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Some Fixed Point Theorems for Self Mappings on Vector b-metric Spaces

Mamta Kamra[1], **Satish Kumar, and Kumari Sarita**
Department of Mathematics, Indira Gandhi University,
Meerpur, Rewari, Haryana-122502, India.

Abstract

The aim of this paper is to prove some fixed point theorems for mappings on a vector b-metric space into itself. Our results are generalizations of some fixed point results for scalar valued metric spaces.

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1. Introduction

Several authors have studied fixed point theorems on various generalized metric spaces. In 1989, Bakhtin [5] introduced the concept of b-metric space, which is a generalization of metric space. Using this idea, many researchers like Czerwik [10], Mehmet Kir [14], Swati Agarwal et al. [2] proved some fixed point theorems in b-metric spaces. Cevik and Altun [7] introduced the concept of vector metric space and proved some fixed point theorems on this space. Petre [16] defined the vector b-metric space, called E-b-metric space. He combined the concepts of b-metric space and vector metric space. Our aim in this paper is to prove some fixed point results on E-b-metric space.

The following definitions and results will be needed in the sequel. For Riesz space, we adopt the notation and terminology of Aliprantis and Border [3]. For definitions and related results on vector metric spaces, one can see Cevik and Altun[7].

Definition 1.1. A partial order is a binary relation $\leq$ on a set X which satisfies the following conditions:

[1]Corresponding Author.

(i) $x \leq x$ (reflexivity)

(ii) $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetry)

(iii) $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitivity)

for all $x, y, z \in X$. A set with a partial order $\leq$ is called a partially ordered set (poset).

A poset $(X, \leq)$ is said to be linearly ordered or totally ordered or a chain if for each pair $x, y \in X$, we have either $x \leq y$ or $y \leq x$.

Definition 1.2. A lattice is a poset in which every set with two elements has a supremum and an infimum.

A lattice in X is said to be complete if every subset has a supremum and an infimum. It is said to be Dedekind complete if every nonempty subset which is bounded above (bounded below), has a supremum (an infimum).

Definition 1.3. A partially ordered vector space is a poset $(E, \leq)$, where E is a real vector space, such that, for all $x, y, z \in E$ and $\lambda > 0$,

(i) $x \leq y$ implies $x + z \leq y + z$.

(ii) $x \leq y$ implies $\lambda x \leq \lambda y$.

Definition 1.4. A partially ordered vector space which is also a lattice under its ordering, is called a Riesz space.

Notation: If $\{x_n\}$ is a decreasing sequence in a Riesz space E such that inf x_n = x, we write $x_n \downarrow x$.

Definition 1.5. A Riesz space E is said to be Archimedean if $\frac{1}{n}x \downarrow 0$ for every $x \in E_+$ where $E_+ = \{x \in E : x \geq 0\}$.

Definition 1.6. A sequence $\{x_n\}$ in a Riesz space E is said to order convergent to x, written as $x_n \xrightarrow{o} x$, if there exists a sequence $\{a_n\}$ in E satisfying $a_n \downarrow 0$ and $|x_n - x| \leq a_n$ for all n, where $|x| = x \vee -x$.

The sequence $\{x_n\}$ is said to be order-Cauchy (or o-Cauchy) if there exists a sequence $\{a_n\}$ in E satisfying $a_n \downarrow 0$ and $|x_n - x_{n+p}| \leq a_n$ for all n and p.

Lemma 1.7. [4] If E is a Riesz space and $a \leq ka$ where $a \in E_+, k \in [0, 1)$, then a = 0.

Remark 1.8. It is well known that $\mathbb{R}^2$ is a Riesz space with coordinatewise ordering defined by

$$(x_1, y_1) \leq (x_2, y_2) \text{ if and only if } x_1 \leq x_2,\ y_1 \leq y_2 \quad \text{for } (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2.$$

Also $\mathbb{R}^2$ is a Riesz space with lexicographical ordering defined by

$(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 < x_2$ or $x_1 = x_2,\ y_1 \leq y_2$ for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

Note that $\mathbb{R}^2$ is Archimedean with coordinatewise ordering but not with lexicographical ordering.

Definition 1.9. Let X be a non-empty set and E be a Riesz space. A function $d : X \times X \to E$ is said to be a vector metric (or E-metric) on X if it satisfies the following properties:

(i) $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) \leq d(x, z) + d(y, z) \quad \forall x, y, z \in X$.

The triple (X, d, E) is said to be a vector metric space.
It is obvious that vector metric spaces generalize metric spaces.
For arbitrary elements x, y, z, w of a vector metric space, the following are true.

(a) $0 \leq d(x, y)$;

(b) $d(x, y) = d(y, x)$;

(c) $|d(x, z) - d(y, z)| \leq d(x, y)$;

(d) $|d(x, z) - d(y, w)| \leq d(x, y) + d(z, w)$.

Example 1.10. A Riesz space E is a vector metric space with $d : E \times E \to E$ defined by

$$d(x, y) = |x - y| \quad \text{for all } x, y \in X.$$

This vector metric is called absolute valued metric on E.

Example 1.11. $\mathbb{R}^n$ is a Riesz space with coordinate wise ordering defined by

$(x_1, x_2, \dots, x_n) \leq (y_1, y_2, \dots, y_n)$ if and only if $x_1 \leq y_1,\ x_2 \leq y_2, \dots,\ x_n \leq y_n$.

Define $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) = (\alpha_1|x_1 - y_1|, \alpha_2|x_2 - y_2|, \dots, \alpha_n|x_n - y_n|)$$

where $\alpha_i, i \leq 1 \leq n$, are non-negative real numbers with $\alpha_1 + \alpha_2 + \cdots + \alpha_n > 0$. Then $(\mathbb{R}^n, d, \mathbb{R}^n)$ is a vector metric space.

Example 1.12. Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ be defined as

$$d(x, y) = (\alpha_1|x - y|, \alpha_2|x - y|)$$

where $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 > 0$. Then d is a vector metric with coordinatewise or lexicographical ordering and $(\mathbb{R}, d, \mathbb{R}^2)$ is a vector metric space.

Definition 1.13. A sequence $\langle x_n \rangle$ in a vector metric space (X, d, E) vectorial converges (or E-converges) to some $x \in E$, written as $x_n \overset{d,E}{\longrightarrow} x$, if there is a sequence $\langle a_n \rangle$ in E such that $a_n \downarrow 0$ and $d(x_n, x) \leq a_n$ for all n.

Definition 1.14. A sequence $\langle x_n \rangle$ in a vector metric space (X, d, E) is called E-Cauchy if there is a sequence $\langle a_n \rangle$ in E such that $a_n \downarrow 0$ and $d(x_n, x_{n+p}) \leq a_n$ for all n and p.

Definition 1.15. A vector metric space X is called E-complete if every E-Cauchy sequence in X E-converges to a limit in X.

Definition 1.16. A subset Y of a vector metric space X is said to be E-closed whenever $\{x_n\} \subseteq Y$ and $x_n \overset{d,E}{\longrightarrow} x$, imply $x \in Y$.

It is easy to see that if $x_n \overset{d,E}{\longrightarrow} x$, then

(i) the limit x is unique.

(ii) every subsequence of $\langle x_n \rangle$ E-converges to x.

(iii) if also $y_n \overset{d,E}{\longrightarrow} y$, then $d(x_n, y_n) \overset{o}{\longrightarrow} d(x, y)$.

When $E = \mathbb{R}$, the concepts of vectorial convergence and convergence in metric coincide. Also when $X = E$ and d is the absolute valued metric, vectorial convergence and convergence in order are the same.

Definition 1.17. [6] Let (X, d, E) and (Y, ρ, F) be vector metric spaces and let $x \in X$. A function $f : X \to Y$ is said to be vectorially continuous at x if $x_n \overset{d,E}{\longrightarrow} x$ in X implies $f(x_n) \overset{\rho,E}{\longrightarrow} f(x)$ in Y. The function f is said to be vectorially continuous if it is vectorially continuous at each point of X.

Definition 1.18. [16] Let X be a nonempty set, E be a Riesz space and $s \geq 1$ be a real number. A function $d : X \times X \to E_+$ is said to be an E-b-metric if, for any $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) \leq s\,[d(x, z) + d(y, z)]$.

The triple (X, d, E) is said to be an E-b-metric space.

Below, we give some examples of E-b-metric spaces:

Example 1.19. Let $X = L^p[0,1]$ with $0 < p < 1$ and $E = \mathbb{R}^2$. Let $d : L^p[0,1] \times L^p[0,1] \to \mathbb{R}^2_+$ be defined by

$$d(f,g) = (\alpha\|f-g\|_p, \beta\|f-g\|_p)$$

where $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$. Then it is easy to see that

$$d(f,h) \leq 2^{\frac{1}{p}}\left[d(f,g) + d(g,h)\right].$$

Hence $(X, d, \mathbb{R}^2)$ is an E-b-metric space with parameter $s = 2^{\frac{1}{p}} > 1$.

Likewise, if $X = l_p, 0 < p < 1$, and $d : l_p \times l_p \to \mathbb{R}^2_+$ is defined as

$$d(x,y) = (\alpha\|x-y\|_p, \beta\|x-y\|_p),$$

where $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$, then $(X, d, \mathbb{R}^2)$ is an E-b-metric space with parameter $s = 2^{\frac{1}{p}} > 1$.

Example 1.20. Let $X = C[0,1] = E$. Define $d : X \times X \to E_+$ as

$$d(f,g) = (f-g)^p, \ p > 1.$$

Then (X, d, E) is an E-b-metric space with parameter $s = 2^{p-1} > 1$. Since the function x^p $(p > 1)$ is convex, we have

$$\left(\frac{1}{2}x + \frac{1}{2}y\right)^p \leq \frac{1}{2}x^p + \frac{1}{2}y^p$$

so that

$$(x+y)^p \leq 2^{p-1}\left(x^p + y^p\right)$$

Therefore

$$\begin{aligned} d(f,h) = (f-h)^p = (f-g+g-h)^p &\leq 2^{p-1}\left[(f-g)^p + (g-h)^p\right] \\ = 2^{p-1}\left[d(f,g) + d(g,h)\right] \end{aligned}$$

Thus the relaxed triangular inequality holds with $s = 2^{p-1} > 1$.

Example 1.21. Let $X = \mathbb{R}^2$, $E = \mathbb{R}^2$ and $d : X \times X \to \mathbb{R}^2_+$ be defined as

$$d((x_1,y_1),(x_2,y_2)) = (\alpha|x_1 - x_2|^2, \beta|y_1 - y_2|^2).$$

where $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$,. Then $(X, d, \mathbb{R}^2)$ is an E-b-metric space with parameter $s = 2 > 1$.

We give below an example of a E-b metric space which is not a metric space.

Example 1.22. Let $X = \{0, 1, 2\}$, $E = \mathbb{R}^2$ and $d : X \times X \to \mathbb{R}^2$ be defined as

$$d(0, 1) = d(1, 0) = (1, 1)$$

$$d(1, 2) = d(2, 1) = (1, 1)$$

$$d(0, 2) = d(2, 0) = (4, 4)$$

Since $d(0, 2) = (4, 4) \not\leq d(0, 1) + d(1, 2)$, (X, d, E) is not a metric space. It is a E-b metric space for $s = 2$.

2. Main Results

We prove some fixed point results on E-b-metric spaces. Our results generalize some of the results of Chatterjea [8], Dubey [11], Kannan [13], Turkoglu et al. [17] and Kir [14].

Lemma 2.1. Let (X, d, E) be a complete E-b-metric space with co-efficient $s \geq 1$ and E-Archimedean. Let $\{x_n\}$ be a sequence in X such that

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \quad \text{for all } n \in N,$$

where $\lambda \in [0, 1)$ and $0 \leq s\lambda < 1$. Then $\{x_n\}$ is a E-Cauchy sequence in X.

Proof. Since

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \quad \text{for all } n \in N,$$

it follows that

$$d(x_n, x_{n+1}) \leq \ \lambda d(x_{n-1}, x_n) \ \leq \lambda^2 d(x_{n-2}, x_{n-1}) \ \leq \cdots \leq \ \lambda^n d(x_0, x_1).$$

Now, for $n > m$, we have

$$\begin{aligned} d(x_m, x_n) &\leq s d(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + \cdots + s^{n-m} d(x_{n-1}, x_n). \\ &\leq s\lambda^m d(x_0, x_1) + s^2 \lambda^{m+1} d(x_0, x_1) + \cdots + s^{n-m} \lambda^{n-1} d(x_0, x_1). \\ &\leq \frac{s\lambda^m}{1 - s\lambda} d(x_0, x_1) \downarrow 0. \end{aligned}$$

This implies that $\{x_n\}$ is a E-Cauchy sequence in X. ■

Theorem 2.2. Let (X, d, E) be a complete E-b-metric space with co-efficient $s \geq 1$ and E-Archimedean. Let T be a self map on X such that

$$d(Tx, Ty) \leq kU(x, y), \quad 0 < k < \frac{1}{s(s + 1)}$$

where

$$U(x, y) \in \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$. Define $x_n = Tx_{n-1},\ n \geq 1$. Then

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq kU(x_{n-1}, x_n)$$

where

$$\begin{aligned} U(x_{n-1}, x_n) &\in \{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\ &= \{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} \\ &= \{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0\} \end{aligned}$$

The following cases arise:

(i) $d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)$

(ii) $d(x_n, x_{n+1}) \leq kd(x_n, x_{n+1})$ which gives $d(x_n, x_{n+1}) = 0$.

(iii) $d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_{n+1}) \leq ks\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right]$ which gives

$$d(x_n, x_{n+1}) \leq \frac{ks}{1-ks} d(x_{n-1}, x_n)$$

(iv) $d(x_n, x_{n+1}) \leq 0$.

Thus $\quad d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n)$ where $\lambda \in \left\{k, \dfrac{ks}{1-ks}\right\} < 1$.

Since $0 < k < \dfrac{1}{s(s+1)}$, it is easy to see that $0 \leq s\lambda < 1$.

Now from lemma 2.1, it follows that $\{x_n\}$ is a E-Cauchy sequence in X. Since X is E-complete, there exists some $z \in X$ such that $x_n \xrightarrow{d,E} z$. So there exists a sequence $\{a_n\} \in E$ such that $a_n \downarrow 0$ and $d(x_n, z) \leq a_n$.

We now show that z is a fixed point of T. Now

$$\begin{aligned} d(Tz, z) &\leq sd(Tz, Tx_n) + sd(Tx_n, z) \\ &\leq skU(z, x_n) + sd(x_{n+1}, z) \end{aligned}$$

where

$$U(z, x_n) \in \{d(z, x_n), d(z, Tz), d(x_n, Tx_n), d(z, Tx_n), d(x_n, Tz)\}$$

We have the following five cases:

Case 1: $U(z, x_n) = d(z, x_n)$
In this case

$$\begin{aligned} d(Tz, z) &\leq skd(z, x_n) + sd(x_{n+1}, z) \\ &\leq ska_n + sa_{n+1} \end{aligned}$$

where $a_n \downarrow 0$.

Case 2: $U(z, x_n) = d(z, Tz)$
In this case

$$\begin{aligned} d(Tz, z) &\leq skd(z, Tz) + sd(x_{n+1}, z) \\ i.e., \quad (1 - sk)d(Tz, z) &\leq sa_{n+1} \\ i.e., \quad d(Tz, z) &\leq \frac{s}{1 - sk} a_{n+1} \downarrow 0. \end{aligned}$$

Case 3: $U(z, x_n) = d(x_n, Tx_n)$
In this case

$$\begin{aligned} d(Tz, z) &\leq skd(x_n, Tx_n) + sd(x_{n+1}, z) \\ &= s^2kd(x_n, z) + s^2kd(z, Tx_n) + sd(x_{n+1}, z) \\ &= s^2ka_n + s^2ka_{n+1} + sa_{n+1} \\ &\leq s(2sk + 1)a_n \downarrow 0. \end{aligned}$$

Case 4: $U(z, x_n) = d(z, Tx_n)$
In this case

$$\begin{aligned} d(Tz, z) &\leq skd(z, Tx_n) + sd(x_{n+1}, z) \\ &= skd(z, x_{n+1}) + sd(x_{n+1}, z) \\ &\leq s(k + 1)a_n \downarrow 0. \end{aligned}$$

Case 5: $U(z, x_n) = d(x_n, Tz)$
In this case

$$\begin{aligned} d(Tz, z) &\leq skd(x_n, Tz) + sd(x_{n+1}, z) \\ &= s^2kd(x_n, z) + s^2kd(z, Tz) + sd(x_{n+1}, z) \\ i.e., \quad (1 - s^2k)d(Tz, z) &\leq s^2ka_n + sa_{n+1} \leq s^2ka_n + sa_n. \\ i.e., \quad d(Tz, z) &\leq \frac{(s^2k + s)a_n}{(1 - s^2k)} \downarrow 0. \end{aligned}$$

This implies $d(Tz, z) = 0$. So $Tz = z$. Hence z is a fixed point of T.
We now show that z is unique. Let, if possible, suppose that z' is another fixed point of T. Then $Tz' = z'$. Now

$$d(z, z') = d(Tz, Tz') \leq kU(z, z')$$

where

$$U(z,z') \in \{d(z,z'), d(z,Tz), d(z',Tz'), d(z,Tz'), d(z',Tz)\} = \{d(z,z'), 0\}$$

This implies $d(z,z') = 0$ and so $z = z'$. Hence T has a unique fixed point. ■

Theorem 2.3. Let (X, d, E) be a complete E-b-metric space with $s \geq 1$ and E-Archimedean. Let T be a self map on X such that

$$d(Tx, Ty) \leq kU(x,y), \quad 0 < k < \frac{2}{s(s+2)}$$

where

$$U(x,y) \in \left\{\frac{1}{2}(d(x,Tx) + d(y,Ty)), \frac{1}{2}(d(x,Ty) + d(y,Tx)), \frac{1}{2}(d(x,Tx) \right.$$
$$\left. + d(x,Ty)), \frac{1}{2}(d(y,Tx) + d(y,Ty))\right\}$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$. Define $x_n = Tx_{n-1},\ n \geq 1$. Then

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq kU(x_{n-1}, x_n)$$

where

$$\begin{aligned} U(x_{n-1}, x_n) \ &\in \ \left\{\frac{1}{2}(d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)), \frac{1}{2}(d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})), \right. \\ &\qquad \left. \frac{1}{2}(d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, Tx_n)), \frac{1}{2}(d(x_n, Tx_n) + d(x_n, Tx_{n-1}))\right\} \\ &= \ \left\{\frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})), \frac{1}{2}d(x_{n-1}, x_{n+1}), \frac{1}{2}(d(x_{n-1}, x_n) \right. \\ &\qquad \left. + d(x_{n-1}, x_{n+1})), \frac{1}{2}d(x_n, x_{n+1})\right\} \end{aligned}$$

The following cases arise:

Case 1. $d(x_n, x_{n+1}) \leq \frac{k}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))$
which gives

$$d(x_n, x_{n+1}) \leq \frac{k}{2-k}\, d(x_{n-1}, x_n) \qquad \text{where } \frac{k}{2-k} < 1.$$

Case 2. $d(x_n, x_{n+1}) \leq \frac{k}{2} d(x_{n-1}, x_{n+1}) \leq \frac{ks}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))$
which gives

$$d(x_n, x_{n+1}) \leq \frac{ks}{2-ks} d(x_{n-1}, x_n) \qquad \text{where } \frac{ks}{2-ks} < 1.$$

Case 3. $d(x_n, x_{n+1}) \leq \frac{k}{2}(d(x_{n-1}, x_n)+d(x_{n-1}, x_{n+1})) \leq \frac{k}{2}(d(x_{n-1}, x_n)+sd(x_{n-1}, x_n)+d(x_n, x_{n+1}))$
which gives

$$d(x_n, x_{n+1}) \leq \frac{k(1+s)}{2-ks}\, d(x_{n-1}, x_n) \qquad \text{where } \frac{k(1+s)}{2-ks} < 1.$$

Case 4. $d(x_n, x_{n+1}) \leq \frac{k}{2} d(x_n, x_{n+1})$
which gives

$$d(x_n, x_{n+1}) = 0.$$

Combining all the cases, we have

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n)$$

where

$$\lambda \in \left\{ \frac{k}{2-k}, \frac{ks}{2-ks}, \frac{k(1+s)}{2-ks} \right\} < 1.$$

Since $0 < k < \frac{2}{s(s+2)}$, we have $0 \leq s\lambda < 1$.
Now from lemma 2.1, it follows that $\{x_n\}$ is a E-Cauchy sequence in X. Since X is E-complete, there exists some $z \in X$ such that $x_n \xrightarrow{d,E} z$. So there exists a sequence $\{a_n\} \in E$ such that $a_n \downarrow 0$ and $d(x_n, z) \leq a_n$.

We now show that z is a fixed point of T. Now

$$\begin{aligned} d(Tz, z) &\leq sd(Tz, Tx_n) + sd(Tx_n, z) \\ &= sd(Tz, Tx_n) + sd(x_{n+1}, z) \\ &\leq skU(z, x_n) + sd(x_{n+1}, z) \end{aligned}$$

where

$$U(z, x_n) \in \left\{ \frac{1}{2}(d(z, Tz) + d(x_n, Tx_n)), \frac{1}{2}(d(z, Tx_n) + d(x_n, Tz)), \frac{1}{2}(d(z, Tz) + d(z, Tx_n)), \frac{1}{2}(d(x_n, Tz) + d(x_n, Tx_n)) \right\}$$

Now following four cases arises:

Case 1: $U(z, x_n) = \frac{1}{2}(d(z, Tz) + d(x_n, Tx_n))$.
In this case

$$\begin{aligned} d(Tz, z) &\leq \frac{sk}{2}(d(z, Tz) + d(x_n, Tx_n)) + sd(x_{n+1}, z) \\ i.e., \left(1 - \frac{sk}{2}\right) d(Tz, z) &\leq \frac{s^2k}{2} d(x_{n+1}, z) + \frac{s^2k}{2} d(x_n, z) + sd(x_{n+1}, z) \\ &\leq \frac{s^2k}{2} a_{n+1} + \frac{s^2k}{2} a_n + sa_{n+1} \\ &\leq s(sk+1)a_n \\ i.e., \quad d(Tz, z) &\leq \frac{s(sk+1)}{(1 - \frac{sk}{2})} a_n \downarrow 0 \end{aligned}$$

Case 2: $U(z, x_n) = \frac{1}{2}(d(z, Tx_n) + d(x_n, Tz))$.
In this case

$$\begin{aligned} d(Tz, z) &\leq \frac{sk}{2}(d(z, Tx_n) + d(x_n, Tz)) + sd(x_{n+1}, z) \\ &\leq \frac{sk}{2} d(x_{n+1}, z) + \frac{s^2k}{2} d(x_n, z) + \frac{s^2k}{2} d(z, Tz) + sd(x_{n+1}, z) \\ i.e., \left(1 - \frac{s^2k}{2}\right) d(Tz, z) &\leq \frac{sk}{2} a_{n+1} + \frac{s^2k}{2} a_n + sa_{n+1} \\ &\leq \left(\frac{sk}{2} + \frac{s^2k}{2} + s\right) a_n \\ i.e., \quad d(Tz, z) &\leq \frac{(sk + s^2k + 2s)}{(2 - s^2k)} a_n \downarrow 0 \end{aligned}$$

Case 3: $U(z, x_n) = \frac{1}{2}(d(z, Tz) + d(z, Tx_n))$.
In this case

$$\begin{aligned} d(Tz, z) &\leq \frac{sk}{2}(d(z, Tz) + d(z, Tx_n) + sd(x_{n+1}, z) \\ i.e., \left(1 - \frac{sk}{2}\right) d(Tz, z) &\leq \frac{sk}{2} a_{n+1} + sa_{n+1} \leq \left(\frac{sk}{2} + s\right) \\ i.e., d(Tz, z) &\leq \frac{sk + 2s}{2 - sk} a_n \downarrow 0 \end{aligned}$$

Case 4: $U(z, x_n) = \frac{1}{2}(d(x_n, Tz) + d(x_n, Tx_n))$.

In this case

$$\begin{aligned} d(Tz,z) &\leq \frac{sk}{2}(d(x_n,Tz)+d(x_n,Tx_n)+sd(x_{n+1},z) \\ &\leq \frac{s^2k}{2}d(x_n,z)+\frac{s^2k}{2}d(z,Tz)+\frac{s^2k}{2}d(x_n,z) \\ &\quad +\frac{s^2k}{2}d(z,Tx_n)+sd(x_{n+1},z) \\ i.e., \left(1-\frac{s^2k}{2}\right)d(Tz,z) &\leq s^2ka_n+\frac{s^2k}{2}d(z,x_{n+1})+sd(x_{n+1},z) \\ &\leq s^2ka_n+\frac{s^2k}{2}a_{n+1}+sa_{n+1} \\ &\leq \left(s^2k+\frac{s^2k}{2}+s\right)a_n \\ i.e., \quad d(Tz,z) &\leq \frac{3s^2k+2s}{2-s^2k}a_n \downarrow 0 \end{aligned}$$

This implies $d(Tz,z)=0$. So $Tz=z$. Hence z is a fixed point of T.

We now show that z is unique. Let, if possible, suppose that z' is another fixed point of T. Then $Tz'=z'$. Now

$$d(z,z')=d(Tz,Tz')\leq kU(z,z')$$

where

$$\begin{aligned} U(z,z') \in \Big\{&\frac{1}{2}(d(z,Tz)+d(z',Tz')),\frac{1}{2}(d(z,Tz')+d(z',Tz)), \\ &\frac{1}{2}(d(z,Tz)+d(z,Tz')),\frac{1}{2}(d(z',Tz')+d(z',Tz))\Big\} \end{aligned}$$

that is,

$$U(z,z')\in\left\{0,d(z,z'),\frac{1}{2}(d(z,z')\right\}$$

This implies $d(z,z')=0$ and so $z=z'$. Hence T has a unique fixed point. ■

Theorem 2.4. Let (X,d,E) be a complete E-b-metric space with $s\geq 1$ and E-Archimedean. Let T be a self map on X such that

$$d(Tx,Ty)\leq a\ \max\{d(x,y),d(x,Tx),d(y,Ty)\}+b\{d(x,Ty)+d(y,Tx)\}$$

where $a,b>0$ and $sa+sb+s^2b<1$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$. Define $x_n = Tx_{n-1},\ n \geq 1$. Then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq a \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ &\quad + b\{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})\} \\ &= a \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} + b\{d(x_{n-1}, x_{n+1})\} \\ &\leq a \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} + bs\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} \end{aligned} \tag{2.1}$$

Two cases arise:

Case 1. $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$
Then (2.4.1) implies

$$d(x_n, x_{n+1}) \leq a\ d(x_{n-1}, x_n) + bs\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}$$

which gives

$$d(x_n, x_{n+1}) \leq \frac{a+bs}{1-bs}\, d(x_{n-1}, x_n) \qquad \text{where} \quad \frac{a+bs}{1-bs} < 1.$$

Case 2. $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$
Then (2.4.1) implies

$$d(x_n, x_{n+1}) \leq a\ d(x_n, x_{n+1}) + bs\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}$$

which gives

$$d(x_n, x_{n+1}) \leq \frac{bs}{1-a-bs}\, d(x_{n-1}, x_n) \qquad \text{where} \quad \frac{bs}{1-a-bs} < 1.$$

Combining two cases, we have

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n)$$

where

$$\lambda \in \left\{\frac{a+bs}{1-bs}, \frac{bs}{1-a-bs}\right\} < 1.$$

Since $0 \leq s\lambda < 1$, it follows from lemma 2.1 that $\{x_n\}$ is a E-Cauchy sequence in X. Since X is E-complete, there exists some $z \in X$ such that $x_n \xrightarrow{d,E} z$. So there exists a sequence $\{a_n\} \in E$ such that $a_n \downarrow 0$ and $d(x_n, z) \leq a_n$.

We now show that z is a fixed point of T. Now

$$\begin{aligned} d(Tz,z) &\le sd(Tz,Tx_n)+sd(Tx_n,z) \\ &= sd(Tz,Tx_n)+sd(x_{n+1},z) \\ &\le sa\max\{d(z,x_n),d(x_n,Tx_n),d(z,Tz)\}+sb\{d(x_n,Tz) \\ &\quad +d(z,Tx_n)\}+sd(x_{n+1},z) \\ &\le sa\max\{d(z,x_n),d(x_n,Tx_n),d(z,Tz)\} \\ &\quad +sb\{s(d(x_n,z)+d(z,Tz))+d(z,Tx_n)\}+sd(x_{n+1},z) \end{aligned}$$

which implies

$$\begin{aligned} (1-s^2b)d(Tz,z) \le sa\max\{d(z,x_n),d(x_n,Tx_n),d(z,Tz)\} \\ +s^2bd(x_n,z)+s(1+b)d(x_{n+1},z) \end{aligned}$$

Three cases arise:

Case 1. $\max\{d(z,x_n),d(x_n,Tx_n),d(z,Tz)\}=d(z,x_n)$. Then, we have

$$(1-s^2b)d(Tz,z)\le sa\;d(z,x_n)+s^2b\;d(x_n,z)+s(1+b)d(x_{n+1},z)$$

which implies

$$\begin{aligned} d(Tz,z) &\le \frac{s(a+bs)}{1-s^2b}\,d(x_n,z)+\frac{s(1+b)}{1-s^2b}\,d(x_{n+1},z) \\ &\le \frac{s(a+bs)}{1-s^2b}\,a_n+\frac{s(1+b)}{1-s^2b}\,a_{n+1} \\ &\le \frac{s(1+a+b+bs)}{1-s^2b}\,a_n\downarrow 0. \end{aligned}$$

This gives $d(Tz,z)=0$. So $Tz=z$. Hence z is a fixed point of T.

Case 2. $\max\{d(z,x_n),d(x_n,Tx_n),d(z,Tz)\}=d(x_n,Tx_n)$
Then, we have

$$\begin{aligned} (1-s^2b)d(Tz,z) &\le sa\;d(x_n,Tx_n)+s^2b\;d(x_n,z)+s(1+b)d(x_{n+1},z) \\ &\le s^2a\;d(x_n,z)+s^2ad(z,Tx_n)+s^2b\;d(x_n,z)+s(1+b)d(x_{n+1},z) \end{aligned}$$

which implies

$$\begin{aligned} d(Tz,z) &\le \frac{s^2(a+b)}{1-s^2b}\,d(x_n,z)+\frac{s(1+b+as)}{1-s^2b}\,d(x_{n+1},z) \\ &\le \frac{s^2(a+b)}{1-s^2b}\,a_n+\frac{s(1+b+as)}{1-s^2b}\,a_{n+1} \\ &\le \frac{s^2(2a+b)+s(1+b)}{1-s^2b}\,a_n\downarrow 0. \end{aligned}$$

This gives $d(Tz, z) = 0$. So $Tz = z$. Hence z is a fixed point of T.

Case 3. $\max\{d(z, x_n), d(x_n, Tx_n), d(z, Tz)\} = d(z, Tz)$
Then, we have

$$(1 - s^2 b)d(Tz, z) \leq sa\ d(z, Tz) + s^2 b\ d(x_n, z) + s(1 + b)d(x_{n+1}, z)$$

which implies

$$\begin{aligned} d(Tz, z) &\leq \frac{s^2 b}{1 - sa - s^2 b}\ d(x_n, z) + \frac{s(1 + b)}{1 - sa - s^2 b}\ d(x_{n+1}, z) \\ &\leq \frac{s^2 b}{1 - sa - s^2 b}\ a_n + \frac{s(1 + b)}{1 - sa - s^2 b}\ a_{n+1} \\ &\leq \frac{s(1 + b + bs)}{1 - sa - s^2 b}\ a_n \downarrow 0. \end{aligned}$$

This gives $d(Tz, z) = 0$. So $Tz = z$. Hence z is a fixed point of T. We now show that z is unique. Let, if possible, suppose that z' is another fixed point of T. Then $Tz' = z'$. Now

$$\begin{aligned} d(z, z') &= d(Tz, Tz') \\ &\leq a \max\left\{d(z, Tz), d(z', Tz'), d(z, z')\right\} + b\left\{d(z, z') + d(z', z)\right\} \\ &\leq (a + 2b)d(z, z') \end{aligned}$$

Since $a + 2b < 1$, we have $d(z, z') = 0$. So $z = z'$. Hence fixed point of T is unique. ■

Theorem 2.5. Let (X, d, E) be a complete E-b-metric space with $s \geq 1$ and E-Archimedean. Let T be a self map on X such that

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(Tx, y) \quad \text{for all } x, y \in X,$$

where $a_i \geq 0$ for all i = 1 to 5 and $k = \max\left(\dfrac{a_1 + a_2 + sa_4}{1 - a_3 - sa_4}, \dfrac{a_1 + a_3 + sa_5}{1 - a_2 - sa_5}\right)$ with $ks < 1$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$. Define $x_n = Tx_{n-1},\ n \geq 1$. Then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq a_1 d(x_{n-1}, x_n) + a_2 d(x_{n-1}, Tx_{n-1}) + a_3 d(x_n, Tx_n) \\ &\quad + a_4 d(x_{n-1}, Tx_n) + a_5 d(x_n, Tx_{n-1}) \\ &= (a_1 + a_2)\, d(x_{n-1}, x_n) + a_3 d(x_n, x_{n+1}) + a_4 d(x_{n-1}, x_{n+1}) \\ &\leq (a_1 + a_2)\, d(x_{n-1}, x_n) + a_3 d(x_n, x_{n+1}) + a_4 s\, \{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\} \end{aligned}$$

which implies

$$d(x_n, x_{n+1}) \leq \frac{a_1 + a_2 + sa_4}{1 - a_3 - sa_4}\, d(x_{n-1}, x_n). \tag{2.2}$$

Also

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1}) \\ &\le a_1d(x_n,x_{n-1})+a_2d(x_n,Tx_n)+a_3d(x_{n-1},Tx_{n-1}) \\ &\quad +a_4d(x_n,Tx_{n-1})+a_5d(x_{n-1},Tx_n) \\ &= (a_1+a_3)\,d(x_{n-1},x_n)+a_2d(x_n,x_{n+1})+a_5d(x_{n-1},x_{n+1}) \\ &\le (a_1+a_3)\,d(x_{n-1},x_n)+a_2d(x_n,x_{n+1})+a_5s\,\{d(x_{n-1},x_n)+d(x_n,x_{n+1})\} \end{aligned}$$

which implies

$$d(x_n,x_{n+1}) \le \frac{a_1+a_3+sa_5}{1-a_2-sa_5}\,d(x_{n-1},x_n). \tag{2.3}$$

Combining (2) and (3), we have

$$d(x_n,x_{n+1}) \le k\; d(x_{n-1},x_n).$$

where

$$k = \max\left(\frac{a_1+a_2+sa_4}{1-a_3-sa_4},\frac{a_1+a_3+sa_5}{1-a_2-sa_5}\right) < 1.$$

Also $sk < 1$. So lemma 2.1 implies that $\{x_n\}$ is a E-Cauchy sequence in X. Since X is E-complete, there exists some $z \in X$ such that $x_n \xrightarrow{d,E} z$. So there exists a sequence $\{b_n\} \in E$ such that $b_n \downarrow 0$ and $d(x_n,z) \le b_n$.

We now show that z is a fixed point of T. Now

$$\begin{aligned} d(Tz,z) &\le sd(Tz,Tx_n)+sd(Tx_n,z) \\ &\le s\,[a_1d(z,x_n)+a_2d(z,Tz)+a_3d(x_n,Tx_n)+a_4d(z,Tx_n) \\ &\quad +a_5d(x_n,Tz)]+sd(x_{n+1},z) \\ &\le sa_1b_n+sa_2d(z,Tz)+sa_3d(x_n,x_{n+1})+sa_4d(x_{n+1},z)+sa_5d(x_n,Tz)+sb_{n+1} \\ &\le sa_1b_n+sa_2d(z,Tz)+sa_3k^nd(x_0,x_1)+sa_4b_{n+1} \\ &\quad +s^2a_5d(x_n,z)+s^2a_5d(z,Tz)+sb_{n+1} \\ &\le sa_1b_n+sa_2d(z,Tz)+sa_3k^nd(x_0,x_1)+sa_4b_n+s^2a_5b_n+s^2a_5d(z,Tz)+sb_n \end{aligned}$$

which implies

$$d(Tz,z) \le \frac{s(1+a_1+a_4+sa_5)}{1-sa_2-s^2a_5}\,b_n+\frac{sa_3}{1-sa_2-s^2a_5}\,k^nd(x_0,x_1) \downarrow 0.$$

This gives $d(Tz,z)=0$. So $Tz=z$. Hence z is a fixed point of T. We now show that z is unique fixed point of T. For this, suppose that z' is another fixed point of T. Then $Tz'=z'$. Now

$$\begin{aligned} d(z,z') &= d(Tz,Tz') \\ &\le a_1d(z,z')+a_2d(z,Tz)+a_3d(z',Tz')+a_4d(z,Tz')+a_5d(z',Tz) \\ &= (a_1+a_4+a_5)d(z,z') \end{aligned}$$

Since $a_1 + a_4 + a_5 < 1$, we have $d(z, z') = 0$. So $z = z'$. Hence T has a unique fixed point. ■

Remark 2.6. In Theorem 2.5, the condition $sk < 1$ where
$k = \max\left(\frac{a_1 + a_2 + sa_4}{1 - a_3 - sa_4}, \frac{a_1 + a_3 + sa_5}{1 - a_2 - sa_5}\right)$ implies

$$s(a_1 + a_2 + sa_4) < 1 - a_3 + a_4 \tag{2.4}$$

$$\text{and} \quad s(a_1 + a_3 + sa_5) < 1 - a_2 + a_5 \tag{2.5}$$

From (4) and (5), we get

$$s(a_1 + a_2 + a_3 + sa_4 + sa_5) < 1 \tag{2.6}$$

The result of Theorem 2.5 also holds if the condition $sk < 1$, is replaced by the condition (6).

Corollary 2.7. Let (X, d, E) be a complete E-b-metric space with $s \geq 1$ and E-Archimedean. Let T be a self map on X such that

$$d(Tx, Ty) \leq \alpha d(x, y) \quad \text{for all } x, y \in X, \text{ where } 0 < \alpha s < 1.$$

Then T has a unique fixed point.

This is Banach Contraction Principle for E-b-metric spaces.

Proof. Result follows by taking $a_1 = \alpha, a_2 = a_3 = a_4 = a_5 = 0$ in the above theorem. ■

Corollary 2.8. Let (X, d, E) be a complete E-b-metric space with $s \geq 1$ and E-Archimedean. Let T be a self map on X such that

$$d(Tx, Ty) \leq \alpha \{d(x, Tx) + d(y, Ty)\} \quad \text{for all } x, y \in X,$$

where $0 < \alpha s < \frac{1}{2}$. Then T has a unique fixed point.

This result is similar to Kannan Type result proved by Kannan [13].

Proof. Result follows by taking $a_1 = a_4 = a_5 = 0$ and $a_2 = a_3 = \alpha$ in the above theorem. ■

Corollary 2.9. Let (X, d, E) be a complete E-b-metric space with $s \geq 1$ and E-Archimedean. Let T be a self map on X such that

$$d(Tx, Ty) \leq \alpha \{d(x, y) + d(x, Tx) + d(y, Ty)\} \quad \text{for all } x, y \in X,$$

where $0 < s\alpha < \frac{1}{3}$. Then T has a unique fixed point.

Proof. Result follows by taking $a_1 = a_2 = a_3 = \alpha$ and $a_4 = a_5 = 0$ in the above theorem. ■

Corollary 2.10. Let (X, d, E) be a complete E-b-metric space with $s \geq 1$ and E-Archimedean. Let T be a self map on X such that

$$d(Tx, Ty) \leq \alpha \{d(x, Ty) + d(y, Tx)\} \quad \text{for all } x, y \in X,$$

. where $0 < s^2\alpha < \frac{1}{2}$. Then T has a unique fixed point.

This result is similar to Chatterjea Type result proved by Chatterjea [8].

Proof. Result follows by taking $a_1 = a_2 = a_3 = 0$ and $a_4 = a_5 = \alpha$ in the above theorem. ■

Example 2.11. Let $X = \mathbb{R},\ E = \mathbb{R}^2$ and

$$d : X \times X \to \mathbb{R}^2$$

be defined as

$$d(x, y) = \left(\alpha(x - y)^2, \beta(x - y)^2\right)$$

where $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$. Then (X, d, E) is an E-b metric space with parameter $s = 2$.
Define

$$T : X \to X \quad \text{as} \quad T(x) = \frac{x}{2} + 1 \quad \forall x \in X.$$

Then

$$d(Tx, Ty) = d\left(\frac{x}{2} + 1, \frac{y}{2} + 1\right) = \frac{1}{4}\left(\alpha(x - y)^2, \beta(x - y)^2\right) = \frac{1}{4}d(x, y)$$

shows that T satisfies the conditions of Cor. 2.6 for $k \in \left[\frac{1}{4}, \frac{1}{2}\right)$ and has $x = 2$ as its unique fixed point. ■

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