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Pricing Barrier Contracts under Heavy-tailed Distributions

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Abstract

Pricing of Exotic options where the underlying asset returns exhibit heavy-tailedness can be a challenging task since the likelihood of over or under estimation tends to be high. This paper focuses on pricing of Barrier options in the case where financial asset returns are likely to evolve similarity to markov jump process like Lévy processes. The Variance Gamma (VG) process has been applied to model variance evolution on a proposed Partial Integro-Differential Equations (PIDE) and solved using the Crank Nicholson numerical method.

AMS subject classification:
Keywords: Barrier Options, Lévy Processes, Variance Gamma (VG), PIDE.

1. Introduction

Barrier Option is an example of a path-dependent exotic option. It's value not only depends on the underlying asset but also on the path the asset took during the life of the option and whether or not the underlying reached a predetermined threshold called the barrier. They are either activated or extinguished if the underlying asset crosses the barrier which then result into knock-in or knock-out barrier contracts. If the barrier level is above the initial asset price, then we have an – up – option and if the barrier level is below the initial asset price we will have a "down" option. Barrier options in some cases have a cash rebate (partial refund) being specified where this amount is paid to the holder

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whenever the barrier is reached for – out – barrier as a cushioning mechanism against the pain of losing the payoff. This rebate can either be paid immediately the barrier is triggered or at the end of the contract.

A barrier can either be in a discrete time or continuous time. The discrete time barriers only come into effect in discrete monitored time for example at close of every market day, every quarter or every month etc. There are analytical formulas which provide methods to price barrier options in continuous time. Pricing discretely monitored barrier options is not as easy as pricing continuously monitored barrier options, since there is essentially no closed solution as explained in [1].

The distribution of financial asset returns should be well understood in risk management. More often, practitioners assume that the returns follows a normal distribution. However empirical evidence suggest that it follows heavy-tailed distribution, where there is a high likelihood of significant deviations from the mean than in the normal distribution.

The famous Black Scholes model was considered a major breakthrough in option pricing where the authors derived a partial differential equation whose solution in the financial market was the price of an option. They assumed that the returns of an underlying asset follows geometric Brownian motion and that the interest rates and the volatility were constants. However these assumptions are incorrect especially that of constant volatility as it could not account for the volatility skews/smiles as evidenced in the option markets. Another limitation to the model is that the log returns distribution tends to be negatively skewed and has fatter tails than the normal distribution as assumed.

To address the drawbacks in the Black Scholes model, several models have been proposed including the jump-diffusion models considered by Merton in [2] and Kou in [3] where the stylized facts of financial assets returns such as volatility smiles and heavy tails have been reflected by these models. The Lévy process has also been introduced to suit the non-normal feature of asset return distributions. Some studies done on Lévy process can be found in the articles [4], [5] and [6] among others.

The valuation of contingent claims under jump diffusion processes in general requires solving partial integro-differential equations (PIDE). There has been interesting research carried out for pricing options numerically under jump diffusion processes. An explicit type approach based on multinomial trees was proposed by Amin in the article [7] however such methods experience time step limitations due to stability considerations and they are accurate only to first order. Another method developed by Xiao in [8] treats the integral term in the PIDE explicitly and the remaining terms implicitly although the method has a setback due to restrictive stability conditions.

The Monte Carlo simulation method has been applied to pricing barrier options under jump diffusion by Steve in [9]. However this method converges very slowly to obtain a more accurate approximation. The Crank Nicholson finite difference method takes the least number of time steps and is more accurate to $O\left((\Delta t)2, (\Delta S)^2\right)$ than other finite difference methods as explained in the article [10].

This paper is organized as follows: In Section 2 we give an introduction and discussion about the Lévy process and the Variance Gamma (VG) process. The VG process for the proposed PIDE for European options to obtain the price of the options is found in Section 3 and in Section 4 we look at the numerical method and example used to approximate the PIDE developed. We finally conclude the paper in Section 5.

2. The Lévy process

This paper considers the pricing of Barrier options where the underlying stock returns follows a LÃ©vy process. In particular we are going to consider the VG process also known as the Laplace motion which is a Lévy process determined by a random time change.

Definition 2.1. (Lévy Process) A right continuous-time stochastic process (X_t) on $(\omega, \mathcal{F}, \mathcal{P})$ with values in $\mathbb{R}$ such that $X_0 = 0$ is called a Lévy process if the following properties are satisfied:

(i) Independent increments: for any increasing sequence of times, $t_0, t_1 \cdots, t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_n-1}$ are independent.

(ii) Stationary increments: $X_{t+h} - X_t$ is independent of the time t.

(iii) Stochastic continuity: $\forall \epsilon > 0, \lim_{h \to 0} \mathcal{P}(|X_{t+h} - X_t| \geq \epsilon) = 0$

2.1. The Variance Gamma Process

The VG process considers a three parameter stochastic process where the parameters are responsible for the control of the volatility, kurtosis and the skewness of the asset returns. Studies found in the articles [11] and [12] only considered a two parameter stochastic processes which controlled the volatility and the kurtosis. To obtain the VG process, the Brownian motion with drift will be evaluated at a random time given by a gamma process. To achieve this, let

$$b(t; \psi, \sigma) = \psi t + \sigma W(t) \tag{2.1}$$

where $W(t)$ is a standard Brownian motion and the process $b(t; \psi, \sigma)$ is a Brownian motion with the drift parameter ψ and volatility σ. The gamma process $\varphi(t; \mu, \nu)$ having the mean rate μ and variance rate ν is the process of independent gamma increments over non-overlapping intervals of time $(t, t+h)$ as explained by Madan in [13]. Consider the change $\delta = \varphi(t+h; \mu, \nu) - \varphi(t; \mu, \nu)$, then the density $f_h(\delta)$ of this increment will be given by the gamma density function with the mean μh and variance νh, that is

$$f_h(\delta) = \left(\frac{\mu}{\nu}\right)^{\frac{\mu^2 h}{\nu}} \frac{\delta^{\frac{\mu^2 h}{\nu} - 1} \exp\left(-\frac{\mu}{\nu}\delta\right)}{\Gamma(\frac{\mu^2 h}{\nu})}, \delta > 0 \tag{2.2}$$

where $\Gamma(\cdot)$ is the gamma function.

The characteristic function $\Phi_{\varphi(t)}(u) = E[\exp(iu\varphi(t;\mu,\nu))]$ of the gamma density can be expressed as

$$\Phi(u) = \left(\frac{1}{1 - \frac{\nu}{\mu} iu}\right)^{\frac{\mu^2 t}{\nu}} . \tag{2.3}$$

The VG process $X(t;\sigma,\nu,\psi)$ is defined in terms of Brownian motion with drift $b(t;\psi,\sigma)$ and the gamma process with a unit mean rate $\varphi(t;1,\nu)$ as

$$X(t;\sigma,\nu,\psi) = b(\varphi(t;1,\nu);\psi,\sigma) . \tag{2.4}$$

The process can also be expressed as a difference of two independent increasing gamma process

$$X(t;\sigma,\nu,\psi) = \varphi_p(t;\mu_p,\nu_p) - \varphi_q(t;\mu_q,\nu_q) \tag{2.5}$$

and the relation between the parameters in equation (2.5) as explained in [13] is given below

$$\mu_p = \frac{1}{2}\sqrt{\psi^2 + \frac{2\sigma^2}{\nu}} + \frac{\psi}{2}$$

$$\mu_q = \frac{1}{2}\sqrt{\psi^2 + \frac{2\sigma^2}{\nu}} - \frac{\psi}{2}$$

$$\nu_p = \left(\frac{1}{2}\sqrt{\psi^2 + \frac{2\sigma^2}{\nu}} + \frac{\psi}{2}\right)^2 \nu$$

$$\nu_n = \left(\frac{1}{2}\sqrt{\psi^2 + \frac{2\sigma^2}{\nu}} - \frac{\psi}{2}\right)^2 \nu$$

The parameters of the VG process are themselves not the volatility, skewness and the kurtosis but are instead linked in the expressions of the moments of the return distribution over an interval of time lapse t. The mean of the process is independent of ν and σ and is given by

$$E[X(t)] = \psi t$$

the variance is given by

$$Var[X(t)] = (\psi^2\nu + \sigma^2)t \tag{2.6}$$

the third central moment (skewness) is given by

$$E[\,(X(t) - E[X(t)])^3\,] = (2\psi^3\nu^2 + 3\sigma^2\psi\nu)t \tag{2.7}$$

and the fourth central moment (kurtosis) is given by

$$E[\,(X(t) - E[X(t)])^4\,] = (3\sigma^4\nu + 12\sigma^2\psi^2\nu^2 + 6\psi^4\nu^3)t + (3\sigma^4 + 6\sigma^2\psi^2\nu + 3\psi^4\nu^2)t^2\,. \tag{2.8}$$

The VG process can be thought of as a time changed Brownian motion and its Lévy measure can be represented as difference of two gamma processes as shown in equation (2.5). The Lévy measure of a gamma process, $\varphi(t;\mu,\nu)$ is given as

$$l_\varphi(y)dy = \begin{cases} \dfrac{\mu^2\exp(-\frac{\mu}{\nu}y)}{\nu y}dy, & \text{for } y > 0. \\ 0, & \text{otherwise.} \end{cases}$$

and thus the Lévy measure for the VG process which is a difference of two gamma processes will be given by

$$l_X(y)dy = \begin{cases} \dfrac{\mu_q^2\exp(-\frac{\mu_q}{\nu_q}|y|)}{\nu_q|y|}dy, & \text{for } y < 0. \\ \dfrac{\mu_p^2\exp(-\frac{\mu_p}{\nu_p}y)}{\nu_p y}dy, & \text{for } y > 0. \end{cases} \tag{2.9}$$

where μ_p and μ_q are the mean rates and ν_p and ν_q are the variance rates. From equation (2.9) above, the Lévy density is divided by the absolute value of the jump size, meaning that the Lévy density is defined with a value of $\dfrac{1}{|y|}$ in the neighbourhood of zero hence as the VG Lévy measure integrates to infinity, the result will be an infinite activity process. We also note that $|y|$ is integrable with respect to the VG Lévy density and thus the process is one of finite variation. This describes the property of the VG process.

2.2. Stock Price Dynamics of the Variance Gamma Process

To obtain the stock price dynamics for the VG process, the geometric Brownian process in the Black Scholes model is replaced by the VG process and the resulting equation is

$$S(t) = S(0)\exp\left(mt + X_{VG}(t;\sigma_S,\nu_S,\psi_S) + \omega_S t\right) \tag{2.10}$$

where m is the statistical mean rate of return. The value of ω is determined as an arbitrage-free condition by evaluating the characteristic function for $X(t)$ at $\mu = \dfrac{1}{i}$ thereby obtaining

$$E(S(t)) = S_0\exp(mt) \Leftrightarrow E[\exp(X(t))] = \exp(-\omega_S t) \tag{2.11}$$

and this will lead to

$$\omega_S = \frac{1}{\nu_S}\ln\left(1 - \psi_S\nu_S - \frac{\sigma^2\nu_S}{2}\right)$$

The stock prices can be discounted at a risk free interest rate to obtain martingales. Therefore under risk neutral probability measure, the expected stock returns is the continuously compounded risk-free interest rate r and hence the risk neutral process will be given by

$$S(t) = S_0 \exp\left[rt + X(t;\sigma, v, \psi) + \omega t\right] \tag{2.12}$$

and similar to the case of risk neutrality, we have

$$\omega_{RN} = \frac{1}{v_{RN}} \ln\left(1 - \psi_{RN} v_{RN} - \frac{\sigma^2 v_{RN}}{2}\right)$$

3. The Variance Gamma PIDE for European Options

The limitations of the Black Scholes model have been addressed by among others in the volatility structure:

(i) The volatility is assumed to be a function of time t and the current level of stock price $S(t)$, i.e, $\sigma \equiv \sigma(t, S(t))$.

(ii) The volatility depends on some random parameter ζ such that $\sigma(t) \equiv \sigma(\zeta(t))$ where $\zeta(t)$ is some random process.

The value of an option say $F(\,\cdot\,)$ which is priced under the VG process which follows the stock price dynamics as shown in equation (2.12) requires additional parameters to be considered, apart from the usual ones considered when the underlying stock follows a geometric Brownian motion. Therefore in addition to the underlying stock price, the strike price, volatility, interest rate and the time to maturity, parameters defining the skewness and kurtosis are put into place. Therefore we can write the value of the option as a function of these parameters as $F(S, K, t, \sigma(t, S(t)), r, \psi, v)$, but for simplicity we will just adopt the notation $F(S, t)$ as the value of the option.

The dynamics of option pricing when the underlying returns are described by a VG process for non-dividend paying stock as described in [14], is given by

$$\begin{aligned} &\frac{\partial F(S,t)}{\partial t} + rS\frac{\partial F(S,t)}{\partial S} \\ &+ \int_{-\infty}^{+\infty} \left[F(S.e^y, t) - F(S,t) - \frac{\partial F(S,t)}{\partial S} S(e^y - 1)\right] \\ &* l(y)dy = rF(S,t) \end{aligned} \tag{3.13}$$

where $l(y)dy$ is the Lévy measure for the VG process.

To solve the term $\displaystyle\int_{-\infty}^{+\infty} (e^y - 1)l(y)dy$ in equation (3.13), we note that from the definition of a Lévy measure, we have that

$$l(y) = \lim_{t \longrightarrow 0} \frac{P(y_t \in dy|_{y_0=0})}{t}$$

and hence the integral will be written as

$$\int_{-\infty}^{+\infty} (e^y - 1) l(y) dy = \frac{1}{t} \int_{-\infty}^{+\infty} (e^y - 1) P(y_t \in dy|_{y_0=0}) dy \,. \tag{3.14}$$

From equation (2.11) it's clear that

$$\int_{-\infty}^{+\infty} e^y . P(y_t \in dy|_{y_0=0}) dy = E(e^{X_t}) = e^{-\omega t}$$

and from the property that

$$\int_{-\infty}^{+\infty} P(y_t \in dy|_{y_0=0}) dy = 1$$

then substituting the above results in equation (3.14) we have that

$$\int_{-\infty}^{+\infty} (e^y - 1) l(y) dy = \frac{e^{-\omega t} - 1}{t} = -\omega \tag{3.15}$$

i.e if we take limits as $t \longrightarrow 0$.

Therefore substituting equation (3.15) into (3.13) we have the following Partial Integro Differential Equation (PIDE),

$$\begin{aligned} &\frac{\partial F(S,t)}{\partial t} + (r + \omega) S \frac{\partial F(S,t)}{\partial S} \\ &+ \int_{-\infty}^{+\infty} \left[F(S.e^y, t) - F(S,t) \right] l(y) dy - rF(S,t) = 0 \,. \end{aligned} \tag{3.16}$$

This PIDE equation describes the behaviour of option prices under the VG process and it is almost similar to the Black Scholes PDE for option pricing where the underlying stock follows geometric Brownian motion.

3.1. PIDE for Barrier Options

Barrier options are known to follow the same dynamics as the plain vanilla options if the underlying stock follows a geometric Brownian motion hence satisfying the Black Scholes PDE but under different boundary conditions. Similarly, the European Barrier options follows the PIDE equation (3.16) which describes the dynamics of European vanilla options under the VG process with specified boundary conditions. This equation can be solved by implementing a finite difference scheme by discretizing the stock and time spaces. The space in which the grid is build for the numerical solution will be

$$[0, T] \times [S_{\min}, S_{\max}]$$

where the minimum and maximum stock values S have to be small and big enough such that there will be no effect on altering this values than a chosen threshold. The problem can be transformed by logarithmic terms by changing the variable

$$x \approx \ln(S)$$

hence obtaining a new function

$$H(x,t) \approx F(S,t)\,.$$

Therefore with the new function, we will have the following relationships

$$\begin{aligned} H(x+y,t) &= F(Se^{y},t) \\ \frac{\partial H(x,t)}{\partial x} &= S\frac{\partial F(S,t)}{\partial S} \\ \frac{\partial H(x,t)}{\partial t} &= \frac{\partial F(S,t)}{\partial t} \end{aligned}$$

The above new relations of the functions can be substituted to the PIDE equation (3.16) we will have

$$\frac{\partial H(x,t)}{\partial t} + (r+\omega)\frac{\partial H(x,t)}{\partial x} + \int_{-\infty}^{+\infty} [H(x+y,t) - H(x,t)]\, l(y)dy - rH(x,t) = 0 \qquad (3.17)$$

3.2. Boundary Conditions

To solve the PIDE in (3.17) above, we need to define the terminal and boundary conditions required to solve the problem for the cases of Down-and-Out Barrier options. The terminal condition dictates that the value of the option at expiration is equal to its payoff as long as the stock price has not touched the barrier level B. For the case of a Down-and-Out call option, the final condition is

$$H(x,T) = \max(e^{x} - K, 0)\,, \quad \text{if } e^{x} > B \;\; \forall x$$

and the boundary conditions are

$$\begin{aligned} H(\ln(B),t) &= R\,,\ \forall t \\ H(+\infty,t) &= e^{x}\,,\ \forall t \end{aligned}$$

where $R \gg 0$ is the rebate value. For the Down-and-Out Put option, the terminal condition is

$$H(x,T) = \max(K - e^{x}, 0)\,, \quad \text{if } e^{x} > B \;\; \forall x$$

and the boundary conditions are

$$\begin{aligned} H(\ln(B),t) &= R\,,\ \forall t \\ H(+\infty,t) &= 0\,,\ \forall t \end{aligned}$$

The figure below describes the final and boundary conditions for the Down-and-Out Call option

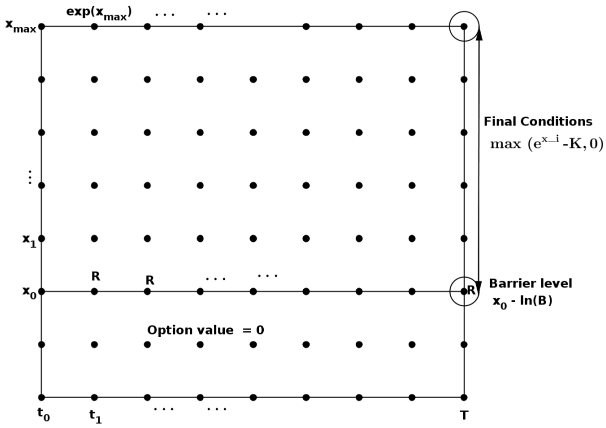


Figure 1: Final and Boundary conditions

3.3. PIDE discretization

The finite difference approximation for the PIDE equation will require discretization of the stock and time space. Since we are considering the knock-out options where the option becomes valueless on hitting the barrier, we will be required to fix the limit of the grid at the barrier bound. Thus for the case where the barrier is below the stock level, i.e, for the down-and-out calls and puts, the range of values to be considered will be

$$[0, T] \times [\ln(B), x_{\max}]$$

where $x_{\max}$ is such that the option value on the boundary is correct upto the level required. If we suppose that the time to maturity for the option is T, then we will have equally spaced time intervals of Δt, i.e, $0, \Delta t, 2\Delta t, \cdots, N\Delta t$ and hence we have $N + 1$ points, where the size of the time step ($t-$ direction) will be

$$\Delta t = \frac{T}{N}.$$

The size of the stock space ($x-$ direction) will be

$$\Delta x = \frac{x_{\max} - \ln(B)}{M}$$

where we will have $M + 1$ points, that is, $0, \Delta x, 2\Delta x, \cdots, M\Delta x$. Therefore the size on the discretization on the (x, t) plane will be of size $(M + 1) \times (N + 1)$. The notation $H(x_m, t_n)$ denotes the value of $H(\cdot)$ at the node (m, n) which can be written as $H(x_m, t_n) = H(m\Delta x, n\Delta t)$ for $m = 0, 1, 2, \cdots M$ and $n = 0, 1, 2, \cdots N$.

To obtain the approximation of the partial derivative with respect to time in equation (3.17), the forward difference is applied and the result will be

$$\frac{\partial H(x, t)}{\partial t} \approx \frac{H(x_m, t_{n+1}) - H(x_m, t_n)}{\Delta t}, \tag{3.18}$$

and for the approximation of the partial derivative with respect to x, the central difference method is applied and the resulting solution will be

$$\frac{\partial H(x,t)}{\partial x} \approx \frac{H(x_{m+1},t_n) - H(x_{m-1},t_n)}{2\Delta x}\,. \tag{3.19}$$

These approximations are then replaced in the PIDE equation (3.17) above thereby obtaining

$$\begin{aligned}&\pi H(x_{m-1},t_n) + (1 + r\Delta t)H(x_m,t_n) - \pi H(x_{m+1},t_n)\\ &- H(x_m,t_n) + \Delta t \int_{-\infty}^{+\infty} \big[H(x_m + y, t_{n+1}) - H(x_m, t_{n+1})\big] *\\ &l(y)dy = 0\end{aligned} \tag{3.20}$$

where

$$\pi = (r+\omega)\frac{\Delta t}{2\Delta x}\,.$$

4. Numerical Method for solving the PIDE

To find the approximation of the PIDE (3.20) we implement the Crank Nicholson numerical scheme. This method is achieved by taking the average of the implicit and the explicit finite difference methods. It is considered over other finite difference methods because of it's unconditional stability and it's accuracy (accurate to $O((\Delta x)^2, (\Delta t)^2)$). This figure below shows the approximation point at which Crank Nicholson method is taken

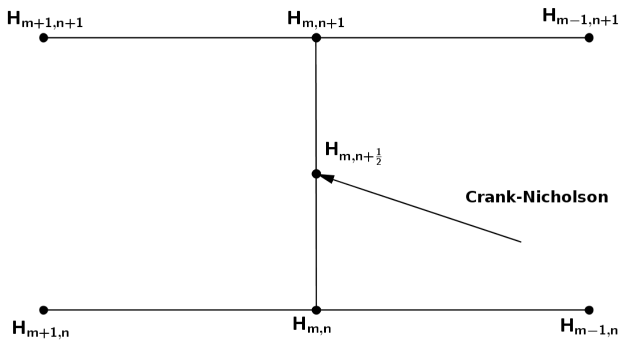


Figure 2: Crank Nicholson scheme

Therefore on taking the average of the implicit and explicit schemes, the resulting equation will be

$$\begin{aligned}
&\frac{1}{2}\big\{H(x_m,t_{n+1})-H(x_m,t_n)\\
&\quad+\pi\left[H(x_{m+1},t_{n+1})-H(x_{m-1},t_{n+1})\right]-r\Delta t H(x_m,t_n)\\
&\quad+\Delta t\int_{-\infty}^{+\infty}\left[H(x_m+y,t_{n+1})-H(x_m,t_{n+1})\right]l(y)dy\\
&\quad-r\Delta t H(x_m,t_n)+H(x_m,t_{n+1})-H(x_m,t_n)\\
&\quad+\pi\left[H(x_{m+1},t_n)-H(x_{m-1},t_n)\right]\\
&\quad+\Delta t\int_{-\infty}^{+\infty}\left[H(x_m+y,t_{n+1})-H(x_m,t_{n+1})\right]l(y)dy\big\}=0\,.
\end{aligned}$$

Rearranging the result above we obtain

$$\begin{aligned}
&\frac{1}{2}\big\{\pi H(x_{m-1},t_n)+2(1+r\Delta t)H(x_m,t_n)-\pi H(x_{m+1},t_n)\\
&\qquad+\pi H(x_{m-1},t_{n+1})-H(x_m,t_{n+1})-\pi H(x_{m+1},t_{n+1})\big\}\\
&\qquad\qquad-\Delta t\int_{-\infty}^{+\infty}\left[H(x_m+y,t_{n+1})-H(x_m,t_{n+1})\right]l(y)dy=0
\end{aligned}$$

The problem in this case will be on how to solve the integral in the PIDE equation above. We need to evaluate it numerically to make the differential equation possible to be written as a linear system of equations. This then will be solved to obtain the price of Barrier options. Here we apply the knowledge of Riemann sums and write the integral as sum of integrals. The different cases of jump sizes, that is, jumps with smaller value than the space step, jumps with bigger value than the space step but is still inside the range and jumps which is so big that the stock finishes outside the range are considered. Therefore the integral can be split into

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\left[H(x_m+y,t_{n+1})-H(x_m,t_{n+1})\right]k(y)dy\\
&\quad=\int_{-\infty}^{x_0-x_m}\left[H(x_m+y,t_{n+1})-H(x_m,t_{n+1})\right]k(y)dy &\text{(I)}\\
&\qquad+\int_{x_0-x_m}^{-\Delta x}\left[H(x_m+y,t_{n+1})-H(x_m,t_{n+1})\right]k(y)dy &\text{(II)}\\
&\qquad+\int_{-\Delta x}^{0}\left[H(x_m+y,t_{n+1})-H(x_m,t_{n+1})\right]k(y)dy &\text{(III)}\\
&\qquad+\int_{0}^{\Delta x}\left[H(x_m+y,t_{n+1})-H(x_m,t_{n+1})\right]k(y)dy &\text{(IV)}
\end{aligned}$$

$$+\int_{\Delta x}^{x_M-x_m}\left[H(x_m+y,t_{n+1})-H(x_m,t_{n+1})\right]k(y)dy \quad \text{(V)}$$

$$+\int_{x_M-x_m}^{+\infty}\left[H(x_m+y,t_{n+1})-H(x_m,t_{n+1})\right]k(y)dy \quad \text{(VI)}$$

The procedure found in [14] is followed to obtain the solution of the integral. The integrals (I) to (IV) is calculated separately and the resulting solution can then be rearranged to obtain

$$\begin{aligned}
&\left\{\beta H_{m-1,n}+\alpha H_{m,n}-\varphi H_{m+1,n}\right\}\\
&\quad=-\frac{1}{2}\left\{\pi H_{m-1,n+1}-H_{m,n+1}-\pi H_{m+1,n+1}\right\}\\
&\quad+\Delta t\Bigg\{\sum_{k=1}^{M-m-1}\frac{1}{v\Delta x}\left[H_{m+k+1,n+1}-H_{m+k,n+1}\right]\\
&\quad*\frac{v_p}{\mu_p}\left(e^{-\frac{\mu_p}{v_p}k\Delta x}-e^{-\frac{\mu_p}{v_p}(k+1)\Delta x}\right)\\
&\quad+\sum_{k=1}^{M-m-1}\frac{1}{v}\left[H_{m+k,n+1}-H_{m,n+1}\right]\\
&\quad-k\left[H_{m+k+1,n+1}-H_{m+k,n+1}\right]\\
&\quad*\left[\mathrm{Ei}\left(\frac{\mu_p}{v_p}k\Delta x\right)-\mathrm{Ei}\left(\frac{\mu_p}{v_p}(k+1)\Delta x\right)\right]\\
&\quad+\sum_{k=1}^{m-1}\frac{1}{v\Delta x}\left[H_{m-k-1,n+1}-H_{m-k,n+1}\right]\\
&\quad*\frac{v_q}{\mu_q}\left(e^{-\frac{\mu_q}{v_q}k\Delta x}-e^{-\frac{\mu_q}{v_q}(k+1)\Delta x}\right)\\
&\quad+\sum_{k=1}^{m-1}\frac{1}{v}\left[H_{m-k,n+1}-H_{m,n+1}\right]\\
&\quad-k\left[H_{m-k-1,n+1}-H_{m-k,n+1}\right]\\
&\quad*\left[\mathrm{Ei}\left(\frac{\mu_q}{v_q}k\Delta x\right)-\mathrm{Ei}\left(\frac{\mu_q}{v_q}(k+1)\Delta x\right)\right]\\
&\quad+\frac{1}{v}\Big\{(e^{x_m}\mathrm{Ei}\left[\left(\frac{\mu_p}{v_p}-1\right)(M-m)\Delta x\right]\\
&\quad-\left[Ke^{-r(T-t_{n+1})}+H_{m,n+1}\right]\mathrm{Ei}\left[\frac{\mu_p}{v_p}(M-m)\Delta x\right]\\
&\quad+\left[R-H_{m,n+1}*\right]\mathrm{Ei}\left(\frac{\mu_q}{v_q}m\Delta x\right)\Big\}\Bigg\}
\end{aligned} \tag{4.21}$$

where we denoted $H(x_m, t_n)$ by $H_{m,n}$ and the constants are:

$$\beta = \frac{\pi}{2} - \frac{\left(1 - e^{-\frac{\mu_q}{v_q}\Delta x}\right)}{\frac{\mu_q}{v_q} v \Delta x} \Delta t$$

$$\alpha = 1 + \left(r + \frac{\left(1 - e^{-\frac{\mu_q}{v_q}\Delta x}\right)}{\frac{\mu_q}{v_q} v \Delta x} + \frac{\left(1 - e^{-\frac{\mu_p}{v_p}\Delta x}\right)}{\frac{\mu_p}{v_p} v \Delta x} \right) \Delta t$$

$$\varphi = \frac{\pi}{2} + \frac{\left(1 - e^{-\frac{\mu_p}{v_p}\Delta x}\right)}{\frac{\mu_p}{v_p} v \Delta x} \Delta t$$

The first three terms of the equation above on the left hand side can be written as a Tri-diagonal matrix for $m = 1, 2, \cdots M$ and $n = 0, 1, 2, \cdots N$.

4.1. Numerical Example

This paper has considered the NSE 20 share index for 503 trading days between January 2015 to December 2016 where the units are in KES. The underlying constants considered are

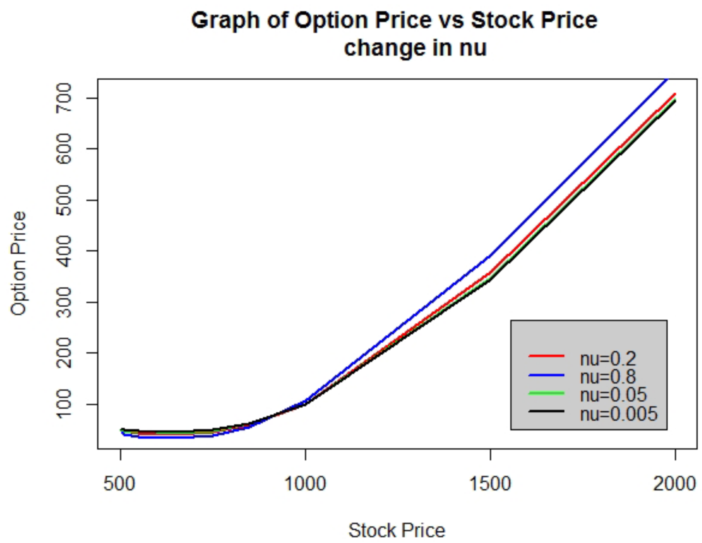
$$\begin{aligned}
S_{max} &= 5000 \\
S_{min} &= 1000 \\
B &= 500\,, \quad \text{Barrier level} \\
R &= 50\,, \quad \text{Rebate value} \\
T &= 1.378\,, \quad \text{time to expiry} \\
K &= 1000\,, \quad \text{strike price} \\
r &= 0.0474\,, \quad \text{interest rate} \\
\sigma &= 0.36\,, \quad \text{volatility} \\
\nu &= 0.2\,, \quad \text{the variance rate} \\
\psi &= -0.34\,, \quad \text{the drift parameter} \\
M &= 1000 \\
N &= 1000\,.
\end{aligned}$$

From the simulations taken, it can be noted that an increase in σ could lead to an increase in the option prices. A similar case is also realised when the interest rate is increased which is a true realization of what the market normally reacts to such situations. The parameters σ, ν and ψ are responsible for the behaviour of the volatility, the kurtosis

ν	Standard Deviation	Skewness	Kurtosis
0.8	0.4712536	−1.491197	6.982661
0.2	0.3907941	−0.4956718	3.767915
0.05	0.3679402	−0.1366368	3.162535
0.005	0.3608019	−0.01411426	3.015133

and the skewness as observed in equations (2.6), (2.7) and (2.8) above. The table below shows how the standard deviation, skewness and the kurtosis responds to changes in ν.

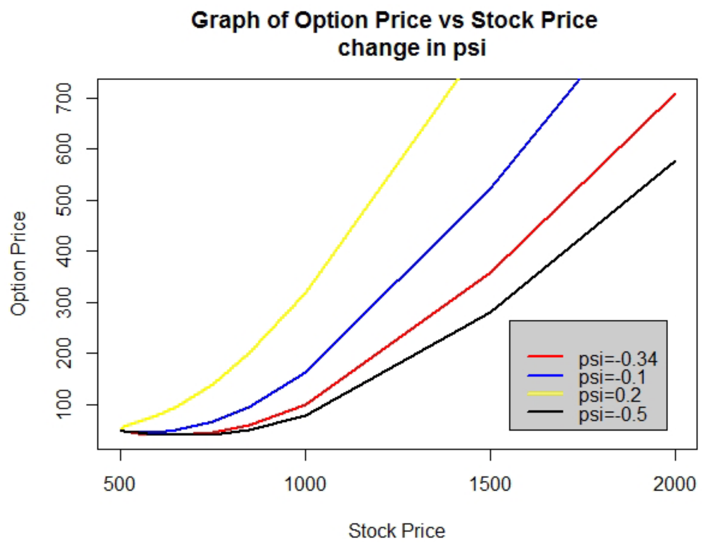
It is observed that a positive increase in ν leads to increase in the standard deviation and the kurtosis while the skewness tends to be more negative and as ψ moves towards zero, the value of the standard deviation and the kurtosis decreases while the skewness increase positively towards zero. This changes then affects the option prices as shown by the graph below where it is observed that an increase in ψ leads to increase in the option prices and vice versa.



The table below shows that as ψ becomes more negative, where the other constants are fixed, the volatility and the kurtosis increases positively while the skewness tends to decrease. On the other hand as ψ increases, the volatility and the kurtosis decrease while the skewness increases.

ψ	Standard Deviation	Skewness	Kurtosis
−0.5	0.4237924	−0.6422021	3.887573
−0.34	0.3907941	−0.4956718	3.767915
−0.1	0.3627671	−0.1645575	3.618099

The graph below shows the effects of the changes in ψ to the option prices. It is noted that as ψ increases the prices of options also increase and vice versa.



The Crank Nicolson method for approximating the PIDE equation of pricing performs better when the step size and the time step considered is small. The table below shows the option prices for the corresponding stock price at various space and time steps

M	N	Stock Price	Option Price
10	10	1255.94	367.11
		2505.94	1387.63
100	100	1255.94	216.33
		2505.94	1108.49
500	500	1255.94	213.10
		2505.94	1103.50
1000	1000	1255.94	213.00
		2505.94	1103.37
1500	1000	1255.94	212.96
		2505.94	1103.31
2000	1000	1255.94	212.95
		2505.94	1103.29
3000	1000	1255.94	212.94
		2505.94	1103.27
3200	1200	1255.94	212.95
		2505.94	1103.30

5. Conclusion

The distribution of the stock returns of the NSE 20 Share Index sample data does not follow a normal distribution. The Shapiro test was conducted where the null hypothesis for the test is that the data is normally distributed. The chosen alpha level is 0.05 and a p-value of 2.2^{-16} was obtained which is less than 0.05, implying that the null hypothesis that the data are normally distributed is rejected. Therefore the data follows a heavy tailed distribution where we chose the variance gamma process in pricing the Barrier Option.

From the examples shown in the table above where two stock prices 1255.94 and 2505.94 were picked, it is clear the smaller the step sizes for Δt and Δx the more the option prices tends to be consistent. The Crank Nicholson method has second order accuracy and hence to obtain more accurate results the step size and the time step chosen i.e the size of M and N should be large enough.

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