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Phase Transitions in a Wide Class of One-Dimensional Models with Unique Ground States

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Abstract

A wide class of one dimensional models with unique ground states that admits phase transitions is introduced. The interaction potential is between the spin variable at a point and the restriction of the configuration to an interval. The spin space is countably infinite.

Keywords: Hamiltonian, Gibbs State, Phase Transition.

1. Introduction

Many authors dealt with the problem of phase transitions with unique ground state models, for example [2-9].

In Kerimov [3,4], the random fields take values in a countably infinite set. Also the potential function of nearest neighbors is symmetric with respect to the two arguments and symmetric with respect to the point x= -1/2 and the external field is symmetric with respect to the point x = 1/2.

In Kerinov and Mallak [7], we constructed a one dimensional model with two spins {0,1} and a unique ground state having infinitely many extreme limit Gibbs states. The interaction potential was between the spin variable at a point and the restriction of the configuration to an interval. The model was interesting since it disproved a conjecture formulated before [3]. In Mallak [8], we considered the case when the spin space is countably infinite. Finally, in Mallak [9], we improved somehow the interaction potential in [7].

In this paper, we construct a wide class of models that admit phase transitions. The interaction potential is between the spin variable at a point and the restriction of the configuration to an interval. The spin space is countably infinite.

In the next section we give some needed mathematical definitions. The class of models is introduced in section 3. We will conclude the paper with some remarks.

2. MATHEMATICAL DEFINITIONS

Definition1.

Let S be a countably infinite set and (Φ, E) any measurable space. A family $(\varphi(x))_{x\in S}$ of random variables which are defined on some probability space and take values in Φ is called a random field , or a spin system. S is called the parameter set, Φ is called the state space, or spin space, and $\varphi(x)$ is called the spin at state x.

Definition2.

Let $\Omega = \Phi^S = \left\{\left(\varphi(x)\right)_{x\in S}: \varphi(x) \in \Phi\right\}$. Then $w \in \Omega$ is called a configuration and Ω is called the set of all possible configurations.

Definition 3.

Let $\Gamma = \{A \subset S: A \neq \varphi, |A| < \infty\}$. An interaction potential is a family $U = (U_A)_{A\in\Gamma}$ of measurable functions (with respect to the product sigma algebra) where $U_A: \Omega \to R$.

For all $A \in \Gamma\ and\ w \in \Omega$, $\mathrm{H}_{\mathrm{A}}^{\mathrm{U}}(\mathrm{w}) = \sum_{\Lambda\in\Gamma, \mathrm{A}\cap\Lambda=\varphi} \mathrm{U}_\Lambda(\mathrm{w})$ is called the total energy of w in A for U. It is also called Hamiltonian.

Definition 4.

On the space $\Omega_\Lambda = \Phi^\Lambda = \{\left(\varphi(x)\right)_{x\in\Lambda}: \varphi(x) \in \Phi\}$ we introduce a probability distribution defining the probability of a configuration by $P_\Lambda(w^\Lambda) = Z_\Lambda^{-1}\exp[-\beta H_\Lambda^\Phi(w^\Lambda)]$ where Z_Λ is a normalizing factor defined by the condition $\sum_{w^\Lambda\in\Omega^\Lambda} P_\Lambda(w^\Lambda) = 1$, that is $Z_\Lambda = \sum_{\mathrm{w}^\Lambda\in\Omega^\Lambda} \exp\left[-\beta H_\Lambda^\Phi(w^\Lambda)\right]$. $\beta = (kT)^{-1}$, where k is a constant we consider it to be 1 and T is the temperature.

This Probability distribution is called a Gibbs probability distribution in Λ

corresponding to the given Hamiltonian.

Definition 5.

Let $\Lambda \in \Gamma$ and $A, B \subseteq \Lambda$ be such that $A \cap B = \varphi\ \& bd(A) \subset B$ where bd(A) denotes the boundary of A. Let $P_\Lambda^A\left(w^{-A} \middle| w^{\sim B}\right) = P_\Lambda^A(\varphi(x) = \varphi^-(x) | \varphi(y) = \varphi^\sim(y), y \in B)$ denote the conditional probability that w^Λ equals w^{-A} on the set A under the condition that its values on the set B equals $w^{\sim B}$. A probability distribution P on the space Ω is said to determine a Gibbs measure (it is also called Gibbs state, Gibbs random field or DLR state) if the conditional distribution $P_{\Lambda}^{A}\left(w^{-A}\middle|w^{\sim B}\right)$ generated by the distribution P coincides with the Gibbs distribution in A with the boundary configuration $w^{\sim bd(A)}$ for arbitrary finite subsets $A, B \subseteq \Lambda\, such\ that\ A \cap B = \varphi\ and\ bd(A) \subset B$. If P is not unique, the given Hamiltonian is said to exhibit phase transition.

Definition 6.

Let $\{P_{\Lambda_i}\}_{i=1}^{\infty}$ be a sequence of probability measures. If $P_{\Lambda_n}(A) \to P(A)$ for each cylinder event A, then P is called the weak limit of the sequence of probability measures.

Definition 7.

An element μ of a convex subset of A is said to be extreme if $\mu \neq \alpha\mu_1 + (1-\alpha)\mu_2, \forall 0 < \alpha < 1, \mu_1, \mu_2 \in A$. An extreme limit Gibbs state is the weak limit of finite volume Gibbs state. It is well-known that the set of all limit Gibbs states coincides with the closed convex hull of the set of weak limits of finite volume Gibbs states.

Definition 8.

A configuration w^{gr} is said to be a ground state if for any finite perturbation $w^{\sim}$ of the configuration w^{gr} the expression $H(w^{\sim}) - H(w^{gr})$ is non negative.

More details can be found in many books, for example [1] and [10].

3. THE CLASS OF MODELS

Let the spin space be $\Phi = \{0,1,2,3,\ldots\} and\ S = Z^{-}$. Define a potential U as follows:

$For\ k \in B_{x+1}$:

$$If\ \frac{number\ of\ zeros\ in\ B_x}{|B_x|} = 1, then$$

$$\mathrm{U_{B_x,k}(\varphi(B_x), \varphi(k) = m) = m, m \neq 0}$$

$$\mathrm{U_{B_x,k}(\varphi(B_x), \varphi(k) = 0) = 0}$$

$$If\ \frac{number\ of\ m_i\ in\ B_x}{|B_x|} \geq \delta > 0, m_i \neq 0, then$$

$$\mathrm{U_{B_x,k}(\varphi(B_x), \varphi(k) = m_i) = \sum_{i=1}^{N_m} m_i + 1}$$

$$\mathrm{U}_{\mathrm{B_x},\mathrm{k}}(\varphi(\mathrm{B_x}),\varphi(\mathrm{k})=\mathrm{l})=\sum_{\mathrm{i}=1}^{\mathrm{N_m}}\mathrm{m_i}+\mathrm{l}+1$$

$$\mathrm{U}_{\mathrm{B_x},\mathrm{k}}(\varphi(\mathrm{B_x}),\varphi(\mathrm{k})=0)=\sum_{\mathrm{i}=1}^{\mathrm{N_m}}\mathrm{m_i}$$

$Finally, if\ \dfrac{number\ of\ m\ in\ B_x}{|B_x|}<\delta, \forall m\neq 0\ \&\ \dfrac{number\ of\ zeros\ in\ B_x}{|B_x|}<1, then$

$\mathrm{U}_{\mathrm{B_x},\mathrm{k}}(\varphi(\mathrm{B_x}),\varphi(\mathrm{k})=\mathrm{m})=\mathrm{m}+2$

$\mathrm{U}_{\mathrm{B_x},\mathrm{k}}(\varphi(\mathrm{B_x}),\varphi(\mathrm{k})=0)=0$

Notice that there exists only a finite number of spins m_i such that $\frac{number\ of\ m_i\ in\ B_x}{|B_x|}\geq\delta>0\ (since\ \sum\frac{\#m_i}{|B_x|}=1)$, call this number N_m where $\delta>0$ is a fixed number.

Let $I_V=[-v,-1]=\cup_{i=1}^{r}B_{-i}$. Suppose the boundary conditions $\varphi^k(x), x\in Z^- - I_V$ are fixed. The Hamiltonian on the subset I_V is given by:

$$H_V\left(\varphi(x)\middle|\varphi^k(x)\right)=\sum_{x=-V}^{-1}U\left(\varphi\left(B_{-n(x)-1}\right),\varphi(x)\right)\qquad(1)$$

It is clear from the construction that $w(x)=0,0,0,\ldots$ is the unique ground state.

Theorem.

For $T\in[\rho,1]$, model (1) has phase transitions.

Proof:

For each $k\in B_{x+1}$, we have

$$p^m\left(\varphi(k)=m\middle|\frac{\#m}{|B_x|}\geq\delta\right)=$$

$$\frac{e^{-\beta(\sum_{i=1}^{Nm}m_i+1)}}{e^{-\beta(\sum_{i=1}^{Nm}m_i+1)}+e^{-\beta(\sum_{i=1}^{Nm}m_i)}+N_m e^{-\beta(\sum_{i=1}^{Nm}m_i+1)}+\sum_{l`}e^{-\beta(\sum_{i=1}^{Nm}m_i+l`+1)}}$$

where $m\in\{1,2,3,\ldots\}, i\in\{1,2,\ldots,N_m\}\ \&\ l`\notin\{m_1,m_2,\ldots,m_{N_m}\}$.

Now,

$$p^m\left(\varphi(k)=m\middle|\frac{\#m}{|B_x|}\geq\delta\right)\geq\frac{1}{1+e^\beta+N_m+1}\geq\frac{1}{2+e^\beta+N_m}\geq h(\delta,\beta)\geq\varepsilon_0>0$$

where $\varepsilon_0<\delta$.

Thus for large volumes $B_{x+1}, k \in B_{x+1}$ and boundary conditions $\frac{\#m}{|B_x|}$, $\forall m \in \{1,2,3, \dots\}$,

$$p^m(\varphi(k) = m) \geq constant > 0$$

Since we have infinite number of m's, finite number of them may coincide.

4. REMARKS

Notice that

$$sup_m N_m \leq \frac{1}{\delta} < \infty \Rightarrow \frac{1}{2 + e^{\beta} + N_m} \geq \frac{1}{2 + e^{\beta} + \frac{1}{\delta}} \geq \epsilon_0$$

Thus for fixed δ & ε_0 $with\ \varepsilon_0 < \delta, we\ choose\ \rho >$ $[\ln\left(\frac{1-2\varepsilon_0-\frac{\varepsilon_0}{\delta}}{\varepsilon_0}\right)]^{-1}$.

Finally we can extend this work to any countable spin space and any countable parameter set.

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