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On the Derivation of the Pricing Equation of Collateralized Deals Using BSDE Approach

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Abstract

The credit crunch of 2007-2008 demonstrated a potential change in derivatives pricing methodologies, mostly the consideration of the counterparty risks in the valuation procedures. The standard pricing theories based on the martingale principles are subject to produce unrealistic results because of not taking the counterparty credit exposure into consideration. One way the problem can be handled is to introduce counterparty risks in the pricing equations by using the effect of collateralization scheme, especially extreme liquid transactions. Assuming a Lipschitz continous driver , this study will investigate a backward stochastic differential equation as a pricing equation based on the market which is weakly efficient. We use these results to ensure the solution of the pricing equations on both cases, collateral dominated in domestic and foreign currencies. We introduce the basis currency in the case of collateral being posted in foreign currency in order to preclude arbitrage opportunities. Overall, We found that the prices differ from each other.

1. Introduction

A decade ago, after the global financial crisis of 2007 – 2008, the quantitative finance experienced unexpected changes particularly in asset pricing. The disregarded factors in the old valuation discounting framework, counterparty credit risks and funding costs, had proved to contribute too much to the aforementioned crisis. However, banks and other financial institutions dealing with derivatives deals have been pushed by this crisis event to carefully review the principles and regulations governing the derivatives contracts traded either in the over-the-counter(OTC) or clearing counterparties, identify new risks embedded in these contracts and hence propose new mitigation tools and new regulatory frameworks against these risks.

The pricing problem in the post crisis concerns about the inclusion of counterparty credit risks and funding costs in the valuation framework. This causes controversial views among practitioners and academics. The most problematic is the introduction of various valuation adjustments, namely funding, debit and credit valuation adjustments, into pricing formula. In addition, the financial crisis had witnessed a bilateral counterparty credit risk in a derivative contract. Since banks can default on their obligations as some banks did during the crisis, the consideration of counterparty risks in the valuation should be bilateral instead of unilateral. It is clear that the counterparty credit risk happens if one party fails to meet her contractual obligations, hence measures to reduce the counterparty credit risks would be primary of importance in the new valuation framework.

As specified in citeISDA, collateralization is widely used in the over-the-counter derivatives contracts and considered as a mitigation tool for counterparty exposures. Hence, the use of collateral is increasingly becoming a standard practice in the derivatives transactions happened in bilateral trading positions. It is worth mentioning that collateral positing is regulated by a Credit Support Annex(CSA) under the International Swaps and Derivatives Association (ISDA) master agreement which specifies the posting and margining rules. A CSA is a bilateral contract that specifies the collateral agreements between counterparties in the OTC derivatives transactions. The counterparty whose market-to-market(MTM) value is negative posts a collateral on the collateral account of her counterparty.

Putting these factors into the valuation discounting framework, one needs to consider the possibility of defaulting by either counterparty. This consideration violates the risk-neutral valuation which is based on the risk free assumption, seemingly unrealistic. In this paper, we are concern about the impact of collateralization on the standard asset pricing theory, specifically the valuation framework of an option-European style. The main motivation of this work is due to the emerging of collateral requirements in the over-the-counter derivatives transactions. In this context, previous and current works are concentrated on assessing the impact of collateralization in derivatives transactions and its inclusion in the standard derivatives pricing theories. This has been investigated recently in [5], [6], and the references therein. The introduction of collateral agreements under CSA into option pricing, Black-Scholes European style, has been discussed by [29]. The author derived a partial differential(PDE) equation representing the dynamics of the value of a derivative which depends on different rates including collateral rate. Although the derivation did not consider the effect of default by any counterparty it showed that the price of the derivative does not depend on the risk free rate.

As it has been shown in the previous works for example, pricing derivatives with partial differential equation (PDE) approach has mostly been considered in the risk neutral valuation. For example the celebrated Black-Scholes formula. Note that the risk neutral valuation does not account for the counterparty risks, and by its name states there is a need to change measures. However, the valuation PDE approach is not a good candidate to value defaultable contingent claims. As discussed by [10], including counterparty risks in a pricing formula is equivalent to forming a backward stochastic differential equation(BSDE). In this context the BSDE representation depends on different interest rates. An important feature of BSDE approach there is no need to switch to another probability measure; that is, the governing equation of a financial deal is formulated under a single probability measure which is the physical measure [28]. In more general settings, BSDEs are suitable for solving hedging problems where niether continuity of pay-off nor a Markovian dynamics is assumed [12]. Once again, a BSDE representation has been identified to mathematically characterize a financial event having a component of counterparty credit risk [9]. All of these features back a BSDE approach to be a best candidate to model defaultable contingent claims.

Adopting the replication strategy we construct a BSDE which represents the dynamics of a European option considering the effects of counterparty exposure. The introduction of collateralization will be treated as a mitigating tool against this exposure. Depending on the CSA agreements, collateralization may require a counterparty to post collateral which dominated in the same currency as the underlying or in a different currency. In this context, we allow counterparties to post collaterals dominated either in the same or different currency from the underlying. The valuation problem here would be solved using no arbitrage conditions and market completeness arguments. These arguments will be given by the existence and uniqueness of the solution for the resulting pricing equation.

2. Backward Stochastic Differential Equation with Jumps

Over the decades, backward stochastic equations have been regarded as a strong tool to solve financial problems, mostly linked to the optimal control of the systems and pricing contingent claims. Researches have been done in line with these problems under different contexts and considerations to find the existence and uniqueness of the resulting equations. This section will focus on the existence and uniqueness of the backward stochastic differential equation driven by the two independent processes, namely Brownian Motion and Poisson random measure. In setting up these mathematical backgrounds, we use the same notations as in Cordoni and Di Persio [8]. Let us fix a terminal time $T > 0$ and a filtered probability space $(\Omega, \mathscr{F}_t, \mathbb{F}, \mathbb{P})$ with $\mathscr{F}_t := (\mathscr{F}_t)_{t\in[0,T]}$, available information at time t generated by both the Brownian motion W and Poisson random measure $N(dt,dx)$ on $\mathbb{R}_+ \times E$ with $E := \mathbb{R}\backslash\{0\}$. Note that the compensated Poisson random measure $\tilde{N}(dx,dt)$ is a martingale and defined as $\tilde{N}(dt,dx) = N(dt,dx) - \lambda(dx)dt$, where $\lambda(dx)dt$ is σ-finite measure on $(E, \mathcal{B}(E))$ satisfying, for more details see Barles et al. [2],

$$\int_E (1 \wedge |x|^2)\lambda(dx) < \infty$$

Throughout this section, we consider the following definitions and spaces in establishing the existence and solutions.

(i) The $L^2(\mathscr{F}_T)$ represents the space of square integrable and $\mathscr{F}_T$-measurable processes.

(ii) The $\mathbb{S}^2(\mathbb{R})$ denotes the set of all progressively measurable processes Y such that $\mathbb{E}\left(\sup_t |Y|^2\right) < \infty$

(iii) The $\mathbb{H}^2(\mathbb{R})$ denotes the set of all predictable processes Z such that $\mathbb{E}\left(\int_0^T |Z|^2\, dt\right) < \infty$

(iv) The $\mathbb{H}^2_N(\mathbb{R})$ represents the set of all $\mathcal{P}\otimes\mathcal{B}(E)-$ measurable processes U such that

$$\mathbb{E}\left(\int_0^T \int_E \left|u_t(x)\right|^2 \lambda(dx)dt\right) < \infty$$

where $\mathcal{P}$ stands for the $\sigma-$algebra of all $\mathbb{F}-$predictable sets of $[0,T]\times\Omega$ and $\tilde{N}(dx,dt)$ the compensated Poisson random measure.

We consider the following linear BSDE with jumps of the form

$$dY_t = -f(t,Y_t,Z_t,U_t) + Z_t dW_t + \int_E U_t(x)\tilde{N}(dt,de),\ Y_T = \xi \tag{2.1}$$

or equivalently

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s,U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_t(x)\tilde{N}(ds,de)$$

The function f is called driver and ξ is the terminal condition and $\tilde{N}(dt,de)$ is a compensated Poisson counting measure, as defined above, that counts the size of the jump occurred at time t. The pair of (f,ξ) is called standard data of the BSDE (2.1) and satisfy certain conditions to have solutions.

Indeed the existence of the solution plays a vital role in the theory of BSDEs, for example for the BSDE of (2.1) to have a meaning. Different directions have been taken into considerations by different researchers to establish the existence results, whether the solution is unique or not. In this direction, much efforts were devoted to find the existence and uniqueness of the solution of (2.1). The most restrictive assumptions are finiteness of the terminal value ξ and the continuity of the driver f. For example the authors in [27], [14] and the reference therein defined the terminal condition ξ to be square integrable (i.e, finite) and assumed the driver f to be Lipschitz continuous with respect to y and z, in the case there are no jumps. We note that all of these conditions and/or assumptions are particularly restrictive to the standard data. The key interesting point here is to find the existence of the solution for the BSDE equation (2.1).

(C1) The terminal condition ξ is square integrable (i.e $\xi \in L^2(\mathscr{F}_T)$).

(**C2**) $f(t,0,0,0) \in L^2(\mathscr{F}_T)$

(**C3**) The driver f is Lipschitz with respect to y, z and u. There exists a constant $C > 0$ such that

$$\left|f(t,y,z,u) - f(t,y',z',u')\right| \leq C\left(\left|y-y'\right| + \left|z-z'\right| + \left\|u-u'\right\|\right) \tag{2.2}$$

Definition 2.1. *A triple of processes* $(Y_t, Z_t, U_t)_{0\leq t\leq t} \in \mathbb{S}^2(\mathbb{R})\times\mathbb{H}^2(\mathbb{R})\times\mathbb{H}^2_N(\mathbb{R})$ *is a solution satisfying*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_t(x)\tilde{N}(ds, de) \tag{2.3}$$

where U is a jump size distribution.

So far we have considered a one-dimensional standard BSDE with Lipschitz driver. The interest here is to find the $\mathscr{F}_t$-adapted processes Y_t, Z_t and U_t such that the equation 2.3 holds. In the favor of applications, the following results tell us that there exits a unique solution for the BSDE 2.3. Before that we would like to present two important lemma that will be used later. The first lemma gives a representation for a jump-diffusion models and the other one establishes the a priori estimates. We mention that representation results are very important to solve problems pertaining to hedging claims in financial mathematics Rachev [30]. In diffusion stochastic processes, every square integrable martingale adapted to the natural filtration of Brownian motion can be written as a stochastic integral with respect to Brownian motion Kallianpur and Karandikar [22] solving hedging problems. The extension to the jump-diffusion processes can be found in the standard books on financial mathematics, for example . In fact, we consider the following result from [12].

Lemma 2.1 (**Martingale representation theorem**). *Let M_t be a square integrable martingale defined on $(\Omega, \mathscr{F}_t, \mathbb{F}, \mathbb{P})$. There exist a predictable process ϕ_t, square integrable process $\phi_t \in \mathbb{H}^2(\mathbb{R})$ and $\psi_t(x) \in \mathbb{H}^2_N(\mathbb{R})$ such that*

$$M_t = M_0 + \int_0^t \phi_s dW_s + \int_0^t \int_E \psi_t(x)\tilde{N}(ds, de). \tag{2.4}$$

where ϕ and ψ are $\mathscr{F}_t$-predictable processes, integrable with respect to W and $\tilde{N}$ respectively.

Note that the predictable processes ϕ and ψ in the equations (2.4) are unique Rémillard and Renaud [31]. The second lemma estimates the solution of the BSDE 2.3 as follows.

Lemma 2.2 (**A priori estimates**). *Let (ξ^i, f^i), $i = 1,2$, be the standard data for the BSDE of 2.3 and the triple (Y^i, Z^i, U^i) the corresponding solutions. Let C be a Lischitz constant for f^1 and define $\Delta Y_t = Y_t^1 - Y_t^2$, $\Delta Z_t = Z_t^1 - Z_t^2$, $\Delta U_t = U_t^1 - U_t^2$ and $\Delta_2 f_t = f^1(t, Y^2, Z^2, U^2) - f^2(t, Y^2, Z^2, U^2)$. For any $\beta > C(2+\mu+\gamma)$ and $\mu, \gamma > C$; then the following a priori estimates hold*

$$\begin{aligned}
\left\|\Delta Y\right\|_\beta^2 &\leq T\left[e^{\beta T}\mathbb{E}\left(\left|\Delta Y_T\right|^2\right) + \frac{1}{\beta - C(2+\mu+\gamma)}\left\|\Delta f\right\|^2\right] \\
\left\|\Delta Z\right\|_\beta^2 &\leq \frac{\mu}{\mu - C}\left[e^{\beta T}\mathbb{E}\left(\left|\Delta Y_T\right|^2\right) + \frac{1}{\beta - C(2+\mu+\gamma)}\left\|\Delta f\right\|^2\right] \\
\left\|\Delta U\right\|_\beta^2 &\leq \frac{\gamma}{\gamma - C}\left[e^{\beta T}\mathbb{E}\left(\left|\Delta Y_T\right|^2\right) + \frac{1}{\beta - C(2+\mu+\gamma)}\left\|\Delta f\right\|^2\right]
\end{aligned} \tag{2.5}$$

Proof. Extending the proposition 2.2 in [3] to the jump-diffusion process will give us results. ■

Theorem 2.3 ([26]). *Suppose that lemma 2.1 is true and assume that the assumptions* **(C1)**, **(C2)** *and* **(C3)** *hold. Then a triplet of the processes (Y, Z, U) uniquely solves the BSDE (2.3).*

Proof. The proof is based on the contraction mapping and martingale representation theorem. We closely follow and apply the theorem 2.1 in [3] to the jump-diffusion process.The adaptedness property allows us to define Y_t as

$$Y_t = \mathbb{E}\left[\xi + \int_t^T f(s, Y_s, Z_s, U_s)ds \middle| \mathscr{F}_t\right], \text{for all} 0 \leq t < \tau \tag{2.6}$$

By Lemma 2.1, there exists a pair of processes (Z, U) such that

$$Y_t = Y_0 + \int_0^t Z_s dW_s + \int_0^t \int_E U_t(x)\tilde{N}(ds, de) \tag{2.7}$$

From the equations (2.6) and (2.7) it follows that

$$\xi = Y_0 + \int_0^T Z_s dW_s + \int_0^T \int_E U_t(x)\tilde{N}(ds, de) - \int_t^T f(s, Y_s, Z_s, U_s)ds \tag{2.8}$$

Thererfore the result follows from extracting the value of Y_0 from the eqaution (2.8) and put into (2.7). Then, we remain to show the existence of Y_t. To do that, it is sufficient to show that the process Y_t is fine. From the BSDE (2.3) we see that the inequality below still holds

$$|Y_t| \leq |\xi| + \left|\int_0^T f(t,Y_t,Z_t,U_t)dt\right| + \left|\int_0^T Z_t dW_t\right| + \left|\int_0^T \int_E U_t(x)\tilde{N}(dt,de)\right| \tag{2.9}$$

On the other hand we know that, by quadratic inequality, $(a+b+c+d)^2 \leq 6a^2 + 6b^2 + 6c^2 + +6d^2$, we have

$$\sup_{t\leq T}|Y_t|^2 \leq 6|\xi|^2 + 6\left|\int_0^T f(t,Y_t,Z_t,U_t)dt\right|^2 + 6\sup_{t\leq T}\left|\int_0^T Z_t dW_t\right|^2 + 6\sup_{t\leq T}\left|\int_0^T \int_E U_t(x)\tilde{N}(dt,de)\right|^2$$

and both by Doob's inequality and taking expection on both sides we get

$$\mathbb{E}\left(\sup_{t\leq T}|Y_t|^2\right) \leq 6\mathbb{E}\left(|\xi|^2\right) + 6T\mathbb{E}\left(\int_0^T |f(t,Y_t,Z_t,U_t)|^2 dt\right) + 24\mathbb{E}\left(\int_0^T |Z_t|^2 dt\right) + 24\mathbb{E}\left(\int_0^T \int_E |U_t(x)|^2 \alpha(dt,de)\right) \tag{2.10}$$

Each term on the right hind side of (4.1) is finite, so does $\mathbb{E}\left(\sup_{t\leq T}|Y_t|^2\right)$. Therefore, there exists a finite and adapted Y_t such that $\mathbb{E}\left(\sup_{t\leq T}|Y_t|^2\right) < \infty$.

We remain now to show that the solution (Y,Z,U) is unique. We only need to show that the mapping that maps to itself is a contraction. Let $Q(y,z,u)$ be a mapping such that $Q(y,z,u) = (Y,Z,U)$. Let (y^1,z^1,u^1) and (y^2,z^2,u^2) be such that $Q(y^1,z^1,u^1) = (Y^1,Z^1,U^1)$ and $Q(y^2,z^2,u^2) = (Y^2,Z^2,U^2)$ where (Y^1,Z^1,U^1) and (Y^2,Z^2,U^2) are the solutions of (2.3). Assume that the estimates in Lemma 2.1 holds. Then we see from the inequality (2.5) that $\Delta_2 f = f^1(y^1,z^1,u^1) - f^2(y^2,z^2,u^2) = f(y^1,z^1,u^1) - f(y^2,z^2,u^2)$ and $\Delta\xi = 0$. These are true because it is involving one equation with two solutions (Y^1,Z^1,U^1) and (Y^2,Z^2,U^2). Taking $C = 0$, the inequality (2.5) will change to

$$\|\Delta Y\|_\beta^2 \leq \frac{T}{\beta}\mathbb{E}\left[\int_0^T e^{\beta s}\left|f(y^1,z^1,u^1) - f(y^2,z^2,u^2)\right|^2 ds\right]$$

$$\|\Delta Z\|_\beta^2 \leq \frac{1}{\beta}\mathbb{E}\left[\int_0^T e^{\beta s}\left|f(y^1,z^1,u^1) - f(y^2,z^2,u^2)\right|^2 ds\right]$$

$$\|\Delta U\|_\beta^2 \leq \frac{1}{\beta}\mathbb{E}\left[\int_0^T e^{\beta s}\left|f(y^1,z^1,u^1) - f(y^2,z^2,u^2)\right|^2 ds\right]$$

We know that $f(y,z,u)$ satisfies the assumption **(C3)**, then it implies that

$$\|\Delta Y\|_\beta^2 + \|\Delta Z\|_\beta^2 + \|\Delta U\|_\beta^2 \leq \frac{(4T+8)C^2}{\beta}\left(\|\Delta y\|_\beta^2 + \|\Delta z\|_\beta^2 + \|\Delta u\|_\beta^2\right) \tag{2.11}$$

The inequality (2.11) shows that the mapping $Q(y,z,u)$ is a contraction with $(4T+8)C^2 < \beta$. Therefore, there exists a fixed point which is a unique solution of the BSDE (2.3). ■

3. Pricing collateralized financial deals

3.1. Pricing domestic deals collateralized in domestic currency

In this section we focuse on credit support annex(CSA) discounting framework where the collateral is posted in the deal's currency. The basic idea of this framework is that collateralized financial deal should be discounted at collateral rate of the currency under consideration. In the sequel we will consider financial deals that are subject to counterparty risks. This has been emphased by the work of [29] in the sense that in real world it is unlikely to get free money. More importantly derivatives contracts involving mutual agreements force parties to collateralize these agreements so that, in any case, any losses incurred should be covered partially or in full depending on the agreements.

Let us start by fixing a terminal time T and a filtered probability space $(\Omega, \mathscr{G}, \mathbb{P})$ equipped with two mutually independent stochastic processes: a unidimensional standard Brownian motion W and a jump process N where the filtration $\mathscr{G}_t$ is generated both by W and N.

We are much concerned in this section on the CSA discounting to determine the fair price of the deal. The below assumptions are credited to [17] and are very useful in setting up our pricing equation:

- **A1:** In this section, the deal and collateral are dominated in the same currency
- **A2:** Each counterparty is subject to post collateral
- **A3:** We assume no funding adjustments involved since the deal is fully collateralized.
- **A4:** We allow repo agreement of the underlying assets.
- **A5:** Collaterals are posted in segregated account(i.e no rehypothecation is allowed)

So far in this section we consider collaterals being posted in deal's currencies. This is important to highlight here because CSA agreements give rights to counterparties to post collaterals dominated in currencies different from the underlying. We will consider this problem in the next section. In general the counterparty with negative exposure posts collateral to the collateral account and the posting depends on the marked-to-market value of the deal.

We consider a financial deal between two dealers, where delear A sells a European option to the dealer B. We assume the that the market is weakly efficient in sense that the price movements of the underlying does not depend on the past values and are governed both by the Brownian motion and Poisson processesIn this section we borrow some important concepts from [17] and assume that the market consists of the following instruments:

- **Stock.** We consider the underlying asset to be risky and driven by two mutually independent processes. Let S_t be the price process of the underlying and follow the SDE of

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t + \int_E S_{t^-} L_t(e)\tilde{N}(ds, de),\ S_0 = s,\ t \in [0, T] \tag{3.1}$$

  where W is a d-dimensional Brownian motion and $\tilde{N}$ compensated Poisson counting measure under physical measure $\mathbb{P}$. The latter counts the unexpected chocks(i.e jumps) in the underlying. The parameters μ_t and σ_t stand for the drift term and volatility of the underlying respectively, and $L_t(e)$ represents the size of jumps. We also allow the underlying to be traded in repo market if need be.

- **Repo account.** Unlike to the classical theories, we suppose instead that a buyer enters into repo agreement in order to hedge uncertain movements of the underlying. This is done when she lends(borrows) money from repo market the same quantity hold in the underlying. We suppose that a repo account, B_t, accrues at repo rate r_t^S and having the dynamics of

$$dB_t = r_t^S B_t dt \tag{3.2}$$

- **The collateral account.** We also consider a collateral account that accrues at collateral rate r_t and obeys the following dynamics

$$dC_t = r_t C_t dt \tag{3.3}$$

In the point of view of the buyer, we form the trading strategy as follows. The trading strategy defined as $\phi := (\alpha_t, \gamma_t)$ is composed by the number of units, α, held in the stock and the number of units, γ, in the repo account. Let Π_t be the wealth process of the holding portfolio of the buyer at time t, then these market instruments will imply the Π_t to be

$$\Pi_t = \alpha_t S_t + B_t + C_t \tag{3.4}$$

From the equation (3.4) we see that the buyer is holding the whole amount of money both at the money account, repo and collateral accounts respectively. Unlike what has been treated in the work of [23], we allow collateral to impact the wealth process of the portfolio.

Definition 3.1. *A collateralized trading strategy ϕ is self-financing if, for $t \in [0,T]$, it holds that*

$$dV_t = \alpha_t dS_t + dB_t - dC_t \tag{3.5}$$

where α_t represents the shares in the underlying at time t.

Remark 3.1. *As explained in CSA manual agreement, the collateral does not implicitly affect the holding portfolio of the collateral holder. We allow the portfolio to depend on the collateral as well for the self-financing condition to make a mathematical sense. Otherwise it will not make sense mathematically such a way that the changes on both sides will be equivalent.*

The negative sign on the right hand side of (3.5) tells us that the collateral holder accounts for the interest rate paid on the collateral amount in his portfolio. Assume that portfolio is self-financing as defined in the definition 3.1. Now we have the following result

Theorem 3.1. *Assume that the trading strategy satisfies the self-financing condition (3.5) and the underlying satisfies the equation (3.1). Let the predictable processes Z_t(resp. K_t) be defined as $\alpha_t\sigma_t S_t$(resp. $\alpha_t L_t(e) S_{t^-}$). Then, for all $t \in [0,T]$, the pre-default value of V_t follows the backward stochastic differential equation of*

$$\begin{cases} dV_t = r_t V_t dt + (\sigma_t + L_t(e))^{-1}\left(\mu - r_t^S\right)(K_t + Z_t)\,dt + ZdW_t + \displaystyle\int_E K_t \tilde{N}(dt, de) \\ V_T = \xi \end{cases} \tag{3.6}$$

And there exists a unique solution (V_t, Z_t, K_t) that solves the BSDE (3.6).

Proof. The pre-default value V_t is seen as the pricing equation of the deal in the point of view of the buyer. We assume that the buyer lends money to the repo market which replicates the amount of money in shares which is equal to $-\alpha S$. By definition 3.1 we have

$$dV_t = \alpha_t\left(\mu_t S_t dt + \sigma_t S_t dW_t + \int_E S_{t^-} L_t(e)\tilde{N}(ds, de)\right) + r_t^S B_t dt - r_t C_t dt \tag{3.7}$$

Add and subtract $r_t V_t dt$ in the right hand side of (3.6), the collateral term will cancel. The result follows from setting $Z_t = \alpha_t\sigma_t S_t$(resp. $K_t = \alpha_t L_t(e) S_{t^-}$) and applying hedging strategies.

∎

In the pricing equation (3.6) we assume that the terminal payoff ξ is a $\mathscr{F}_T$-measurable. This is because the default of either counterparty does not imply a jump in the underlying.

Define an adapted process $\zeta_t = (\sigma_t + L_t(e))^{-1}\left(\mu - r_t^S\right)$. The BSDE (3.6) becomes a linear BSDE of the form

$$dV_t = r_t V_t dt + \zeta_t (K_t + Z_t) dt + Z dW_t + \int_E K_t \tilde{N}(dt,de),\; V_T = \xi, \tag{3.8}$$

If we define the driver $f(t,V_t,Z_t,K_t) = -r_t V_t - \zeta_t (K_t + Z_t)$, then we have the BSDE of the same form as the equation (2.3) but in differential form

$$dV_t = -f(t,V_t,Z_t,K_t)dt + Z dW_t + \int_E K_t \tilde{N}(dt,de) \tag{3.9}$$

In addition to the aforementioned assumptions in the secion 2 we suppose that r_t and ζ_t are square integrable and bounded (i.e $\int_0^T |r_t|^2\, dt < \infty$ and $\int_0^T \left|\zeta_t\right|^2 dt < \infty$) to ensure the solution of the BSDE (3.9). By the theorem 2.3, a triple of processes (V_t, Z_t, K_t) uniquely solves the BSDE (3.9). In the BSDE of (3.9), one can see that the value of the derivative is discounted by the collateral rate which is contrary to the classical theory where the risk-free is the discounting rate. In this section we determined a pricing equation in a single currency where the collateral is dominated in the same currency as the deal. As the CSA agreement gives a possibility to post different currency from the deal's one, we will determine a pricing equation pertaining to this optionality in the next section.

3.2. Pricing domestic deals collateralized in foreign currency

In the previous section we considered a CSA discounting framework where deals only allow counterparties to post collateral in domestic currency. . In this section, we will address deals backed by multi-currency CSA whereby one of counterparties is allowed to post collateral from a set of eligible currencies. This gives counterparties a right to post cheapest currency available in the predefined set of eligible currencies. In fact if the deal is dominated in USA dollar and Euro is the cheapest to deliver, counterparty will post Euro as a collateral.

As discussed by [15] and [16], deals backed by multi-currency CSA lead to different discounting rates and involve a foreign exchange rate. This poses a complexity in valuation since what is cheapest today may not be in the future, consequently there will be a change of discounting rate relevant to the cheapest-to-deliver option[32]. In the subsequent paragraphs we will consider this optionality in determining a BSDE representing this scenario.

Let (d, f) be a set of currencies where d represents domestic currency and f foreign currency. Consider a collateral account, C_t^f, which is dominated in foreign currency f with a foreign accrual rate r_t^f. The complexity here is that the deal has to be discounted at corresponding collateral rate while is in the domestic currency d. In the point of view of collateral taker, there is a foreign exchange in place to change the foreign currency into domestic one so that her replicating portfolio will still hold even if collateral is posted in foreign currency. As it stands, it changes the foreign currency into the domestic currency.

Unlike the dynamics of foreign exchange considered in [7], we keep it fixed throughout the life of the contract such that the collateral provider enters into a foreign swap contract. We can describe this scenario as: Suppose counterpaty B buys a financial deal and pays V_t dominated in d and counterparty A posts $\frac{V_t}{X_t^{(d,f)}}$ dominated in f. A buys $\frac{V_t}{X_t^{(d,f)}}$ dominated in currency f at spot rate $\frac{1}{X_t^{(d,f)}}$ and agrees to resell back at forward rate $X_{t+dt}^{(d,f)}$. This will eliminated any uncertain movement of $X_t^{(d,f)}$ because it is fixed at the inception of the FX swap contract [34].

It does not matter if the default event occurs before the maturity. In our disposal we have a financial deal in place currency backed by multi-currency CSA. Obviously counterparties are allowed to post collateral dominated in foreign currency f as mentioned earlier. Let C_t^d and C_t^f be collateral accounts dominated in domestic currency and foreign currency respectively, then there exists a foreign exchange rate $X_t^{(d,f)}$such that

$$C_t^d = X_t^{d,f} C_t^f \tag{3.10}$$

The exchange rate $X_t^{d,f}$ in the relation (3.10) expresses the unit of domestic amounts in terms of foreign amounts. This implies that there is no problem in changing the foreign interests as CSA stipulates that the collateral account must accrue at relevant collateral rate which is in foreign currency. In order to preclude any arbitrage opportunities stemming from exchanging currencies, we extract a relation between collateral rates from [17] such that the two collateral yields the same return. There exist a cross currency basis when changing from domestic currency to foreign currency f such t hat

$$(r_t - b_t) C_t dt = r_t^f C_t^f dt \tag{3.11}$$

We write r_t(resp. C_t) to denote the collateral rate(resp. account) in domestic currency at time t. Substituting the relation (3.11) into the BSDE of (3.6) we get the following result

Theorem 3.2. *Assume that the trading strategy satisfies the self-financing condition (3.5) and S_t satisfies the equation (3.1), then the multi-currency pricing equation of the deal follows the backward stochastic differential equation of the form*

$$dV_t = (r_t - b_t)V_t dt + \zeta_t (K_t + Z_t) dt + Z dW_t + \int_E K_t \tilde{N}(dt, de),\ V_T = \xi, \tag{3.12}$$

And there exists a unique solution (V_t, Z_t, K_t) that solves the BSDE (3.12).

Proof. The proof can be done similarly as in theorem 3.1. ■

As the CSA agreements stipulate, we can see that both of the pricing equations (3.8) and 3.12 are discounting at their respective collateral rates which is different from the classical theory. We note that the two pricing equation are different from one another. There is an immediate effect of posting foreign currency which is captured by the basis currency b_t at time t.

4. The Stability of the Solution

In this section we want to establish the stability of the solution of the BSDE (3.12) with the Lipschitz driver. As the underlying conditions and assumptions of the BSDE (3.8) are almost the same as of the BSDE (3.12), the stability of the BSDE (3.12) will definitely imply the stability of the BSDE (3.8).

The stability for the solution of the equation can be established whether its solution is known or not. Initially the Lyapunov functions were used to determine the stability of stochastic differential equations, especially linear stochastic system driven by Brownian motion Mao [24]. This has been established in different perspectives including stochastic stability and almost sure stability. The monographs on the application of Lyapunov function about the stability can be found in [25] and the references therein. The other direction is to consider a sequence of solution that converges to the unique solution for the respective BSDE, therefore the solution is stable. For more details, the interested reader can consult [1], [18] and the references therein. In the sequel we aim at constructing a Cauchy sequence that has a limit as unique solution of the BSDE in question, hence the stability for the solution.

Let ξ be a $\mathscr{F}_T-$measurable. We maintain the same assumptions and the probability space as in the section 3. Now we consider the following BSDE

$$dV_t = -f(t,V_t,Z_t,K_t)dt + ZdW_t + \int_E K_t\tilde{N}(dt,de),\ V_T = \xi, \tag{4.1}$$

We start by fixing $t \geq 0$ as in the previous sections. For each $n \in \mathbb{N}$, let $\xi^n \in L^2(\mathscr{F}_T)$ and the triple (Y^n, Z^n, U^n) solve the following BSDE

$$dV_t^n = -f^n(t,V_t^n,Z_t^n,K_t^n)dt + Z^n dW_t + \int_E K_t^n\tilde{N}(dt,de),\ V_T^n = \xi^n, \text{ for all } t \in [0,T] \tag{4.2}$$

As established in [19], for each $n \in \mathbb{N}$ we have the following assumption:

- **A6:** There exists a sequence (ξ^n, C^n) that converges almost surely to the limit (ξ, C).
- **A7:** There exists a sequence $(S_t^n)_{n\geq 1}$ that converges almost surely to the limit S_t.
- **A8:** The driver $f^n(t,V_t^n,Z_t^n,K_t^n)$ converges to $f(t,V_t,Z_t,K_t)$ almost surely

Note that as posting collateral depends on the net exposure therefore the convergence of collateral account will depend as well the convergence of net exposure. We are now in position to establish the stability of the solution for the BSDE (4.2).

Theorem 4.1. *Assume the assumptions* **A1-A8** *hold. Then, for each* $n \in \mathbb{N}$ *and all* $t \in [0,T]$*, there exists a triple of solution* $(V_t^n, Z_t^n, K_t^n)_{n\geq 1}$ *such that*

$$\mathbb{E}\left[\left|V_t^n - V_t\right|^2\right] + \mathbb{E}\left[\int_0^T \left|Z_t^n - Z_t\right|^2 dt\right] + \mathbb{E}\left[\int_0^T \int_E \left|K_t^n - K_t\right|^2 \lambda(dt,de)\right] \longrightarrow 0, as\, n \longrightarrow \infty \tag{4.3}$$

Proof. It suffices to show that the sequence $(V_t^n, Z_t^n, K_t^n)_{n\geq 1}$ is a Cauchy sequence. By the relation (2.11), there exists a sequence $(V_t^{n+1}, Z_t^{n+1}, K_t^{n+1})$ such that

$$\left\|V_t^{n+1} - V_t^n\right\|^2 + \left\|Z_t^{n+1} - Z_t^n\right\|^2 + \left\|K_t^{n+1} - K_t^n\right\|^2 \leq \frac{(4T+8)C^2}{\beta}\left[\left\|V_t^{n+1} - V_t^n\right\|_\beta^2 + \left\|Z_t^{n+1} - Z_t^n\right\|_\beta^2 + \left\|K_t^{n+1} - K_t^n\right\|_\beta^2\right]$$

We know by the principle of contraction mapping that for any n and $0 < \alpha < 1$,

$$\left\|V_t^{n+1} - V_t^n\right\|^2 \leq \alpha\left\|V_t^n - V_t^{n-1}\right\|^2 \leq \cdots \leq \alpha^n\left\|V_t^1 - V_t^0\right\|^2$$

This implies that

$$\left\|V_t^{n+1} - V_t^n\right\|^2 + \left\|Z_t^{n+1} - Z_t^n\right\|^2 + \left\|K_t^{n+1} - K_t^n\right\|^2 \leq \left(\frac{(4T+8)C^2}{\beta}\right)^n\left[\left\|V_t^1 - V_t^0\right\|_\beta^2 + \left\|Z_t^1 - Z_t^0\right\|_\beta^2 + \left\|K_t^1 - K_t^0\right\|_\beta^2\right] \tag{4.4}$$

Since $(4T+8)C^2 < \beta$ the right hand side of (4.4) gets closer to 0 as $n \to \infty$. Therefore, the sequence $(V_t^n, Z_t^n, K_t^n)_{n\geq 1}$ is Cauchy. Hence, the sequence $(V_t^n, Z_t^n, K_t^n)_{n\geq 1} \longrightarrow (V_t, Z_t, K_t)$. ∎

5. Conclusion

In this paper we derived a pricing equation of a collateralized financial deals, usually called CSA discounting framework, using backward stochastic equation(BSDE) and we found that the market under consideration is stable. The pre-default pricing equations are uniquely represented in the sense that their respective solutions exist and are unique. This BSDE can be extended to the equation whose terminal time is random but finite by taking into consideration CSA regulations regarding default events.

In order to mitigate counterparty exposure embedded in a financial deal we allow each party in the contract to post collateral according to what is the cheapest to deliver for her.

This optionality has to deal with the dominance either in the domestic or foreign currency. In this perspective, we put much emphasis on one collateralization scheme under which collateral is posted in the form of cash. Other collateralization schemes like bonds can be similarly used as well but with few differernces. Indeed, the aforementioned pricing equations do not give the explicit solutions rather implicit. In continued work we are exploiting numerical methods in order to find the solutions.

References

[1] Antonelli, F. (1996). Stability of backward stochastic differential equations. *Stochastic processes and their applications*, 62(1):103–114.

[2] Barles, G., Buckdahn, R., and Pardoux, E. (1997). Backward stochastic differential equations and integral-partial differential equations. *Stochastics: An International Journal of Probability and Stochastic Processes*, 60(1-2):57–83.

[3] Biais, B., Björk, T., Cvitanic, J., El Karoui, N., Jouini, E., and Rochet, J. C. (2006). *Financial mathematics: lectures given at the 3rd session of the Centro Internazionale Matematico Estivo (CIME) held in Bressanone, Italy, July 8-13, 1996*. Springer.

[4] Bichuch, M., Capponi, A., and Sturm, S. (2016). Arbitrage-Free Pricing of XVA-Part I: Framework and Explicit Examples.

[5] Brigo, D., Capponi, A., and Pallavicini, A. (2014). Arbitrage-Free Bilateral Counterparty Risk Valuation Under Collateralization and Application to Credit Default Swaps. *Mathematical Finance*, 24:125–146.

[6] Brigo, Damiano and Morini, Massimo and Pallavicini, Andrea (2013). *Counterparty credit risk, collateral and funding: with pricing cases for all asset classes*. John Wiley & Sons.

[7] Castagna, Antonio (2012). Pricing of collateralized derivatives contracts when more than one currency are involved: Liquidity and funding value adjustments.

[8] Cordoni, F. and Di Persio, L. (2014). Backward stochastic differential equations approach to hedging, option pricing, and insurance problems. *International Journal of Stochastic Analysis*, 2014.

[9] Cordoni, F. and Di Persio, L. (2016). A BSDE with delayed generator approach to pricing under counterparty risk and collateralization. *International Journal of Stochastic Analysis*, 2016.

[10] Crépey, S. (2011). A BSDE approach to counterparty risk under funding constraints. *Available at grozny. maths. univ-evry. fr/pages perso/crepey*.

[11] Darling, R. W., Pardoux, E., et al. (1997). Backwards SDE with random terminal time and applications to semilinear elliptic PDE. *The Annals of Probability*, 25(3):1135–1159.

[12] Delong, Ł. (2013). *Backward stochastic differential equations with jumps and their actuarial and financial applications*. Springer.

[13] Dumitrescu, R., Quenez, M.-C., and Sulem, A. (2016). Bsdes with default jump. *arXiv preprint arXiv:1612.05681*.

[14] El Karoui, N and Quenez, MC (1997). Non-Linear Pricing Theory and Backward Stochastic Differential Equations. In *Financial Mathematics*, pages 191–246. Springer.

[15] Fujii, M., Shimada, Y., Takahashi, A., et al. (2010). A note on construction of multiple swap curves with and without collateral. *FSA Research Review*, 6(139-157).

[16] Fujii, M. and Takahashi, A. (2013). Derivative pricing under asymmetric and imperfect collateralization and CVA. *Quantitative Finance*, 13(5):749–768.

[17] García Muñoz, L. M., de Lope Contreras, F., and Palomar Burdeus, J. E. (2015). Pricing Derivatives in the New Framework: OIS Discounting, CVA, DVA & FVA.

[18] Hu, Y. and Peng, S. (1997a). A stability theorem of backward stochastic differential equations and its application. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 324(9):1059–1064.

[19] Hu, Y. and Peng, S. (1997b). A stability theorem of backward stochastic differential equations and its application. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 324(9):1059–1064.

[20] International Swaps and Derivatives Association(ISDA) (2014). ISDA Margin Survey 2014.

[21] International Swaps and Derivatives(ISDA) (2009). Collateral Use for OTC Derivatives.

[22] Kallianpur, G. and Karandikar, R. L. (2012). *Introduction to option pricing theory*. Springer Science & Business Media.

[23] Liu, Q. (2015). A consistent framework for valuation under collateralization, credit risk and funding costs.

[24] Mao, X. (1994). *Exponential stability of stochastic differential equations*. Marcel Dekker.

[25] Mao, X. (2007). *Stochastic differential equations and applications*. Elsevier.

[26] Pardoux E. (1990). Adapted solution of backward stochastic differential equation. *Systems & Control Letters*, 14(1):55–61.

[27] Peng, Shige (1993). Backward stochastic differential equations and applications to optimal control. *Applied mathematics & optimization*, 27(2):125–144.

[28] Pham, H. (2015). Feynman-Kac representation of fully nonlinear PDEs and applications. *Acta Mathematica Vietnamica*, 40(2):255–269.

[29] Piterbarg, V. (2010). Funding beyond discounting: collateral agreements and derivatives pricing. *Risk*, 23(2):97.

[30] Rachev, S. T. (2003). *Handbook of heavy tailed distributions in finance: Handbooks in finance*, volume 1. Elsevier.

[31] Rémillard, B. and Renaud, J.-F. (2011). A martingale representation for the maximum of a lévy process. *Communications on Stochastic Analysis*, 5(4):5.

[32] Sawyer, Nick (01 Mar 2011). Multi-currency CSA chaos behind push to standardised CSA. *Risk magazine*.

[33] Somasundaram, D. (2001). *Ordinary differential equations: a first course*. CRC Press.

[34] Youmbi, D. K. (2014). Derivatives pricing after the 2007-2008 crisis: How the crisis changed the pricing approach.