpha), q_\alpha = \theta^{-1}(k_\alpha)$

Let

$\hat{p},\hat{q} \in (I^I)^X$ be such that $\hat{p}(x)(\alpha) = p_\alpha(x)$ and $\hat{q}(x)(\alpha) = q_\alpha(x)$, for every $x \in X$.

$\therefore \hat{\rho}_1(\hat{p},\hat{q}) = 0 \,(1)$

and $\hat{p} = \theta^{-1}(\hat{h})$ and $\hat{q} = \theta^{-1}(\hat{k})$

$\therefore$ (1) $\Rightarrow \hat{\rho}_1(\theta^{-1}(\hat{h}), \theta^{-1}(\hat{k})) = 0$

$\therefore\ \theta : (X,\hat{\rho}_1) \to (Y,\hat{\rho}_2)$ is a second order AM1-fuzzy proximity map.

CONCLUSION

In this paper a second order fuzzy proximity is defined by extending the first order fuzzy proximity of Artico and Moresco [1]. It is proved that the association $\rho \rightarrow \hat{\rho}$ is functorial. Every second order fuzzy proximity $\hat{\rho}$ induces a second order fuzzy topology $\widehat{\delta}(\hat{\rho})$. The map $\hat{\rho} \rightarrow \widehat{\delta}(\hat{\rho})$ is a functor.

REFERENCES

[1] Artico. G. and Moresco. R., 1984, Fuzzy Proximities and Totally Bounded Fuzzy Uniformities, J. Math. Anal. Appl., vol.99, pp. 320 – 337.

[2] Artico. G. And Moresco. R., 1987,Fuzzy Proximities Compatible withLowen Fuzzy Uniformities, Fuzzy Sets and Systems, vol.21, pp.85 – 98.

[3] Artico. G. And Moresco. R., 1988, α* - compactness of the fuzzy unit interval, Fuzzy Sets and Systems, vol.25,pp.243 – 249.

[4] Artico. G. And Moresco. R., 1989, Fuzzy Uniformities induced by Fuzzy Proximities, Fuzzy Sets and Systems, vol.31,pp.111 – 121.

[5] Chang. C.L., 1968, Fuzzy Topological Spaces, J. Math. Anal. Appl., vol.24, pp.182 – 190.

[6] Kalaichelvi. A., 2007, Second order fuzzy topological spaces- I, ActaCienciaIndica, Vol. XXXIII M, No. 3,

pp. 819-826.

[7] Zadeh. L.A.,1965, Fuzzy Sets, Information and Control, vol.8, pp.338 – 353.