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A Collocation Technique for Hybrid Block Methods with a Constructed Orthogonal basis for Second Order Ordinary Differential Equations

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Abstract

The numerical solution of Initial Value Problem (IVP) of general second order Ordinary Differential Equation (ODE) was the focus of this work. To achieve this, an orthogonal polynomial valid in the interval [0,1] and with respect to weight function $w(x) = x$ was constructed and employed as basis function for the development of some continuous hybrid schemes in a collocation and interpolation technique. The main scheme obtained was implemented together with the block formulae for the numerical solution of second order ordinary differential equations.

The schemes were analyzed using appropriate existing theorems to investigate their stability, consistency, convergence and the investigation shows that the developed schemes are consistent, zero-stable and hence, convergent. Numerical examples are presented to illustrate the accuracy and effectiveness of the method. The results obtained when compared with existing method are favourable.

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1. Introduction

Many fields of application notably in Science and Engineering yields Initial Value Problems (IVPs) of Second Order Ordinary Differential Equations (ODEs) of the form

$$y^{''} = f(x, y, y^{'}), y(a) = y_0, y^{'}(a) = z_0 \;\; x \in [a, b] \tag{1}$$

where f is continuous in $[a, b]$. Some of these problems may not be easily solved analytically, hence numerical schemes are developed to approximate the solution. The method of reducing (1) to a system of first order differential equations has been reported to increase the dimension of the problem and therefore results in more computations (see Bun(1992)). Some researchers have attempted the solution by directly using Linear Multtistep Methods (LMMs) without reduction to system of first order differential equations, they include: Onumanyi(1999), Yusuph and Onumanyi (2002) and Muhammed (2010).

Block Methods for solving ODEs were initially proposed by Milne (1953). The Milne's idea of proceeding in blocks was developed by Rosser (1967) for Runge-Kutta method. Many researchers have used block methods, they include: Adeniyi, Alabi and Folaranmi (2008), Awoyemi and Kayode (2008), Awari (2013), Umaru and Adeniyi (2014). While Anake (2011) used power series as the basis function, Adeyefa (2014) adopted Chebyshev as the basis function. Some researchers constructed orthogonal polynomials in certain interval for different weight functions. Haruna and Adeniyi (2014) constructed weight function of $w(x) = x^2$ for interval $[-1, 1]$, Bamgbola and Adeniyi (2014) used weight function $w(x) = x^2$ with interval [0,1] and Ekundayo and Adeniyi (2014) used weight function x for interval [0,1]. All these works were for first order differential equations.

The present study is an extension of those works to second order ODEs for the Initial Value Problem (1) by adopting an orthogonal polynomial constructed for the interval [0,1] with respect to the weight function $w(x) = x$.

2. Methodology

2.1. Derivation of the Orthogonal Polynomial

Consider the equation below

$$\int_a^b w(x)\phi_m(x)\phi_n(x)dx = h_n\delta_{mn} \tag{2}$$

with

$$\delta_{mn} = \begin{cases} 0, m \neq n \\ 1, m = 1 \end{cases}$$

where the weight function $w(x)$ is continuous and positive in the interval [a,b] such that the moments

$$\mu = \int_a^b w(x)x^n dx, \quad n = 0, 1, 2, \ldots \tag{3}$$

exists. The integral

$$\langle\phi_m,\phi_n\rangle = \int_a^b w(x)\phi_m(x)\phi_n(x)dx \tag{4}$$

is the inner product of the polynomials ϕ_m and ϕ_n.
For orthogonality,

$$\langle\phi_m,\phi_n\rangle = \int_a^b w(x)\phi_m(x)\phi_n(x)dx = 0, \;\; m \neq n, \; [-1,1]. \tag{5}$$

We shall now consider the requirements for the construction of orthogonal polynomials valid in [0,1] with respect to $w(x) = x$ for our basis function $\phi_n(x), \;\; n = 1,2,3,\ldots$ of the approximant

$$y_n(x) = \sum_{j=0}^{n} a_j\phi_j(x) \approx y(x). \tag{6}$$

For this purpose, let $\phi_n(x)$ be a polynomial of the nth order defined by

$$\phi_n(x) = \sum_{j=0}^{n} c_j^{(n)}x^j. \tag{7a}$$

The requirements for the construction are that

$$\phi_n(1) = 1 \tag{7b}$$

and

$$\int_o^1 w(x)\phi_m\phi_n(x)dx = 0; \;\; m \neq n.$$

Using the conditions above, the following orthogonal polynomial were generated

$$\phi_o(x) = 1$$

$$\phi_1(x) = 3x - 2$$

$$\phi_2(x) = 10x^2 - 12x + 3$$

$$\phi_3(x) = 35x^3 - 60x^2 + 30x - 4$$

$$\phi_4(x) = 126x^4 - 280x^3 + 210x^2 - 60x + 5$$

$$\phi_5(x) = 462x^5 - 1260x^4 + 1260x^3 - 560x^2 + 105x - 6$$

$$\phi_6(x) = 1716x^6 - 5544x^5 + 6930x^4 - 4200x^3 + 1260x^2 - 168x + 7$$

$$\phi_7(x) = 6435x^7 - 24024x^6 + 36036x^5 + 27720x^4 + 11550x^3 - 2520x^2 + 252x - 8.$$

2.2. Development of the Method

We shall consider the derivation of the proposed one-step continuous multistep method which will be used to generate the main method and other methods required to set the main method. In this study, we shall employ the generated polynomial as basis function for the derivation of our method. For this purpose, we shall approximate the analytical solution of (1) with an approximant of the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j \phi_j(x) \tag{8}$$

where $a_j's$ are constants to be determined, r is the number of collocation points and s is the number of interpolation points, on the partition

$$a = x_o < x_1 < \cdots < x_n < x_{n+1} < \cdots < x_N = b$$

of the integration interval [a,b] with a constant step size h, given by $h = x_{n+1} - x_n$, $n = 0, 1, \ldots, N-1$. The function $\phi_j(x)$ is the jth degree generated polynomial valid in the range of integration of (1), that is, in [a,b]. The second derivative of (8) is given by

$$y''(x) = \sum_{j=0}^{r+s-1} a_j \phi_j''(x) \tag{9}$$

where $x \in [a, b]$ and $r + s$ is the sum of the number of collocation and interpolation points.

We shall interpolate at at least two points to be able to approximate (1), and make this happen, we proceed by selecting some points x_{n+v} where $v \in (0, n)$. Then equation (8) is interpolated at x_{n+s} and its second derivative is collocated at x_{n+r}, so as to obtain a system of equations which will be solved by Gaussian Elimination Method. We shall consider this method as hybrid method.

2.3. Development of One-Step Method with one Off-Step Point

In deriving this method, we set $s = 2$ and $r = 3$ in (8) and (9) so as to obtain a system of five equations each of degree four as follows:

$$\sum_{j=0}^{4} a_j \phi_j(x) = y(x) \tag{10}$$

$$\sum_{j=0}^{4} a_j \phi_j''(x) = f(x, y, y''). \tag{11}$$

We now collocate (11) at $x = x_{n+s}$, $s = 0, \frac{1}{2}$ and 1, and interpolate (10) at $x = x_{n+r}$, $r = 0$ and $\frac{1}{2}$; from these we get a system of equations written in matrix form $AX = B$ as

$$\begin{bmatrix} 1 & -2 & 3 & -4 & 5 \\ 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{8} & \frac{3}{8} \\ 0 & 0 & 20 & -120 & 420 \\ 0 & 0 & 20 & -15 & -42 \\ 0 & 0 & 20 & 90 & 252 \end{bmatrix} \begin{bmatrix} a_o \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+\frac{1}{2}} \\ h^2 f_n \\ h^2 f_{n+\frac{1}{2}} \\ h^2 f_{n+1} \end{bmatrix}. \tag{12}$$

Equation (12) above is solved by Gaussian Elimination method to obtain the values of the unknown parameters a_j, $j = 0, 1, 2, 3...4$ as follows

$$\left.\begin{aligned} a_0 &= \frac{1}{360}(480y_{n+\frac{1}{2}} - 120y_n + 3h^2 f_n + h^2 f_{n+1} + 26h^2 f_{n+\frac{1}{2}}) \\ a_1 &= \frac{1}{5040}(3360y_{n+\frac{1}{2}} - 3360y_n + 75h^2 f_n + 19h^2 f_{n+1} + 494h^2 f_{n+\frac{1}{2}}) \\ a_2 &= \frac{h^2}{140}(f_{n+1} + 6f_{n+\frac{1}{2}}) \\ a_3 &= -\frac{h^2}{1890}(7f_n - 11f_{n+1} + 4f_{n+\frac{1}{2}}) \\ a_4 &= \frac{h^2}{756}(f_n + f_{n+1} - 2f_{n+\frac{1}{2}}). \end{aligned}\right\} \tag{13}$$

Substituting (13) in (10) yields a continuous implicit one-step method in the form

$$y(x) = \alpha_o(x)y_n + \alpha_1(x)y_{n+1} + h^2 \sum_{j=0}^{2} \beta_j(x) f_{n+j} \tag{14}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients. From (14), we get the parameters $\alpha_j(x)$ and $\beta_j(x)$ as

$$\left.\begin{aligned} \alpha_o &= \frac{(h - 2x + 2x_n)}{h} \\ \alpha_{\frac{1}{2}} &= \frac{2(x - x_n)}{h} \\ \beta_o &= \frac{((x - x_n)(h - 2x + 2x_n)(7h^2 - 10hx + 10hx_n + 4x^2 - 8x(x_n) + 4x_n^2))}{48h^2} \\ \beta_{\frac{1}{2}} &= \frac{((x - x_n)(h - 2x + 2x_n)(3h^2 - 6hx + 6hx_n - 4x^2 + 8x(x_n) - 4x_n^2))}{24h^2} \\ \beta_1 &= \frac{((x - x_n)(h - 2x + 2x_n)(h^2 + 2hx - 2hx_n - 4x^2 + 8x(x_n) - 4x_n^2))}{48h^2}. \end{aligned}\right\} \tag{15}$$

By evaluating (14) at x_{n+1}, the main method is obtained as

$$y_{n+1} = 2y_{n+\frac{1}{2}} - y_n + \frac{h^2}{48}\left\{f_n + 10f_{n+\frac{1}{2}} + f_{n+1}\right\}. \tag{16}$$

The block methods are derived by evaluating the first derivative of (14) in order to obtain additional equations needed to couple with (16).

Differentiating (14), we obtain

$$y'(x) = \sum_{j=0}^{1} \alpha'_j(x) y_{n+j} + \sum_{j=0}^{2} \beta'_j(x) f_{n+j} \tag{17}$$

where

$$\left.\begin{aligned}
\alpha'_o &= -\frac{2}{h}\\
\alpha'_{\frac{1}{2}} &= \frac{2}{h}\\
\beta'_o &= -\frac{(7h^3 - 48h^2x + 48h^2x_n + 72hx^2 - 144hx(x_n) + 72hx_n^2 - 32x^3 + 96x^2x_n - 96xx_n^2 + 32x_n^3)}{48h^2}\\
\beta'_{\frac{1}{2}} &= -\frac{(3h^3 - 48hx^2 + 96hx(x_n) - 48hx_n^2 + 32x^3 - 96x^2(x_n) - 96x(x_n)^2 - 32x_n^3)}{24h^2}\\
\beta'_1 &= \frac{((h - 4x + 4x_n)(h^2 + 4hx - 4hx_n - 8x^2 + 16x(x_n) - 8x^2))}{48h^2}.
\end{aligned}\right\} \tag{18}$$

Evaluating (17) at x_n, $x_{n+\frac{1}{2}}$ and x_{n+1}, we get the following discrete derivative schemes:

$$hy'_n = -2y_n + 2y_{n+\frac{1}{2}} - \frac{h^2}{48}(7f_n - f_{n+1} + 6f_{n+\frac{1}{2}}) \tag{19}$$

$$hy'_{n+\frac{1}{2}} = -2y_n + 2y_{n+\frac{1}{2}} + \frac{h^2}{48}(3f_n - f_{n+1} + 10f_{n+\frac{1}{2}}) \tag{20}$$

$$hy'_{n+1} = -2y_n + 2y_{n+\frac{1}{2}} + \frac{h^2}{48}(f_n + 9f_{n+1} + 26f_{n+\frac{1}{2}}). \tag{21}$$

Solving equations (16), (19), (20) and (21) simultaneously, we obtain the following explicit results

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{2}y'_n + \frac{h^2}{96}(7f_n + 6f_{n+\frac{1}{2}} - f_{n+1}) \tag{22}$$

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{6}(f_n + 2f_{n+\frac{1}{2}}) \tag{23}$$

$$y'_{n+\frac{1}{2}} = y'_n + \frac{h}{24}(5f_n + 8f_{n+\frac{1}{2}} - f_{n+1}) \tag{24}$$

$$y'_{n+1} = y'_n + \frac{h}{6}(f_n + 4f_{n+\frac{1}{2}} + f_{n+1}). \tag{25}$$

Solving (29) by Gaussian Elimination yields the a's as follows:

$$\left.\begin{aligned}
a_o &= \frac{1}{22680}(45360y_{n+\frac{1}{3}} - 22680y_n + 208h^2f_n + 23h^2f_{n+1} + 2211h^2f_{n+\frac{1}{3}} + 708h^2f_{n+\frac{2}{3}})\\
a_1 &= \frac{1}{45360}(45360y_{n+\frac{1}{3}} - 45360y_n + 397h^2f_n + 77h^2f_{n+1} + 4155h^2f_{n+\frac{1}{3}} + 1923h^2f_{n+\frac{2}{3}})\\
a_2 &= \frac{h^2}{560}(f_{n+1} + 9f_{n+\frac{1}{3}} + 18f_{n+\frac{2}{3}})\\
a_3 &= \frac{h^2}{420}(f_{n+1} - 5f_{n+\frac{1}{3}} + 4f_{n+\frac{2}{3}})\\
a_4 &= \frac{h^2}{3696}(4f_n + 7f_{n+1} - f_{n+\frac{1}{3}} - 10f_{n+\frac{2}{3}})\\
a_5 &= -\frac{3h^2}{6160}(f_n - f_{n+1} - 3f_{n+\frac{1}{3}} + 3f_{n+\frac{2}{3}}).
\end{aligned}\right\} \tag{30}$$

Substituting (30) into (26) yields the continuous implicit hybrid one step method in the form

$$\alpha_{\frac{2}{3}}y_{n+\frac{2}{3}} + \alpha_1 y_{n+1} = h^2(\beta_o f_n + \beta_{\frac{1}{3}}f_{n+\frac{1}{3}} + \beta_{\frac{2}{3}}f_{n+\frac{2}{3}} + \beta_1 f_{n+1}) \tag{31}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients. From (31), we get the parameters $\alpha_j(x)$ and $\beta_j(x)$ as

$$\left.\begin{aligned}
\alpha_o &= \frac{1}{h}(h - 3x + 3x_n)\\
\alpha_{\frac{1}{3}} &= \frac{3(x - x_n)}{h}\\
\beta_o &= -\frac{1}{1080h^3}((x - x_n)(h - 3x + 3x_n)(97h^3 - 249h^2x + 249h^2x_n + 243hx^2 - 486hx(x_n) +\\
&\quad 243hx_n^2 - 81x^3 + 243x^2(x_n) - 243x(x_n)^2 + 81x_n^3))\\
\beta_{\frac{1}{3}} &= -\frac{1}{360h^3}((x - x_n)(h - 3x + 3x_n)(38h^3 + 114h^2x - 114h^2x_n - 198hx^2 + 396hx(x_n)\\
&\quad -198hx_n^2 + 81x^3 - 243x^2(x_n) + 243x(x_n)^2 - 81x_n^3))\\
\beta_{\frac{2}{3}} &= \frac{1}{360h^3}((x - x_n)(h - 3x + 3x_n)(13h^3 + 39h^2x - 39h^2x_n - 153hx^2 + 306hx(x_n) -\\
&\quad 153hx_n^2 + 81x^3 - 243x^2(x_n) + 243x(x_n)^2 - 81x_n^3))\\
\beta_1 &= -\frac{1}{1080h^3}((x - x_n)(h - 3x + 3x_n)(2h - 3x + 3n)(4h^2 + 18hx - 18hxn - 27x^2 + 54x(x_n)\\
&\quad -27x_n^2)).
\end{aligned}\right\} \tag{32}$$

Evaluating (31) at $x_{n+\frac{2}{3}}$ and x_{n+1}, the main method is obtained as

$$y_{n+\frac{2}{3}} = y_n + 2y_{n+\frac{1}{3}} + \frac{h^2}{108}(f_n + 10f_{n+\frac{1}{3}} + f_{n+\frac{2}{3}}) \tag{33}$$

Solving equations (33), (34), (37), (38), (39) and (40) simultaneously, the following explicit results are obtained.

$$y_{n+\frac{1}{3}} = y_n + \frac{h}{3}y_n^{'} + \frac{h^2}{3240}(97f_n + 114f_{n+\frac{1}{3}} - 39f_{n+\frac{2}{3}} + 8f_{n+1}) \quad (41)$$

$$y_{n+\frac{2}{3}} = y_n + \frac{2h}{3}y_n^{'} + \frac{h^2}{405}(28f_n + 66f_{n+\frac{1}{3}} - 6f_{n+\frac{2}{3}} + 2f_{n+1}) \quad (42)$$

$$y_{n+1} = y_n + hy_n^{'} + \frac{h^2}{120}(13f_n + 36f_{n+\frac{1}{3}} + 9f_{n+\frac{2}{3}} + 2f_{n+1}) \quad (43)$$

$$y_{n+\frac{1}{3}}^{'} = y_n^{'} + \frac{h}{72}(9f_n + 19f_{n+\frac{1}{3}} - 5f_{n+\frac{2}{3}} + f_{n+1}) \quad (44)$$

$$y_{n+\frac{2}{3}}^{'} = y_n^{'} + \frac{h}{9}(f_n + 4f_{n+\frac{1}{3}} + f_{n+\frac{2}{3}}) \quad (45)$$

$$y_{n+1}^{'} = y_n^{'} + \frac{h}{8}(f_n + 3f_{n+\frac{1}{3}} + 3f_{n+\frac{2}{3}} + f_{n+1}). \quad (46)$$

3. Analysis of the Method

The basic properties of this method such as order, error constant, zero stability and consistency are analyzed here under.

Equations (16), (33) and (34) derived are discrete schemes belonging to the class of LMMs of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}. \quad (47)$$

Following Futunla (1988) and Lambert (1973), we define the Local Truncation Error(LTE) associated with (47) by difference operator;

$$L[y(x):h] = \sum_{j=0}^{k} [\alpha_j y(x_n + jh) - h^2 \beta_j F(X_n + jh)] \quad (48)$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a,b]$. Expanding (48) in Tayloy Series about the point x, we obtain the expression

$$L[y(x):h] = c_o y(x) + c_1 hy^{'}(x) + \cdots c_{p+2}h^{p+2}y^{p+2}(x)$$

$$B = \begin{bmatrix} \frac{6}{96} & -\frac{1}{96} \\ \frac{2}{6} & 0 \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \end{bmatrix}$$

$$d = \begin{bmatrix} 0 & \frac{7}{96} \\ 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix}$$

The first characteristic polynomial of the block hybrid method is given by

$$\rho(R) = det(RA^o - A^1) \qquad (49)$$

where

$$A^o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A^1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Substituting A^o and A^1 in equation (49) and solving for R, the values of R are obtained as 0 and 1. According to Fatunla (1988, 1991), the block method equations (22)-(25) are zero-stable, since from (49), $\rho(R) = 0$, satisfy $|R_j| \leq 1$, $j = 1$ and for those roots with $|R_j| = 1$, the multiplicity does not exceed two.

Equations (41)–(46) give

$$A^o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[y_{n+\frac{1}{2}}, y_{n+\frac{2}{3}}, y_{n+1} \right]'$$

$$A^1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \left[y_{n-2}, y_{n-1} \right], y'_n$$

$$B = \begin{bmatrix} \frac{114}{3240} & -\frac{39}{3240} & \frac{8}{3240} \\ \frac{66}{405} & -\frac{6}{405} & \frac{2}{405} \\ \frac{36}{120} & \frac{9}{120} & \frac{2}{120} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{bmatrix}$$

$$d = \begin{bmatrix} 0 & 0 & \frac{97}{3240} \\ 0 & 0 & \frac{28}{405} \\ 0 & 0 & \frac{13}{120} \end{bmatrix} \begin{bmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}$$

4. Numerical Examples

We consider here the application of the derived scheme on three test problems.

Example 4.1. (Special linear problem)

$$y^{''} - 100y = 0, 0 \leq x \leq 0.12$$

$$y(0) = 1, y^{'}(0) = -10, h = 0.01$$

Exact solution: $y(x) = e^{-10x}$
Areo et al. (2013).

Numerical results for this example are presented in Table 4.1 and Table 4.4.

Example 4.2. (Variable Coefficient Nonlinear Problem)

$$y^{''} - x(y^{'})^2 = 0$$

$$y(0) = 1, y^{'}(0) = \frac{1}{2}, h = 0.0025$$

Exact Solution: $y(x) = 1 + \frac{1}{2} ln \frac{2+x}{2-x}$
Awoyemi et al. (2012). See Table 4.2 and Table 4.5 for the numerical results for this problem.

Example 4.3. (Special Nonhomogenous problem)

$$y^{''} = y + xe^{3x}, y(0) = -\frac{3}{32},$$

$$y^{'}(0) = -\frac{5}{32} \;\; h = 0.0025.$$

Exact solution: $y(x) = \frac{(4x-3)}{32e^{-3x}}$
Adesanya et al. (2009).
See Table 4.3 and Table 4.6 for the numerical results for this problem.

Table 3: Numerical Results of OMOOP for Example 4.3

X	Exact method	One Offstep Method	Error
0.0025	-0.094140915761849	-0.094140915761852	2.997602166487923e-015
0.0050	-0.094532404142339	-0.094532404142346	6.994405055138486e-015
0.0075	-0.094924451608388	-0.094924451608399	1.100508573159686e-014
0.0100	-0.095317044390700	-0.095317044390716	1.600108934241007e-014
0.0125	-0.095710168480981	-0.095710168481000	1.899869150889799e-014
0.0150	-0.096103809629113	-0.096103809629137	2.400857290751901e-014
0.0175	-0.096497953340316	-0.096497953340344	2.800537579616957e-014
0.0200	-0.096892584872264	-0.096892584872296	3.200217868482014e-014
0.0225	-0.097287689232184	-0.097287689232220	3.599898157347070e-014
0.0250	-0.097683251173920	-0.097683251173960	3.999578446212126e-014

Table 4: Numerical Results of OMTOP for Example 4.1

X	Exact method	Two Offstep Method	Error
0.01	0.904837418035960	0.904837418109995	7.403500035252364e-011
0.02	0.818730753077982	0.818730753366770	2.887879935187243e-010
0.03	0.740818220681718	0.740818221314804	6.330860280456818e-010
0.04	0.670320046035639	0.670320047133962	1.098322988113409e-009
0.05	0.606530659712633	0.606530661390881	1.678247985026360e-009
0.06	0.548811636094027	0.548811638462824	2.368796936380591e-009
0.07	0.496585303791409	0.496585306959364	3.167954953919860e-009
0.08	0.449328964117222	0.449328968182859	4.075636994560483e-009
0.09	0.406569659740599	0.406569664834220	5.093621024965245e-009
0.10	0.367879441171442	0.367879447396920	6.225478021981701e-009
0.11	0.332871083698080	0.332871091174631	7.476550989427011e-009
0.12	0.301194211912202	0.301194220766149	8.853946964482162e-009

Comparison of the Absolute Error for the Methods

Table 7: Comparing Absolute Error of the Methods for Example 4.1

One Offstep method	Two Offstep Method	Areo(2013)
-7.236727095349949e-008	-7.403500035252364e-011	3.0×10^{-10}
-8.389529798646933e-008	-2.887879935187243e-010	1.0×10^{-09}
-9.574455300809248e-008	-6.330860280456818e-010	2.8×10^{-09}
-1.080829660216054e-007	-1.098322988113409e-009	3.0×10^{-09}
-1.210786619854076e-007	-1.678247985026360e-009	5.1×10^{-09}
-1.349020959939828e-007	-2.368796936380591e-009	6.4×10^{-09}
-1.497281689744057e-007	-3.167954953919860e-009	9.4×10^{-09}
-1.210786619854076e-007	-4.075636994560483e-009	1.15×10^{-08}
-1.080829660221654e-007	-5.093621024965245e-009	1.49×10^{-08}
-1.349020959939828e-007	-6.225478021981701e-009	1.77×10^{-08}
-1.497281689744057e-007	-7.476550989427011e-009	2.3×10^{-08}
-1.497281689744057e-007	-8.853946964482162e-007	2.7×10^{-08}

Table 8: Comparing Absolute Error of the Methods for Example 4.2

One Offstep method	Two Offstep Method	Awoyemi(2012)
1.998401444325282e-015	1.998401444325282e-015	0.498272253e-10
3.108624468950438e-015	3.108624468950438e-015	0.41043058e-09
2.886579864025407e-015	2.886579864025407e-015	0.14285815e-08
1.998401444325282e-015	1.998401444325282e-015	0.35242687e-08
5.995204332975845e-015	5.995204332975845e-015	0.7243532e-08
5.995204332975845e-015	5.995204332975845e-015	0.13335597e-07
1.088018564132653e-014	1.088018564132653e-014	0.22872871e-07
1.487698852997710e-014	1.487698852997710e-014	0.37447019e-07
1.887379141862766e-014	1.887379141862766e-014	0.59503708e-07

Ph.D. Thesis (Unpublished) Covenant University, Ota, Nigeria (2011).

[5] E.A. Areo, A Self-Starting Linear Multistep Method for Direct Solution of Initial Value Problem of Second Order Ordinary Differential Equations, International Journal of Pure and Applied Mathematics, 85(2013), 345–364.

[6] Y.S. Awari, Derivation and Application of Six Point Linear Multistep Numerical Methods for Solution of second Order Initial Value Problems. *ISOR Journal of Mathematics*, 7(2013), 23–29.

[7] D.O. Awoyemi, S.J. Kayode, A Minimal Order Collocation Method for Direct Solution of Initial Value Problems of General Second Order Ordinary Differential Equations. In proceedings of the conference organized by National Mathematical Centre, Abuja, Nigeria (2008).

[8] D.O. Awoyemi, T.A. Anake, A.O. Adesanya, A One Step Method for Solution of General Second Ordinary Differential Equations, 2(2012), 159–163.

[9] M.O. Bamgbola, R.B. Adeniyi, Formulation of Discrete and Continuous Hybrid Method, Using Orthogonal Polynomials as Basis Function. *Journal of Nigerian Association of Mathematical Physics.* (accepted).

[10] R.A. Bun, Y.D. Vsilyev, A Numerical Method for Solving Differential Equations of any orders. *Comp. Math. Physics*, 32(1992), 317–330.

[11] G. Dahlquist, Some Properties of Linear Multistep and One Leg Method for Ordinary differential Equations. Department of Computer science, Royal institute of Technology, Stockholm (1979).

[12] G. Dahlquist, On One-leg Multistep Method. *SIAM Journal on Numerical Analysis*, 20(1983), 1130–1146.

[13] F. Ekundayo, R.B. Adeniyi, (2014). An Orthogonal Collocation Approach for Continuous Formulation of Hybrid Methods for Initial Value Problems in Ordinary Differential Equations. *Journal of Nigerian Association of Mathematical Physics* (2014)(accepted).

[14] S.O. Fatunla, Numerical Methods for Initial Value Problems in Ordinary Differential Equations, Academic Press inc. Harcourt Brace, Jovanovich Publishers, New York (1988).

[15] S.O. Fatunla, Block Methods for Second Order Initial Value Problem (IVP), International Journal of Computer Mathematics, 41(1991), 55–63.

[16] Y. Haruna, R.B. Adeniyi, A Collocation Technique Based on Orthogonal Polynomial for Construction of Continuous Hybrid Methods. *Ilorin Journal of Science* (Accepted).

[17] J.D. Lambert, Computational Methods in Ordinary Differential System, *John Wiley and Sons, New York* (1973).