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Exact Solution for Camassa-Holm Equations which Desribe Pseudo-Spherical Surface

Qasem Ahmad Al-Haj Abdullah
Mathematics Department, Zarqa University, Jordan.

Gharib Musa Ibrahim Gharib
Mathematics Department, Zarqa University, Jordan.

Abstract

In this paper, we study traveling wave solutions for Camassa-Holm equation, these solutions are obtained by using Tan-Cot method, Wu's elimination method, an infinite number of conserved quantities for above equation are also obtained by solving a set of coupled Riccati equations.

INTRODUCTION

Many partial differential equations which continue to be investigated due to their role in mathematics and physics exhibit interrelationships with the geometry of surfaces, or sub manifolds, immersed in a three-dimensional space [1]. In particular, it has been known for a while that there is a relationship between surfaces of a constant negative Gaussian curvature in Euclidean three-space, the Camassa-Holm equation and Bäcklund transformations which are relevant to the given equation [7]. It is well known that nonlinear complex physical phenomena are related to nonlinear partial differential equations (NLPDEs) which are involved in many fields from physics to Biology, Chemistry, Mechanics, etc.

As mathematical models of the phenomena, the investigation of exact solutions to the NLPDEs reveals to be very important for the understanding of these physical problems. Many mathematicians and physicists have well understood this importance when they decided to pay special attention to the development of sophisticated methods for constructing exact solutions to the NLPDEs [2].

Many methods for solving nonlinear partial differential equations such as Bäcklund transformation [3]; it was extended by M. Wang, Y. Wang and Y. Zhou [4, 5] , which give the exact solution. It is based on generalized trigometric functions, (Tan - Cot) and their properties [11].

The aim of this work is to establish exact solutions of distinct physical structures, solitons and kink waves solutions for the nonlinear partial differential equations.

MAIN RESULTS

The structure (Equations of pseudo-spherical type) was introduced by S.S. Chern and K.Tenenblat in 1986 [8], motivated by the fact that [9] generic solutions of equations integrable by the Ablowitz, Kaup, Newell and Segur (AKNS) inverse scattering scheme determine-whenever their associated linear problems are real- pseudo-spherical surfaces, that is Riemannian surfaces of constant Gaussian curvature $k = -1$.

A scalar differential equation $U\left(x, t, u, u_x, \ldots, u_{x^n t^m}\right) = 0$ in two independent variables *x, t* is of pseudo-spherical type (or, it is said to describe pseudo-spherical surfaces) if there exist one-forms $\omega^\alpha \neq 0$,

$$\omega^\alpha = f_{\alpha 1}(x, t, u, \ldots, u_{x^r t^p})dx + f_{\alpha 2}(x, t, u, \ldots, u_{x^s t^q})dt \;, \quad \alpha = 1, 2, 3 \qquad (1)$$

Whose coefficients $f_{\alpha\beta}$ are differential functions, such that the one-forms $\overline{\omega}^\alpha = \omega^\alpha(u(x,t))$ satisfy the structure equations

$$d\overline{\omega}^1 = \overline{\omega}^3 \wedge \overline{\omega}^2, \quad d\overline{\omega}^2 = \overline{\omega}^1 \wedge \overline{\omega}^3, \quad d\overline{\omega}^3 = \overline{\omega}^1 \wedge \overline{\omega}^2$$

Whenever $u = u(x, t)$ is a solution to $U = 0$.

We recall that a differential function is a smooth function which depends on *x, t*, and a finite number of derivatives of *u* [6]. We sometimes use the expression“PSS equation instead of “equation of pseudo-spherical type”. Also, we exclude from our considerations the trivial case when the functions $f_{\alpha\beta}$ all depend only on *x, t*.

Theorem:[10]

The Camassa–Holm equation

$$m = u_{xx} - u \,, \quad m_t = -m_x u - 2m u_x$$

describe pseudo-spherical surfaces.

Proof : We consider one-forms $\sigma^{\alpha}, \alpha = 1,2,3$, given by

$$\sigma^1 = (m - \beta + \varepsilon\eta^{-2}(\beta - 1))dx + (-u_x \beta\eta^{-1} - \beta\eta^{-2} - u\, m - 1 + u\,\beta + u_x \eta^{-1} + \eta^{-2})dt$$
$$\sigma^2 = \eta dx + (-\beta\eta^{-1} - \eta u + \eta^{-1} + u_x)dt$$
$$\sigma^3 = (m+1)dx + (\varepsilon u \eta^{-2}(\beta - 1) - um + \eta^{-2} + \frac{u_x}{\eta} - u - \frac{\beta}{\eta^2} - \frac{u_x \beta}{\eta})dt$$

in which the parameters η and β are constrained by the relation

$$\eta^2 + \beta^2 - 1 = \varepsilon \left[\frac{\beta - 1}{\eta} \right]^2$$

It is not hard to check that the structure equations (1) are satisfied whenever $u(x,t)$ is a solution of (Camassa–Holm) (if $\varepsilon = 1$ and $m = u_{xx} - u$).

Tan-Cot Method:**[11]**

The method is applied to find out an exact solution of a nonlinear ordinary differential equation. Consider the following PDE,

$$P(u, u_x, u_t, u_{xx}, u_{xxx}, \ldots) = 0 \tag{2}$$

Where P is a polynomial of the variable u and its derivatives. If we consider $u(x,t) = U(\xi)$, $\xi = k(x - \lambda t)$, where k and λ are real constants. so that we can use the following changes:

$$\frac{\partial}{\partial t} = -k\lambda \frac{d}{d\xi}, \; \frac{\partial}{\partial x} = k \frac{d}{d\xi}, \; \frac{\partial^2}{\partial x^2} = k^2 \frac{d^2}{d\xi^2}, \; \frac{\partial^3}{\partial x^3} = k^3 \frac{d^3}{d\xi^3}$$

and so on, then (2) becomes an ordinary differential equation

$$Q(U, U', U'', U''', \ldots) = 0 \tag{3}$$

With Q being another polynomial form of its argument, which will be called the reduced ordinary differential equation of (3). Integrating (3) as long as all terms contain derivatives, the integration constants are considered to be zeros in view of the localized solutions. However, the nonzero constants can be used and handled as well [6].

Now finding the traveling wave solutions to (2) is equivalent to obtaining the solution to the reduced ordinary differential equation (3). Applying tan-cot method, the solutions of nonlinear equations can be expressed as:

$f(\xi) = a\tan^b(\mu\xi)$ $,|\xi| \le \dfrac{\pi}{2\mu}$ or $f(\xi) = a\cot^b(\mu\xi)$ $,|\xi| \le \dfrac{\pi}{2\mu}$

Where a , μ, and b are parameters to be determined, μ is the wave number. We use $f(\xi)$ and their derivative:

$$f(\xi) = a\tan^b(\mu\xi)$$
$$f'(\xi) = ab\,\mu\left(\tan^{b-1}(\mu\xi) + \tan^{b+1}(\mu\xi)\right)$$
$$f''(\xi) = ab\,\mu^2\left((b-1)\tan^{b-2}(\mu\xi) + 2b\tan^b(\mu\xi) + (b+1)\tan^{b+2}(\mu\xi)\right) \qquad (4)$$

and their derivative. Or use

$$f(\xi) = a\cot^b(\mu\xi)$$
$$f'(\xi) = -ab\,\mu\left(\cot^{b-1}(\mu\xi) + \cot^{b+1}(\mu\xi)\right)$$
$$f''(\xi) = ab\,\mu^2\left((b-1)\cot^{b-2}(\mu\xi) + 2b\cot^b(\mu\xi) + (b+1)\cot^{b+2}(\mu\xi)\right) \qquad (5)$$

and so on. We substitute (4) or (5) into the reduced equation (3), balance the terms of the tan functions when (4) are used, or balance the terms of the cot functions when (5) are used, and solve the resulting system of algebraic equations by using computerized symbolic packages. We next collect all terms with the same power in $tan^k(\mu\xi)$ or $cot^k(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknown's a , μ and b, and solve the subsequent system.

Cassama-Holm equation:

$$u_t + 2ru_x - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0 \qquad (6)$$

Let $u(x,t)=u(\xi)$ and by using the wave variable $\xi=k(x - \lambda)t$ where k and λ are real constants. equation (6) turns to be the following ordinary differential equation:

$$(2r-\lambda)ku' + \lambda k^3 u''' + 3kuu' - 2k^3 u'u'' - k^3 uu''' = 0$$

$$(2r-\lambda)u' + \lambda k^2 u''' + 3uu' - 2k^2 u'u'' - k^2 uu''' = 0 \qquad (7)$$

by integrating equation (7) once with zero constant, we have

$$(2r-\lambda)u + \lambda k^2 u'' + \frac{3}{2}u^2 - \frac{1}{2}k^2(u')^2 - k^2 uu'' = 0 \qquad (8)$$

Applying Tan-Cot Method, the solutions of nonlinear equations can be expressed as:

$f(\xi) = a\tan^b(\mu\xi)$ $,|\xi| \le \dfrac{\pi}{2\mu}$ where a , μ, and b are parameters to be determined, μ is the wave number.

We use $f\ (\xi)$ and their derivative:

$$f\ '(\xi) = ab\ \mu\left(\tan^{b-1}(\mu\xi) + \tan^{b+1}(\mu\xi)\right)$$

$$f\ ''(\xi) = ab\,\mu^2\left((b-1)\tan^{b-2}(\mu\xi) + 2b\ \tan^{b}(\mu\xi) + (b+1)\tan^{b+2}(\mu\xi)\right)$$

So the equation (8) becomes:

$$\begin{aligned}&(-\alpha k\,\lambda + 2rk\,)a\tan^{b}(\mu\xi) + \\ &(\lambda k^{3} - k^{3}a\tan^{b}(\mu\xi))\,(ab\,\mu^2\left((b-1)\tan^{b-2}(\mu\xi) + 2b\ \tan^{b}(\mu\xi) + (b+1)\tan^{b+2}(\mu\xi)\right)) \\ &+\frac{3}{2}\theta k a^2\tan^{2b}(\mu\xi) - \frac{1}{2}k^{3}a^2b^2\ \mu^2\left(\tan^{b-1}(\mu\xi) + \tan^{b+1}(\mu\xi)\right)^2 = 0\end{aligned}$$

$$\begin{aligned}&(2r-\lambda)a\tan^{b}(\mu\xi) + \lambda k^{2}ab\,\mu^2\left((b-1)\tan^{b-2}(\mu\xi) + 2b\ \tan^{b}(\mu\xi) + (b+1)\tan^{b+2}(\mu\xi)\right) \\ &+\frac{3}{2}a^2\tan^{2b}(\mu\xi) - \frac{1}{2}k^{2}\left(ab\ \mu\left(\tan^{b-1}(\mu\xi) + \tan^{b+1}(\mu\xi)\right)\right)^2 \\ &-k^{2}a\tan^{b}(\mu\xi)\left(ab\,\mu^2\left((b-1)\tan^{b-2}(\mu\xi) + 2b\ \tan^{b}(\mu\xi) + (b+1)\tan^{b+2}(\mu\xi)\right)\right) = 0\end{aligned} \tag{9}$$

$$\begin{aligned}&a(-1+b)bk^2\lambda\mu^2\mathrm{Tan}[\mu\xi]^{-2+b} + a(2r-\lambda)\mathrm{Tan}[\mu\xi]^{b} + 2ab^2k^2\lambda\mu^2\mathrm{Tan}[\mu\xi]^{b} + \\ &\tfrac{3}{2}a^2\mathrm{Tan}[\mu\xi]^{2b} - 3a^2b^2k^2\mu^2\mathrm{Tan}[\mu\xi]^{2b} + ab(1+b)k^2\lambda\mu^2\mathrm{Tan}[\mu\xi]^{2+b} - \\ &a^2(-1+b)bk^2\mu^2\mathrm{Tan}[\mu\xi]^{-2+2b} - \tfrac{1}{2}a^2b^2k^2\mu^2\mathrm{Tan}[\mu\xi]^{-2+2b} - \\ &\tfrac{1}{2}a^2b^2k^2\mu^2\mathrm{Tan}[\mu\xi]^{2+2b} - a^2b(1+b)k^2\mu^2\mathrm{Tan}[\mu\xi]^{2+2b} = 0\end{aligned} \tag{10}$$

Equating the exponents and the coefficients of each pair of the tan functions we find the following algebraic system:

$$2b-2 = b \Rightarrow b = 2$$

Substituting the value of b into (9):

$$\begin{aligned}&2\lambda k^{2}a\mu^2 + ((2r-\lambda)a + 8\lambda k^{2}a\mu^2 - 4k^{2}a^2\ \mu^2)\tan^2(\mu\xi) \\ &+(6\lambda k^{2}a\mu^2 + \frac{3}{2}a^2 - 12k^{2}a^2\ \mu^2)\tan^4(\mu\xi) - 8k^{2}a^2\ \mu^2\ \tan^6(\mu\xi) = 0\end{aligned} \tag{11}$$

then we get the following system of equations:

$$\begin{aligned}&2\lambda k^{2}a\mu^2 = 0 \\ &(2r-\lambda)a + 8\lambda k^{2}a\mu^2 - 4k^{2}a^2\ \mu^2 = 0 \\ &6\lambda k^{2}a\mu^2 + \frac{3}{2}a^2 - 12k^{2}a^2\ \mu^2 = 0 \\ &-8k^{2}a^2\ \mu^2 = 0\end{aligned} \tag{12}$$

By solving (12) we find:

$$\lambda = \frac{2r}{1-8k^2\mu^2} \quad , \quad a = \frac{24k^2\mu^2 r}{(1-8k^2\mu^2)(24k^2\mu^2-3)} \qquad (13)$$

Then the exact soliton solution of equation (6) can be written in the form:

$$u(x,t) = \frac{24k^2\mu^2 r}{(1-8k^2\mu^2)(24k^2\mu^2-3)} \tan^2\left(\mu k\,(x - \frac{2r}{1-8k^2\mu^2})t \right) \qquad (14)$$

For $\mu=k=r=1$, (14) becomes: $u(x,t) = \frac{-24}{147}\tan^2\left(x + \frac{2}{7}t \right)$

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REFERENCES

[1] Rogers, C., Schief, W.F. (2002), Bäcklund and Darboux Transformations, Geometry and Modern Applications in Soliton Theory. In: Ablowitz, M.J., Eds., **Cambridge Texts in Applied Mathematics**, Cambridge University Press, Cambridge.

[2] Bracken, P. (2010), Surfaces Specified by Integrable Systems of Partial Differential Equations Determined by StructureEquations and Lax Pair**. Journal of Geometry and Physics**, vol60: 562-569

[3] Ren Y., Zhang H.(2006), New generalized hyperbolic functions and auto-Backlund transformation to find new exact solutions of the (2+1)dimensional NNV equation, **Phys. Lett. A**, 357: 438-448.

[4] Wang M., Wang Y. M. (2001), A new Backlund transformation and multisoliton solutions to the KdV equation with general variable coefficients, **Phys. Lett. A,** 287: 211-216.

[5] Wang M., Wang Y. M. and Zhou Y.(2002), An auto-Backlund transformation and exact solutions to a generalized KdV equation with variable coefficients and their applications, **Phys. Lett. A**,303: 45-51.

[6] Stenger F.(2000), Summary of Sinc numerical methods, **Appl. Math. Comput**. 121:379-420.

[7] WazwazA. M.(2007), New solitary wave solutions to the modified formsof Degasperis-prcesi and Camassa-Holm equations, **Appl. Math. Comput**. 186: 130-141.

[8] ChernS.S., TenenblatK.(1986), Pseudo-spherical surfaces and evolution equations, **Stud. Appl. Math**. 74, 55-83.

[9] Sasaki R.(1979), Soliton equations and pseudospherical surfaces, **Nucl. Phys. B**,154: 343–357.

[10] Reyes E. G.(2002), The Soliton Content of the Camassa–Holmand Hunter–Saxton Equations, **Proceedings of Institute of Mathematics of NAS of Ukraine**, 43: Part 1, 201–208

[11] Kumar R., Kumar M. and Kumar A.,(2013), Some Soliton Solutions of Non Linear Partial Differential Equations by Tan-Cot Method**, Journal of Mathematics**, Issue 6 :23-28