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Study of pairwise ω-compact spaces

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Deparment of mathematics.

Abstract

In [9] the author, introduce the notion of ω-compactness and investigated its fundamental properties. In this paper we introduce and study a pairwise ω-compact spaces in bitopological spaces, and we investigate some more properties of this type of compact spaces.

AMS subject classification:
Keywords: Bitopological spaces, $p-\omega$-closed compact spaces, $p-\omega$-closed lindelöf $p-\omega$-continuous, $p-M-\omega$-compact, p-strongly-ω-continuous function.

1. Introduction

From (1963), when Kelly [9] introduced the concept of bitopological space, several topological property in single topology are generalised into bitopological spaces, such as compactness, paracompactness, seperation axiomes, connected, types of functions and other topices. We shall use p-to denote pairwise, e.g. p-compact stands for pairwise compact. If (X, τ_1, τ_2) is a bitopological space and $A \subseteq X$, $cl_1(A)$ and $cl_2(A)$ will denote the closure of A with respect to τ_1 and $\tau_2$respectively. Let (X, τ) be a topological space and let A be a subset of X. A point $x \in (X, \tau_1, \tau_2)$ is called a condensation point of A, if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. In 1982, Hdeib defined ω-closed sets and ω-open sets as follows. A is called ω-closed if it contains all its condensation points. The complement of an ω-closed set is called ω-open. also $cl^{\omega} A$ will denote the intersection of all ω-closed sets which contains A. The family of all ω-open sets in (X, τ) is denoted by $W(\tau)$. Compactness and properties closely related to compactness play an important role in applications of General Topology to Real Analysis and Functional Analysis. Throughout this paper, spaces (X, τ_1, τ_2) (or simply X) always means a bitopological spaces on which no separation axioms are assumed unless explicitly stated. In [8] the author, introduce the notion of ω-compactness and investigated its fundamental properties. In this paper we introduce and study a pairwise ω-compact spaces in bitopological spaces, and we investigate some more properties of this type of compact spaces.

2. Preliminaries

Definition 2.1. [3] A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called p-continous, if $f_1 : (X, \tau_1) \to (Y, \sigma_1)$ and $f_2 : (X, \tau_2) \to (Y, \sigma_2)$ are continous functions.

Definition 2.2. [7] A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called p-closed, if $f_1 : (X, \tau_1) \to (Y, \sigma_1)$ and $f_2 : (X, \tau_2) \to (Y, \sigma_2)$ are closed functions, i.e. F_1 is closed in τ_1, then $f(F_1)$ is closed in σ_1, and if F_2 is closed in τ_2, then $f(F_2)$ is closed in σ_2.

Definition 2.3. [7] A cover U of the bitopological space $(X, \tau_1, \tau_2)$is called $\tau_1\tau_2$-open if $U \subset \tau_1 \cup \tau_2$.If in addition, U contains at least one-nonempty member of τ_2,it is called p-open.

Definition 2.4. [7] A bitopological space is called a p-compact, if every p-open cover of the space has a finite subcover.

Definition 2.5. [7] A bitopological space is called a s-compact, if every $\tau_1\tau_2$-open cover of the space has a finite subcover.

Definition 2.6. [7] A bitopological space is called a p-lindelöf, if every p-open cover of the space has a countable subcover.

Definition 2.7. [7] A bitopological space is called a s-lindelöf, if every $\tau_1\tau_2$-open cover of the space has a countable subcover.

Definition 2.8. [1] A subset A of a bitopological space $(X, \tau_1, \tau_2)$is pairwise-ω-open, (simply $p - \omega$-open) if for each$x \in A$ there exists a pairwise-open subset U_x containing x such that $U_x - A$ is a countable set. The complement of a pairwise ω-open is said to be pairwise-ω-closed set(simply $p - \omega$-closed). The family of all pairwise ω-open (respectively pairwise ω-closed) subsets of a space (X, τ_1, τ_2) is denoted by $p - \omega - BO(X)$, (respectively $p - \omega - BC(X)$). Also the family of all pairwise-ω-open sets of (X, τ_1, τ_2) containing x is denoted by$p - \omega - BO(X; x)$.

Definition 2.9. [1] A subset A of a bitopological space $(X, \tau_1, \tau_2)$is semi-ω-open, (simply $s - \omega$-open) if for each $x \in A$ there exists a $\tau_1\tau_2$-open subset U_x containing x such that $U_x - A$ is a countable set. The complement of a semi-ω-open is said to be semi-ω-closed set(simply $s - \omega$-closed). The family of all semi-ω-open (respectively semi-ω-closed) subsets of a space (X, τ_1, τ_2) is denoted by $s - \omega - BO(X)$, (respectively $s - \omega - BC(X)$). Also the family of all semi-ω-open sets of (X, τ_1, τ_2) containing x is denoted by$s - \omega - BO(X; x)$.

Definition 2.10. [1] A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called a pairwise-ω-closed function, if it functions pairwise closed sets onto pairwise-ω-closed sets.

Definition 2.11. [1] A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called a semi-ω-closed function, if it functions semi closed sets onto semi-ω-closed sets.

Definition 2.12. [1] A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be p-weakly continuous function if for every p-open set $U \subset Y$, $f^{-1}(U)$ is $p - \omega$-open.

Definition 2.13. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be p-strongly-ω-continuous function if for every $p - \omega$-open set $U \subset Y$, $f^{-1}(U)$ is p-open.

Definition 2.14. [2] A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called $p - \omega$-continuous at point $x \in (X, \tau_1, \tau_2)$, if for every p-open set V containing $f(x)$, there is $p - \omega$-open set U contaning x such that $f(U) \subset V$. If f is $p - \omega$-continuous at each point of (X, τ_1, τ_2), then f is said to be $p - \omega$-continuous on (X, τ_1, τ_2).

3. Properties of pairwise ω-compact spaces

Definition 3.1. A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is called $p - \omega$-continuous (resp. $p - \omega$-irresolute) if, $f_1 : (X, \tau_1) \rightarrow (Y, \sigma_1)$ and $f_2 : (X, \tau_2) \rightarrow (Y, \sigma_2)$ are ω-continuous (resp. ω-irresolute) functions.

Definition 3.2. A family $\hat{A}$ of subsets of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2 - \omega$-open if $\hat{A} \subset W(\tau_1) \cup W(\tau_2)$. If, in addition $\hat{A} \cap W(\tau_1) \neq \phi$ and $\hat{A} \cap W(\tau_2) \neq \phi$ then $\hat{A}$ is called pairwise ω-open. (simply $p.\omega.o$).

Definition 3.3. A bitopological space (X, τ_1, τ_2) is said to be pairwise-ω-compact, simply $p.\omega.c.$ (resp. pairwise M-ω-compact, simply $p.M - \omega.c.$) if each $p.\omega.$open (resp. $\tau_1\tau_2 - \omega$-open) cover of X has a finite subcover. Clearly every $p.M - \omega.c.$ space is $p.\omega.c.$, and we can easily show that the converse may not be true.

Definition 3.4. A space (X, τ_1, τ_2) is said to be $p - \omega$-lindelöf if every $p - \omega$-open cover of (X, τ_1, τ_2) has a countable subcover.

Theorem 3.5. [9] If $A \subseteq B \subseteq (X, \tau)$ where A is ω-open relative to B and B is open in (X, τ), then A is ω-open in X.

Theorem 3.6. Every p-open subset of a $p - M - \omega$-compact space is $p - M - \omega$-compact.

Proof. Let A be a p-open subset of a $p - M - \omega$-compact space (X, τ_1, τ_2). If $\{V_\alpha : \alpha \in \Gamma\}$ is a $\tau_1\tau_2 - \omega$-open cover of (A, τ_A, τ_A), then each V_α is $\tau_1\tau_2 - \omega$-open in (X, τ_1, τ_2), for each $\alpha \in \Gamma$. Therefore $\{V_\alpha : \alpha \in \Gamma\} \cup \{X - A\}$ is $\tau_1\tau_2 - \omega$-open cover of (X, τ_1, τ_2). Since (X, τ_1, τ_2) is $p - M - \omega$-compact space, there exists a finite subset $\Gamma_0 \subset \Gamma$ such that $\{V_\alpha : \alpha \in \Gamma_0\}$ covers A. Hence A is $p - M - \omega$-compact. ■

Theorem 3.7. A space (X, τ_1, τ_2) is $p - M - \omega$-compact if and only if it is $p - \omega$-compact and (X, τ_1), (X, τ_2) are ω-compact.

Proof. $\Rightarrow$ Let (X, τ_1, τ_2) be $p - M - \omega$-compact, then it is $p - \omega$-compact. To show that (X, τ_1) is ω-compact, let $\underset{\sim}{U}$ be a cover of X by ω-open sets in τ_1 then $\underset{\sim}{U} \subseteq W(\tau_1) \subseteq$

$W(\tau_1)\cup W(\tau_2)$. So $\underset{\sim}{U}$ is a $\tau_1\tau_2-\omega$-open cover of the $p-M-\omega$-compact X. Thus $\underset{\sim}{U}$ has a finite subcover. Hence (X,τ_1) is ω-compact. Simillaraly (X,τ_2) is ω-compact.

$\Leftarrow$ Let (X,τ_1,τ_2) be $p-\omega$-compact and (X,τ_1), (X,τ_2) are ω-compact spaces, and let $\underset{\sim}{U}$ be a cover of $\tau_1\tau_2-\omega$-open cover of X. Then $\underset{\sim}{U}\subseteq W(\tau_1)\cup W(\tau_2)$, we have if there exist U_1 and U_2 such that $U_1\in W(\tau_1)$ and $U_2\in W(\tau_2)$, then $\underset{\sim}{U}$ is a $p-\omega$-open cover of X. Hence $\underset{\sim}{U}$ has a finite subcover. If $\underset{\sim}{U}\subseteq W(\tau_1)$ or $\underset{\sim}{U}\subseteq W(\tau_2)$ then $\underset{\sim}{U}$ has a finite subcover since (X,τ_1), (X,τ_2) are ω-compact. Hence (X,τ_1,τ_2) is $p-M-\omega$-compact. ■

Theorem 3.8. A space (X,τ_1,τ_2) is $p-M-\omega$-compact if and only if every $\tau_1\tau_2-\omega$-closed family of subsets of X having the finite intersection property, has no a non-empty intersection.

Theorem 3.9. A space (X,τ_1,τ_2) is $p-\omega$-compact if and only if every $p-\omega$-closed family of subsets of X having the finite intersection property, has no a non-empty intersection.

Definition 3.10. A subset A of a bitopological space (X,τ_1,τ_2) is said to be pairwise $-\omega$-compact (resp.pairwise $-\omega$-lindelöf) relative to (X,τ_1,τ_2) if every pairwise $-\omega$-cover of A has a finite (resp. countable) subcover, where the ω-open sets belong to $W(\tau_1)\cup W(\tau_2)$.

Theorem 3.11. A bitopological space (X,τ_1,τ_2) is pairwise $-\omega$-compact if and only if each proper $\tau_i-\omega$-closed subset of (X,τ_1,τ_2) is ω-compact relative to (X,τ_j),where $i,j=1,2,i\neq j$.

Proof. $\Rightarrow$) Let H be any proper $\tau_i-\omega$-closed subset of pairwise $-\omega$-compact bitopological space (X,τ_1,τ_2).Then $X-H$ is a nonempty $\tau_i-\omega$-open set. Let $\{V_\alpha:\alpha\in\Gamma\}$ be a cover of H by $\tau_j-\omega$-open sets, then $\{V_\alpha:\alpha\in\Gamma\}\cup\{X-H\}$ is a $p-\omega$-open cover of X and hence it has a finite subcover say $\left\{V_{\alpha_i}:i=1,2,\dots\right\}\cup\{X-H\}$. Therefore, we have $H\subset\left\{V_{\alpha_i}:i=1,2,\dots\right\}$. Hence H is ω-compact relative to (X,τ_j), for $i,j=1,2,i\neq j$.

$\Leftarrow$) Let $\{V_\alpha:\alpha\in\Gamma\}$ be an infinite $p-\omega$-open cover of X. Let $\Gamma_i=\{\alpha\in\Gamma:V_\alpha\in W(\tau_i)\}$ for $i=1,2$. We have two cases to explained:

Case 1: If $\cup\left\{V_\alpha:\alpha\in\Gamma_j\right\}=X$, then choose $\alpha_0\in\Gamma_i$ such that $U_{\alpha_0}\neq\phi$. Since $\left\{V_\alpha:\alpha\in\Gamma_j\right\}$ is a $\tau_j-\omega$-open cover of the proper $\tau_i-\omega$-closed subset of $X-V_{\alpha_0}$ therefore it has a finite a subcover say $\left\{V_\alpha:\alpha\in\Gamma_j^*\right\}$, so (X,τ_1,τ_2) is pairwise $-\omega$-compact.

Case 2: If $\cup\left\{V_\alpha:\alpha\in\Gamma_j\right\}\neq X, H=X-\left\{V_\alpha:\alpha\in\Gamma_j\right\}$ is a proper $\tau_j-\omega$-closed subset of X, and $H\subset\cup\{V_\alpha:\alpha\in\Gamma_i\}$,so there is a finite subcover of $\{V_\alpha:\alpha\in\Gamma_i\}$ say $\left\{V_\alpha:\alpha\in\Gamma_i^*\right\}$. Thus $H\subset\cup\left\{V_\alpha:\alpha\in\Gamma_i^*\right\}$ and so $X-\left\{V_\alpha:\alpha\in\Gamma_i^*\right\}\subset X-H=$

$\cup \left\{ V_\alpha : \alpha \in \Gamma_j \right\}$.

Hence $X - \left\{ V_\alpha : \alpha \in \Gamma_i^* \right\} \subset \cup \left\{ V_\alpha : \alpha \in \Gamma_j^* \right\}$ and $X = \left\{ V_\alpha : \alpha \in \Gamma_i^* \cup \Gamma_j^* \right\}$. ■

Theorem 3.12. A bitopological space (X, τ_1, τ_2) is pairwise $-\omega$-lindelöf if and only if each proper $\tau_i - \omega$-closed subset of (X, τ_1, τ_2) is ω-lindelöf relative to (X, τ_j), where $i, j = 1, 2, i \neq j$.

Proof. In a similar way of the previous theorem. ■

Definition 3.13. A subset C of a bitopological space (X, τ_1, τ_2) is called pairwise $-\omega - F_\sigma$ set if $C = \cup \{C_i : i = 1, 2, \dots\}$ where C_i is a $\tau_1 - \omega$-closed or a $\tau_2 - \omega$-closed set for $i = 1, 2, \dots$.

Theorem 3.14. If (X, τ_1, τ_2) is pairwise $-\omega$-lindelöf then every pairwise $-\omega - F_\sigma$ subset of X is pairwise $-\omega$-lindelöf relative to (X, τ_1, τ_2).

Proof. Let $C = \cup \{C_i : i = 1, 2, \dots\}$ be a pairwise $-\omega - F_\sigma$ subset of X and $\{V_\alpha : \alpha \in \Gamma\}$ be a pairwise $-\omega$-cover of C where $V_\alpha \in W(\tau_1) \cup W(\tau_2)$ for each $\alpha \in \Gamma$. Then $\{V_\alpha : \alpha \in \Gamma\}$ is a pairwise $-\omega$-cover of C_i for each i. Since C_i is pairwise $-\omega$-lindelöf, so for each i we have $C_i \subset \cup \{V_\alpha : \alpha \in \Gamma\}$ and so $\left\{ V_\alpha : \alpha \in \left\{ \cup \Gamma_i^* : i = 1, 2, \dots \right\} \right\}$ is a countable subcover of $\{V_\alpha : \alpha \in \Gamma\}$ for C. Hence the result. ■

Theorem 3.15. A $p - \omega$-continuous image of a $p - \omega$-compact space is p-compact.

Proof. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $p - \omega$-continuous function from a $p - \omega$-compact space X onto a (Y, σ_1, σ_2). Let $\{A_\alpha : \alpha \in \Gamma\}$ be a p-open cover of Y. Then $\left\{ f^{-1}(A_\alpha) : \alpha \in \Gamma \right\}$ is a $p - \omega$-open cover of X. Since X is $p - \omega$-compact space then it has a finite subcover $\left\{ f^{-1}(A_{\alpha_1}), f^{-1}(A_{\alpha_2}), \dots, f^{-1}(A_{\alpha_n}), : \alpha \in \Gamma \right\}$. Since f is onto $\left\{ A_{\alpha_1}, A_{\alpha_2}, \dots, A_{\alpha_n}, : \alpha \in \Gamma \right\}$ is a cover of Y. Hence Y is p-compact. ■

Remark 3.16. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $p - \omega$-continuous function and A is $p - \omega$-compact relative to (X, τ_1, τ_2), the $f(A)$ is p-compact in (Y, σ_1, σ_2).

Theorem 3.17. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $p - \omega$-irresolute and surjective function and (X, τ_1, τ_2) is $p - \omega$-compact space then (Y, σ_1, σ_2) is $p - \omega$-compact space.

Proof. By using the simillar technique in the previous theorem. ■

Proposition 3.18. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $p - \omega$-irresolute function subset B is $p - \omega$-compact relative to (X, τ_1, τ_2), then the image $f(B)$ is $p - \omega$-compact relative to (Y, σ_1, σ_2).

Proof. Let $\{A_\alpha : \alpha \in \Gamma\}$ be a $p - \omega$-open cover of Y such that $f(B) \subseteq \cup A_\alpha$ for each $\alpha \in \Gamma$. Since f is $p - \omega$-irresolute function, then $B \subseteq \cup f^{-1}(A_\alpha)$ for each $\alpha \in \Gamma$. Since

X is $p-\omega$-compact space then it has a finite $\Gamma_0 \subset \Gamma$such that $B \subseteq \cup f^{-1}(A_\alpha)$ for each $\alpha \in \Gamma_0$, therefore $f(B) \subseteq \cup A_\alpha$ for each $\alpha \in \Gamma_0$, so $f(B)$ is $p-\omega$-compact relative to (Y, σ_1, σ_2). ■

Theorem 3.19. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is p-strongly-ω-continuous onto function, where (X, τ_1, τ_2) is p-compact space, then (Y, σ_1, σ_2) is $p-\omega$-compact space.

Proof. Let $\{A_\alpha : \alpha \in \Gamma\}$ be a $p-\omega$-open cover of (Y, σ_1, σ_2), then $\left\{f^{-1}(A_\alpha) : \alpha \in \Gamma\right\}$ be a p-open cover of X, since f is p-strongly-ω-continuous function. Since X is p-compact space then it has a finite subcover $\left\{f^{-1}(A_{\alpha_1}), f^{-1}(A_{\alpha_2}), \ldots, f^{-1}(A_{\alpha_n}), : \alpha \in \Gamma\right\}$. Since f is onto $\left\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}, : \alpha \in \Gamma\right\}$ is a cover of Y. Hence Y is $p-\omega$-compact space. ■

Definition 3.20. A space (X, τ_1, τ_2) is said to be $p-\omega$-closed compact if every $p-\omega$-closed cover of (X, τ_1, τ_2) has a finite subcover.

Definition 3.21. A space (X, τ_1, τ_2) is said to be $p-\omega$-closed lindelöf if every $p-\omega$-closed cover of (X, τ_1, τ_2) has a countable subcover.

Theorem 3.22. Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $p-\omega$-continuous surjection function. Then the following statements hold:

(1) If (Y, σ_1, σ_2) is $p-\omega$-closed lindelöf, then (X, τ_1, τ_2) is p-lindelöf,

(2) If (Y, σ_1, σ_2) is $p-\omega$-closed compact, then (X, τ_1, τ_2) is p-compact.

Proof. Let $\underset{\sim}{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a $p-\omega$-open cover of (X, τ_1, τ_2), since f is $p-\omega$-continuous surjection function, then $\forall y \in (Y, \sigma_1, \sigma_2)$, there exists a countable subsets Λ_y, Λ_y of Λ,

s.t $f^{-1}(y) \subseteq \bigcup\limits_{\alpha \in \Lambda_y} \{V_\alpha : \alpha \in \Lambda_y\} \bigcup \bigcup\limits_{\alpha \in \Lambda_y^*} \{W_\alpha : \alpha \in \overset{*}{\Lambda}_y\}$, where $\{V_\alpha : \alpha \in \Lambda_y\}$ is $\tau_1-\omega$-open, $\{W_\alpha : \alpha \in \overset{*}{\Lambda}_y\}$ is $\tau_2-\omega$-open. Let $O_y = Y - f(X - \bigcup\limits_{\alpha \in \Lambda_y} V_\alpha)$ is a $\sigma_1-\omega$-open set containing y, and $O_y^* = Y - f(X - \bigcup\limits_{\alpha \in \overset{*}{\Lambda}_y} W_\alpha)$ is a $\sigma_2-\omega$-open set containing y, where $f^{-1}(O_y) \subseteq \bigcup\limits_{\alpha \in \Lambda_y} V_\alpha$, $f^{-1}(O_y^*) \subseteq \bigcup\limits_{\alpha \in \overset{*}{\Lambda}_y} W_\alpha$. Let $\underset{\sim}{O} = \{O_y : y \in Y\} \bigcup \{O_y^* : y \in Y\}$ is a $p-\omega$-open cover of Y.

Since Y is $p-\omega$-closed lindelöf, $Y \subseteq \bigcup\limits_{i=1}^{\infty}(O_{y_i}) \bigcup \bigcup\limits_{i=1}^{\infty}(\overset{*}{O}_{y_j})$.

Thus, $f^{-1}(Y) \subseteq \bigcup_{i=1}^{\infty} f^{-1}(O_{y_i}) \bigcup \bigcup_{j=1}^{\infty} f^{-1}(\overset{*}{O}_{y_j}) \subseteq$ union of countable of $\underset{\sim}{U}$, i.e (X, τ_1, τ_2) is p-lindelöf.

(2) Similar to (1). ■

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