Fractional Complex Transform for Solving the Fractional Differential Equations

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Abstract

In this paper, fractional complex transform (FCT) with help of New Iterative Method (NIM) is used to obtain numerical and analytical solutions for the fractional Fokker-planck equation, Fractional Nonlinear Gas Dynamics equation and the nonlinear time-fractional Fisher’s equation and fractional telegraph equation. Fractional complex transform (FCT) is proposed to convert fractional differential equations to its differential partner and then applied NIM to the new obtained equations. Several examples are given and the results are compared to exact solutions. The results reveal that the method is very effective and simple.

Keywords: Fractional complex transform, New iterative method, fractional Fokker-planck equation, fractional nonlinear Gas Dynamics equation and the nonlinear time-fractional Fisher’s equation and fractional telegraph equation.

1. INTRODUCTION

Fractional models have been shown by many scientists to adequately describe the operation of variety of physical and biological processes and systems. Consequently, considerable attention has been given to the solution of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest. Since most fractional differential equations do not have exact analytic solutions, approximation and numerical techniques, therefore, are used extensively ([9]-[11], [14]-[15]). Numerical and analytical methods have included finite difference method ([7], [17]), Adomian decomposition method ([23]-[24]).
Transform is an important method to solve mathematical problems. Many useful transforms for solving various problems were appeared in open literature, such as the travelling wave transform [1], the Laplace transform [16], the Fourier transform [8], the Bücklund transformation [18], the integral transform [19], and the local fractional integral transforms [29].

Very recently the fractional complex transform ([12]-[13], [31]-[36]) was suggested to convert fractional order differential equations with modified Riemann-Liouville derivatives into integer order differential equations, and the resultant equations can be solved by advanced calculus.

The time-fractional Fokker-Planck equation serves as a mathematical model for a number of problems in physical and biological sciences. It arises from a diffusion approximation of some stochastic processes regarded as Markovian and continuous. It is a generalized diffusion equation governing the evolution of the probability density in time. For the two-variable case, to which attention is restricted here [6].

It is commonly known that the equation of Gas dynamics is the mathematical expressions of conservation laws which exist in engineering practices such as conservation of mass, conservation of momentum, conservation of energy etc. The nonlinear equations of ideal gas dynamics are applicable for three types of nonlinear waves like shock fronts, rarefactions, and contact discontinuities. In 1981, Steger and Warming addressed that the conservation-law form of the inviscid gas dynamic equation possesses a remarkable property by virtue of which the nonlinear flux vectors are homogeneous functions of degree one which permits the splitting of flux vectors into subvectors by similarity transformations [5].

The time-fractional Fisher’s equation (TFFE), which is a mathematical model for a wide range of important physical phenomena, is a partial differential equation obtained from the classical Fisher equation by replacing the time derivative with a fractional derivative of order $\alpha$.

The telegraph equation developed by Oliver Heaviside in 1880 is widely used in Science and Engineering. Its applications arise in signal analysis for transmission and propagation of electrical signals and also modelling reaction diffusion. In recent years, great interest has been developed in fractional differential equation because of its frequent appearance in fluid mechanics, mathematical biology, electrochemistry, and physics. A space-time fractional telegraph equation is obtained from the classical telegraph equation by replacing the time and space derivative terms by fractional derivatives and complex transform method ([33]-[36]).

The paper is organized as follows: In section 2, we provide the Basic Complex Transform. Section 3, Basic Idea of New Iterative Method (NIM). Sections 4, Applications. Sections 5, Conclusion.
2. FRACTIONAL COMPLEX TRANSFORM (FCT)[20]

Consider the following general fractional differential equation

\[ f(u, u^{(\alpha)}_x, u^{(\beta)}_y, u^{(\gamma)}_z, u^{(2\beta)}_x, u^{(2\gamma)}_y, u^{(2\lambda)}_z, \ldots) = 0 \]  

(2.1)

where \( u^{(\alpha)} = \frac{\partial^\alpha}{\partial t^\alpha} u(x, y, z, t) \) denotes the modified Riemann-Liouville derivative. \( 0 < \alpha \leq 1, 0 < \beta \leq 1, 0 < \gamma \leq 1, 0 < \lambda \leq 1 \).

Introducing the following transforms

\[ T = \frac{qt^\alpha}{\Gamma(1+\alpha)}, \]  

(2.2)

\[ X = \frac{pt^\beta}{\Gamma(1+\beta)}, \]  

(2.3)

\[ Y = \frac{kt^\gamma}{\Gamma(1+\gamma)}, \]  

(2.4)

\[ Z = \frac{lt^\lambda}{\Gamma(1+\lambda)}, \]  

(2.5)

where \( p; q; k \) and \( l \) are constants.

Using the above transforms, we can convert fractional derivatives into classical derivatives:

\[ \frac{\partial^\alpha u}{\partial t^\alpha} = q \frac{\partial u}{\partial T} \]  

(2.6)

\[ \frac{\partial^\beta u}{\partial x^\beta} = p \frac{\partial u}{\partial X} \]  

(2.7)
\[
\frac{\partial^\gamma u}{\partial x^\gamma} = k \frac{\partial u}{\partial Y}
\]  \hspace{1cm} (2.8)

\[
\frac{\partial^\lambda u}{\partial z^\lambda} = l \frac{\partial u}{\partial Z}
\]  \hspace{1cm} (2.8)

Therefore, we can easily convert the fractional differential equations into partial differential equations, so that everyone familiar with advanced calculus can deal with fractional calculus without any difficulty. For example, consider a fractional differential equation

\[
\frac{\partial^\alpha u}{\partial t^\alpha} + 2u \frac{\partial^\beta u}{\partial x^\beta} + 4 \frac{\partial^\gamma u}{\partial y^\gamma} + 5 \frac{\partial^\delta u}{\partial z^\delta} = 0.
\]  \hspace{1cm} (2.9)

By using the above transformations we get:

\[
q \frac{\partial u}{\partial T} + 2up \frac{\partial u}{\partial X} + 2k \frac{\partial u}{\partial Y} + 5k \frac{\partial u}{\partial Z} = 0.
\]  \hspace{1cm} (2.10)

which can be solved by New Iterative method.

### 3. NEW ITERATIVE METHOD (NIM)[4]

To describe the idea of the NIM, consider the following general functional equation ([2]-[3], [22]-[23], [27]-[28], [30]):

\[
u(x) = f(x) + N(u(x)),
\]  \hspace{1cm} (3.1)

where \( N \) is a nonlinear operator from a Banach space \( B \to B \) and \( f \) is a known function. We are looking for a solution \( u \) of (3.1) having the series form

\[
u(x) = \sum_{i=0}^{\infty} u_i
\]  \hspace{1cm} (3.2)

The nonlinear operator \( N \) can be decomposed as follows

\[
N \left( \sum_{i=0}^{\infty} u_i \right) = N(u_0) + \sum_{j=0}^{\infty} \left\{ N \left( \sum_{j=0}^{\infty} u_j \right) - N \left( \sum_{j=0}^{\infty} u_j \right) \right\}
\]  \hspace{1cm} (3.3)

From Eqs. (3.2) and (3.3), Eq. (3.1) is equivalent to

\[
\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=0}^{\infty} \left\{ N \left( \sum_{j=0}^{\infty} u_j \right) - N \left( \sum_{j=0}^{\infty} u_j \right) \right\}
\]  \hspace{1cm} (3.4)
We define the recurrence relation:

\( u_0 = f, \quad (3.5a) \)

\[ u_i = N(u_0), \quad (3.5b) \]

\[ \ldots \]

\[ u_{n+1} = N(u_0 + u_1 + \ldots + u_n) - N(u_0 + u_1 + \ldots + u_{n-1}), \quad n = 1, 2, 3, \ldots \quad (3.5c) \]

Then:

\( u_0 + u_1 + \ldots + u_{n+1} = N(u_0 + u_1 + \ldots + u_n), \quad n = 1, 2, 3, \ldots \)

\[ u = \sum_{i=0}^{\infty} u_i = f + N\left( \sum_{i=0}^{\infty} u_i \right), \quad (3.6) \]

If \( N \) is a contraction, i.e.

\[ \|N(x) - N(y)\| \leq k\|x - y\|, \quad 0 < k < 1, \]

\[ \|u_{n+1}\| = \|N(u_0 + u_1 + \ldots + u_n) - N(u_0 + u_1 + \ldots + u_{n-1})\| \]

\[ \leq k\|u_n\| \leq \ldots k^n\|u_0\| \quad n = 1, 2, 3, \ldots, \quad (3.7) \]

and the series \( \sum_{i=0}^{\infty} u_i \) absolutely and uniformly converges to a solution of (2.1), which is unique, in view of the Banach fixed point theorem [3].

The \( k \)-term approximate solution of (2.1) and (2.2) is given by \( \sum_{i=0}^{k-1} u_i \).

### 3.1 Reliable Algorithm

After the above presentation of the NIM, we introduce a reliable algorithm for solving nonlinear partial differential equations using the NIM. Consider the following nonlinear partial differential equation of arbitrary order:

\[ D^n = A(u, \partial u) + B(x, t), \quad n \in N \quad (3.8a) \]

with the initial conditions

\[ \frac{\partial^m}{\partial t^m} u(x, 0) = h_m(x), \quad m = 0, 1, 2, \ldots, n - 1, \quad (3.8b) \]

where \( A \) is a nonlinear function of \( u \) and \( \partial u \) (partial derivatives of \( u \) with respect to \( x \) and \( t \)) and \( B \) is the source function. In view of the integral operators, the initial
value problem (3.8a) and (3.8b) is equivalent to the following integral equation

\[ u(x,t) - \sum_{m=0}^{n-1} h_m(x) \frac{t^m}{m!} + I^n_t B(x,t) + I^n_t A = f + N(u), \]  

(3.9)

Where

\[ f = \sum_{m=0}^{n-1} h_m(x) \frac{t^m}{m!} + I^n_t B(x,t), \]  

(3.10)

and

\[ N(u) = I^n_t A \]  

(3.11)

where \( I^n_t \) is an integral operator of \( n \) fold.
We get the solution of (3.9) by employing the algorithm (3.5).

4. APPLICATIONS
In this section, We apply the new iterative method approach to study five examples

4.1 Example[6]
We consider the Fractional Fokker-planck equation:

\[ \frac{\partial^\alpha u}{\partial t^\alpha} - (1 + x) \frac{\partial u}{\partial x} - \left( x^2 e^t \right) \frac{\partial^2 u}{\partial x^2} = 0 \]  

(4.1)

subject to the initial condition

\[ u(x,0) = 1 + x, \]  

(4.2)

To apply FCT to Eq.(4.1), we use the above transformations:

\[ \frac{\partial^\alpha u}{\partial t^\alpha} = q \frac{\partial u}{\partial T} \]

So we have the following partial differential equation:

\[ q \frac{\partial u}{\partial T} = (1 + x) \frac{\partial u}{\partial x} + \left( x^2 e^T \right) \frac{\partial^2 u}{\partial x^2} \]  

(4.3)

For simplicity we set \( q = 1 \), so we get

\[ \frac{\partial u}{\partial T} = (1 + x) \frac{\partial u}{\partial x} + \left( x^2 e^T \right) \frac{\partial^2 u}{\partial x^2} \]  

(4.4)
Fractional Complex Transform for Solving the Fractional Differential Equations

Now, we solve Eq. (4.4) by means of NIM. To apply NIM to (4.4), we construct the correction functional as follows:

\[ u_n(x,t) = u_{n-1} + I_T[(1 + x) \frac{\partial u_{n-1}}{\partial x} + (x^2 e') \frac{\partial^2 u_{n-1}}{\partial x^2}] \]

\[ u_0(x,t) = u_0 = 1 + x \]

\[ u_1(x,t) = I_T[(1 + x) \frac{\partial u_0}{\partial x} + (x^2 e') \frac{\partial^2 u_0}{\partial x^2}] = I_T[(1 + x)(1)] \]

\[ u_1 = (1 + x)T \]

\[ u_2(x,t) = I_T[(1 + x) \frac{\partial u_1}{\partial x} + (x^2 e') \frac{\partial^2 u_1}{\partial x^2}] = I_T[(1 + x)T] \]

\[ u_2 = (1 + x) \frac{T^2}{2} \]

\[ u_3(x,t) = I_T[(1 + x) \frac{\partial u_2}{\partial x} + (x^2 e') \frac{\partial^2 u_2}{\partial x^2}] = I_T[(1 + x) \frac{T^2}{2}] \]

\[ u_3 = (1 + x) \frac{T^3}{6} \]

\[ \vdots \]

\[ u_n = (1 + x) \frac{T^n}{n!} \]

\[ u(x,t) = \sum_{n=0}^{n-1} (1 + x) \frac{T^n}{n!} \]

By the fractional complex transform

\[ T = \frac{\Gamma(1 + x)}{\Gamma(1 + \alpha)} \]

We have

\[ u_1 = (1 + x) \frac{\Gamma(1 + x)}{\Gamma(1 + \alpha)} \]

\[ \vdots \]
Fig. 1. the exact solution of $u(x,t)$ at $\alpha = 1$

Fig. 2. the approximate solution of $u(x,t)$ at $\alpha = 0.5$
4.2 Example [25]

We consider the following nonlinear time-fractional gas dynamics equation of the form:

\[ D_t^\alpha + \frac{1}{2}(u^2)_x - u(1-u) = 0 \quad t > 0 \quad 0 < \alpha \leq 1 \]

subject to the initial condition

\[ u(x,0) = e^{-x} \quad (4.6) \]

To apply FCT to Eq.(4.5), we use the above transformations:

\[ \frac{\partial^\alpha u}{\partial t^\alpha} = q \frac{\partial u}{\partial T}, \]

so we have the following partial differential equation:

\[ q \frac{\partial u}{\partial T} = u - u^2 - \frac{1}{2}(u^2)_x \quad (4.7) \]
For simplicity we set \( q = 1 \), so we get

\[
\frac{\partial u}{\partial T} = u - u^2 - \frac{1}{2}(u^2)_x \tag{4.8}
\]

Now, we solve Eq. (4.8) by means of NIM. To apply NIM to (4.8), we construct the correction functional as follows:

\[
u_n(x, t) = u_{n-1} + I_T[(u_n - u_n^2 - \frac{1}{2}(u_n^2)_x)]
\]

\[
u_0(x, t) = u_0 = e^{-x}
\]

\[
u_1(x, t) = I_T[(u_0 - u_0^2 - \frac{1}{2}(u_0^2)_x)]
\]

\[
I_T[e^{-x} - e^{-2x} - \frac{1}{2}(-2e^{-2x})]
\]

\[
u_1 = e^{-x}T
\]

\[
u_2(x, t) = I_T[u_1 - u_1^2 - \frac{1}{2}(u_1^2)_x]
\]

\[
= I_T[e^{-x}T - e^{-2x}T^2 - \frac{1}{2}(-2e^{-2x})]
\]

\[
u_2 = e^{-x} \frac{T^2}{2}
\]

\[
u_3(x, t) = I_T[u_2 - u_2^2 - \frac{1}{2}(u_2^2)_x]
\]

\[
= I_T[(1 + x)\frac{T^2}{2}]
\]

\[
u_3 = e^{-x} \frac{T^3}{6}
\]

\[
\ldots
\]

\[
u_n = e^{-x} \frac{T^n}{n!}
\]

\[
u(x, t) = \sum_{n=0}^{\infty} \frac{e^{-x}T^n}{n!}
\]
By the fractional complex transform

\[ T = \frac{t^\alpha}{\Gamma(1+\alpha)} \]

We have

\[ u_i = e^{-x} \frac{t^\alpha}{\Gamma(1+\alpha)} \]

\[ \vdots \]

*Fig. 4. The exact solution of \( u(x,t) \) at \( \alpha = 1 \)
Fig. 5. The approximate solution of $u(x,t)$ at $\alpha = 0.5$

Fig. 6. The approximate solution of $u(x,t)$ at $\alpha = 0.9$
4.3 Example[26]

We consider the following

The time-fractional Fisher’s equation (TFFE) of the form:

\[ D_t^\alpha u(x,t) = u_{xx} + 6u(1-u) \quad t > 0 \quad 0 < \alpha \leq 1 \]  (4.10)

subject to the initial condition

\[ u(x,0) = \frac{1}{(1+e^x)^2} \]  (4.11)

To apply FCT to Eq.(4.10), we use the above transformations:

\[ \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = q \frac{\partial u}{\partial T} \]

so we have the following partial differential equation:

\[ q \frac{\partial u}{\partial T} = u_{xx} + 6u - 6u^2 \]  (4.11)

For simplicity we set \( q = 1 \), so we get

\[ \frac{\partial u}{\partial T} = u_{xx} + 6u - 6u^2 \]  (4.12)

Now, we solve Eq. (4.12) by means of NIM. To apply NIM to (4.12), we construct the correction functional as follows:

\[ u_n(x,t) = u_{n-1} + I_T[u_{nx} + 6u_n - 6u_n^2] \]  (4.13)

\[ u_0(x,t) = u_0 = \frac{1}{(1+e^x)^2} \]

\[ u_1(x,t) = I_T[u_{0xx} + 6u_0 - 6u_0^2] \]

\[ I_T[-2e^x(1+e^x)^{-3} + 6e^{2x}(1+e^x)^{-4} + 6(1+e^x)^{-2} - 6(1+e^x)^{-4}] \]

\[ u_1 = [-2e^x(1+e^x)^{-3} + 6e^{2x}(1+e^x)^{-4} + 6(1+e^x)^{-2} - 6(1+e^x)^{-4}]T \]

\[ u_2(x,t) = I_T[u_{1xx} + 6u_1 - 6u_1^2] \]

\[ = I_T[-144e^{3x}(1+e^x)^{-5} + 42e^{2x}(1+e^x)^{-4} - 2e^x(1+e^x)^{-3} + 120e^{4x}(1+e^x)^{-6}]T \]

\[ + (6(-2e^x(1+e^x)^{-3} + 6e^{2x}(1+e^x)^{-4} + 6(1+e^x)^{-2} - 6(1+e^x)^{-4})T - 6(76e^{2x}(1+e^x)^{-6} \]

\[ - 24e^{3x}(1+e^x)^{-7} + 24e^x(1+e^x)^{-7} - 24e^{5x}(1+e^x)^{-5} - 72e^{3x}(1+e^x)^{-8} + 36e^{4x}(1+e^x)^{-8} \]

\[ + 36(1+e^x)^{-4} - 72(1+e^x)^{-6} + 36(1+e^x)^{-8})T^2] \]
\( u_2 = \left[ (-144e^{-3}(1+e^x)^{-5} + 42e^{-2x}(1+e^x)^{-4} - 2e^{-x}(1+e^x)^{-3} + 120e^{4x}(1+e^x)^{-6}) \frac{T^2}{2} \right] \\
+ 3\left[ (-2e^{-x}(1+e^x)^{-3} + 6e^{2x}(1+e^x)^{-4} + 6(1+e^x)^{-2} - 6(1+e^x)^{-4})T^2 \right] \\
- \left[ 2(76e^{2x}(1+e^x)^{-6} - 24e^{2x}(1+e^x)^{-7} + 24e^x(1+e^x)^{-7} - 24e^x(1+e^x)^{-5} - 72e^{2x}(1+e^x)^{-8} + 36e^{4x}(1+e^x)^{-8} + 36(1+e^x)^{-3} - 72(1+e^x)^{-6} + 36(1+e^x)^{-8})T^3 \right] \\

By the fractional complex transform

\( T = \frac{t^\alpha}{\Gamma(1+\alpha)} \)

we have

\( u_i = \left[ -2e^{-x}(1+e^x)^{-3} + 6e^{2x}(1+e^x)^{-4} + 6(1+e^x)^{-2} - 6(1+e^x)^{-4} \right] \frac{t^\alpha}{\Gamma(1+\alpha)} \)

\( \text{Fig. 7. the exact solution of } u(x,t) \text{ at } \alpha = 1 \)
Fig. 8. the approximate solution of $u(x,t)$ at $\alpha = 0.5$

Fig. 9. the approximate solution of $u(x,t)$ at $\alpha = 0.9$
4.4 Example[21]

We consider the time-fractional telegraph equation:

\[ D^\alpha_t u = \frac{\partial^2 u}{\partial t^\alpha} + \frac{\partial u}{\partial t} + u \]  

(4.14)

subject to the initial condition

\[ u(x,0) = e^{-t} \]  

(4.15)

To apply FCT to Eq.(4.1), we use the above transformations:

\[ \frac{\partial^\alpha u}{\partial x^\alpha} = p \frac{\partial u}{\partial X} \]

so we have the following partial differential equation:

\[ p \frac{\partial u}{\partial X} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u \]  

(4.16)

For simplicity we set \( p = 1 \), so we get

\[ \frac{\partial u}{\partial X} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u \]  

(4.17)

Now, we solve Eq. (4.17) by means of NIM. To apply NIM to (4.17), we construct the correction functional as follows:

\[ u_n(x,t) = u_{n-1} + I_x \left[ \frac{\partial^2 u_n}{\partial t^2} + \frac{\partial u_n}{\partial t} + u_n \right] \]

\[ u_0(x,t) = u_0 = e^{-t} \]

\[ u_1(x,t) = I_x \left[ \frac{\partial^2 u_0}{\partial t^2} + \frac{\partial u_0}{\partial t} + u_0 \right] = I_x [e^{-t} - e^{-t} + e^{-t}] = e^{-t}X \]

\[ u_2(x,t) = I_x \left[ \frac{\partial^2 u_1}{\partial t^2} + \frac{\partial u_1}{\partial t} + u_1 \right] = I_x [e^{-t}X - e^{-t}X + e^{-t}X] = e^{-t} \frac{X^2}{2} \]
\[ u_3(x,t) = I_x \left[ \frac{\partial^2 u_2}{\partial t^2} + \frac{\partial u_2}{\partial t} + u_2 \right] \]
\[ = I_x \left[ e^{-\gamma} \frac{X^2}{2} - e^{-\gamma} \frac{X^2}{2} + e^{-\gamma} \frac{X^2}{2} \right] \]
\[ u_3 = e^{-\gamma} \frac{X^3}{6} \]

By the fractional complex transform
\[ X = \frac{x^\alpha}{\Gamma(1+\alpha)} \]
we have
\[ u_1 = e^{-\gamma} \frac{x^\alpha}{\Gamma(1+\alpha)} \]
\[ \vdots \]

*Fig. 10. The exact solution of \( u(x,t) \) at \( \alpha = 1 \)*
Fig. 11. the approximate solution of $u(x,t)$ at $\alpha = 0.5$

Fig. 12. the approximate solution of $u(x,t)$ at $\alpha = 0.9$
5. CONCLUSION

In this paper, we have successfully developed FCT with help of NIM to obtain approximate solution of the fractional differential equations. The fractional complex transform can easily convert a fractional differential equations to its differential partner, so that its New Iterative algorithm can be simply constructed. The fractional complex transform is extremely simple but effective for solving fractional differential equations. The method is accessible to all with basic knowledge of Advanced Calculus and with little fractional calculus. It may be concluded that FCT NIM is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of fractional differential equations.

REFERENCES


365, 345–350.


