

Hybrid pairs of nonself multi-valued mappings in metrically convex metric spaces

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Abstract

Employing the concept of weakly compatible mappings between set-valued mappings and single valued mappings due to Ćirić and Ume [8], we prove some results on common fixed points for two hybrid pairs of mappings satisfying a generalized contraction type condition on a complete metrically convex metric spaces which generalize relevant results due to Assad and Kirk [4], Assad [5], Ćirić and Ume [7, 8], Itoh [17], Khan [23], Rhoades [26] and others. As an application of our main result, we also prove a common fixed point theorem in Banach spaces besides furnishing an illustrative examples.

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1. Introduction

Assad and Kirk [4] initiated the study of nonself multi-valued mappings and also discussed fixed point theorems for such mappings. In recent years, several fixed point theorems for such mappings were proved which include relevant results due to Assad [5], Ćirić and Ume [7, 8], Hadžić [10], Hadžić and Gajic [11], Itoh [17], Khan [23] and others. On the other hand there exists fixed point theorems for hybrid pairs of self mappings which include Hicks and Rhoades [12], Imdad and Ahmad [14], Kaneko and Sessa [19] and others. Combining these two ideas, Ahmad and Imdad [1,2], Imdad and Khan [13, 14, 16] and Khan and Imdad [21] have recently proved results on fixed

and coincidence points of generalized hybrid contractions for compatible mappings, pointwise R -weakly commuting mappings as well as weakly compatible mappings.

The purpose of this paper is to extend and generalize a fixed point theorem due to Ćirić and Ume [8] proved for nonself multi-valued mappings to two hybrid pairs of weakly compatible mappings which is either partially or completely generalizes the results due to Assad and Kirk [4], Assad [5], Ćirić and Ume [7, 8], Itoh [17], Khan [23], Rhoades [26] and others. Here for the sake of completeness, we state the main theorem of Ćirić and Ume [8] which runs as follows:

Theorem 1.1. Let (X, d) be a complete metrically convex metric space and K be a nonempty closed subset of X . Let S, T be mappings of K into $CB(X)$ such that

$$H(Sx, Ty) \leq \alpha d(x, y) + \beta \max \{D(x, Sx), D(y, Ty)\} \\ + \gamma \max \{D(x, Sx) + D(y, Ty), D(x, Ty) + D(y, Sx)\} \quad (1.1)$$

for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ are such that $\alpha + 2\beta + 3\gamma + \alpha\gamma < 1$.

If $Sx \subseteq K$ and $Tx \subseteq K$ for each $x \in \delta K$ (The boundary of K), then there exists a point $u \in K$ such that $u \in Su, u \in Tu$ and $Su = Tu$.

Notice that $D(x, Sx) = \inf \{d(x, y) : y \in Sx\}$.

2. Preliminaries

Let (X, d) be a metric space. We denote

- (i) $CB(X) = \{A : A \text{ is nonempty closed and bounded subset of } X\}$,
- (ii) for nonempty subsets A, B of $X, x \in X$,
 $d(x, A) = \inf \{d(x, a) : a \in A\}$ and $H(A, B) = \max [\{\sup d(a, B) : a \in A\}, \{\sup d(A, b) : b \in B\}]$.

It is well known (cf. Kuratowski [20]) that $CB(X)$ is a metric space with the distance function H which is known as Hausdorff-Pompeiu metric on X .

Definition 2.1. ([24]) Two self mappings F and T of a metric space (X, d) are T -weak compatible iff the following limits exist and satisfy:

- (iii) $\lim_{n \rightarrow \infty} d(TFx_n, FTx_n) \leq \lim_{n \rightarrow \infty} d(FTx_n, Fx_n)$, and
- (iv) $\lim_{n \rightarrow \infty} d(TFx_n, Tx_n) \leq \lim_{n \rightarrow \infty} d(FTx_n, Fx_n)$,

whenever $\{x_n\}$ is a sequence in X such that $Tx_n \rightarrow t, Fx_n \rightarrow t$ for some $t \in X$.

Definition 2.2. ([24]) Mappings $F : X \rightarrow CB(X)$ and $T : X \rightarrow X$ are T -weak compatible if $TFx \in CB(X)$ for all $x \in X$ and the following limits exist and satisfy:

$$(v) \lim_{n \rightarrow \infty} H(TFx_n, FTx_n) \leq \lim_{n \rightarrow \infty} H(FTx_n, Fx_n), \text{ and}$$

$$(vi) \lim_{n \rightarrow \infty} H(TFx_n, Tx_n) \leq \lim_{n \rightarrow \infty} H(FTx_n, Fx_n),$$

whenever $\{x_n\}$ is a sequence in X such that $Tx_n \rightarrow t \in A$ and $Fx_n \rightarrow A \in CB(X)$.

Motivated by the definition of weak compatibility due to Pathak [24], Ćirić and Ume [6] adopted the same to nonself setting.

Definition 2.3. ([6]) Let K be a nonempty subset of a metric space (X, d) . Mappings $F : K \rightarrow CB(X)$ and $T : K \rightarrow X$ are said to be weakly compatible on K if for any sequences $\{x_n\}$ and $\{y_n\}$ in K such that $Tx_n \in K, Fx_n \cap K \neq \phi$, the following limits exist and satisfy:

$$(vii) \limsup_{n \rightarrow \infty} d(Ty_n, FTx_n) \leq \limsup_{n \rightarrow \infty} H(FTx_n, Fx_n), \text{ and}$$

$$(viii) \limsup_{n \rightarrow \infty} d(Ty_n, Tx_n) \leq \limsup_{n \rightarrow \infty} H(FTx_n, Fx_n),$$

whenever $y_n \in Fx_n \cap K$ and $\lim_{n \rightarrow \infty} d(y_n, Tx_n) = 0$.

Notice that if $F : X \rightarrow X$ and $T : X \rightarrow X$ are self mappings of X and \limsup is replaced by \lim , then this definition reduces to Definition 2.2 of Pathak [24].

Definition 2.4. ([13]) Let K be a nonempty subset of a metric space (X, d) , $T : K \rightarrow X$ and $F : K \rightarrow CB(X)$. The pair (F, T) is said to be quasi-coincidentally commuting if for all coincidence points 'x' of (F, T) , $TFx \subset FTx$ whenever $Fx \subset K$ and $Tx \in K$ for all $x \in K$.

Definition 2.5. ([13]) A mapping $T : K \rightarrow X$ is said to be coincidentally idempotent w.r.t mapping $F : K \rightarrow CB(X)$, if T is idempotent at the coincidence points of the pair (F, T) .

Definition 2.6. ([21]) A mapping $T : K \rightarrow X$ is said to be occasionally coincidentally idempotent w.r.t mapping $F : K \rightarrow CB(X)$, if there exists a point $z \in K$ such that T is idempotent at the coincidence points of the pair (F, T) .

Definition 2.7. ([4]) A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

Lemma 2.8. ([4]) Let K be a nonempty closed subset of a metrically convex metric space (X, d) . If $x \in K$ and $y \notin K$ then there exists a point $z \in \delta K$ (the boundary of K) such that

$$d(x, z) + d(z, y) = d(x, y).$$

Lemma 2.9. ([7]) Let $A, B \in CB(X)$ and $a \in A$, then for any positive number $q < 1$ there exists $b = b(a)$ in B such that $q d(a, b) \leq H(A, B)$.

3. Main Results

Theorem 3.1. Let (X, d) be a complete metrically convex metric space and K a nonempty closed subset of X . Let $F, G : K \rightarrow CB(X)$ and $S, T : K \rightarrow X$ satisfying:

$$(ix) \quad \delta K \subseteq SK \cap TK, FK \cap K \subseteq SK, GK \cap K \subseteq TK,$$

$$(x) \quad Tx \in \delta K \Rightarrow Fx \subseteq K, Sx \in \delta K \Rightarrow Gx \subseteq K, \text{ and}$$

$$H(Fx, Gy) \leq \alpha d(Tx, Sy) + \beta \max\{d(Tx, Fx), d(Sy, Gy)\} \\ + \gamma \max\{d(Tx, Fx) + d(Sy, Gy), d(Tx, Gy) + d(Sy, Fx)\} \quad (3.1)$$

for all $x, y \in X$ with $x \neq y$, where $\alpha, \beta, \gamma \geq 0$, $q < 1$ such that $\alpha + 2\beta + 3\gamma + \alpha\gamma < q < 1$,

$$(xi) \quad (F, T) \text{ and } (G, S) \text{ are weakly compatible pairs,}$$

$$(xii) \quad T \text{ and } S \text{ are continuous on } K.$$

Then there exists a point $z \in K$ such that $z = Tz = Sz \in Fz \cap Gz$.

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way. Let $x \in \delta K$ then due to $\delta K \subseteq TK$ there exists a point $x_0 \in K$ such that $x = Tx_0$. From $Tx \in \delta K \Rightarrow Fx \subseteq K$, one concludes that $Fx_0 \subseteq FK \cap K \subseteq SK$. Let $x_1 \in K$ be such that $y_1 = Sx_1 \in Fx_0 \subseteq K$. Since $y_1 \in Fx_0$, there exists a point $y_2 \in Gx_1$ such that

$$q d(y_1, y_2) \leq H(Fx_0, Gx_1).$$

Suppose $y_2 \in K$. Then $y_2 \in GK \cap K \subseteq TK$, which implies that there exists a point $x_2 \in K$ such that $y_2 = Tx_2$. Otherwise, if $y_2 \notin K$, then there exists a point $p \in \delta K$ such that

$$d(Sx_1, p) + d(p, y_2) = d(Sx_1, y_2).$$

Since $p \in \delta K \subseteq TK$, there exists a point $x_2 \in K$ such that $p = Tx_2$ and so

$$d(Sx_1, Tx_2) + d(Tx_2, y_2) = d(Sx_1, y_2).$$

Let $y_3 \in Fx_2$ be such that $q d(y_2, y_3) \leq H(Gx_1, Fx_2)$.

Thus, repeating the foregoing arguments, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$(xiii) \quad y_{2n} \in Gx_{2n-1}, y_{2n+1} \in Fx_{2n}$$

$$(xiv) \quad y_{2n} \in K \Rightarrow y_{2n} = Tx_{2n} \text{ or } y_{2n} \notin K \Rightarrow Tx_{2n} \in \delta K, \text{ and}$$

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n})$$

(xv) $y_{2n+1} \in K \Rightarrow y_{2n+1} = Sx_{2n+1}$ or $y_{2n+1} \notin K \Rightarrow Sx_{2n+1} \in \delta K$, and

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1})$$

(xvi) $q d(y_{2n-1}, y_{2n}) \leq H(Gx_{2n-1}, Fx_{2n-2})$ and $q d(y_{2n}, y_{2n+1}) \leq H(Fx_{2n}, Gx_{2n-1})$.

We denote

$$P_o = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} = y_{2i}\}, P_1 = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} \neq y_{2i}\},$$

$$Q_o = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} = y_{2i+1}\} \text{ and}$$

$$Q_1 = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} \neq y_{2i+1}\}.$$

One can note that $(Tx_{2n}, Sx_{2n+1}) \notin P_1 \times Q_1$ and $(Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1$.

Now we distinguish the following three cases.

Case 1. If $(Tx_{2n}, Sx_{2n+1}) \in P_o \times Q_o$, then

$$\begin{aligned} q d(Tx_{2n}, Sx_{2n+1}) &= q d(y_{2n}, y_{2n+1}) \leq H(Fx_{2n}, Gx_{2n-1}) \\ &\leq \alpha d(Tx_{2n}, Sx_{2n-1}) + \beta \max\{d(Tx_{2n}, Fx_{2n}), d(Sx_{2n-1}, Gx_{2n-1})\} \\ &\quad + \gamma \max\{d(Tx_{2n}, Fx_{2n}) + d(Sx_{2n-1}, Gx_{2n-1}), d(Tx_{2n}, Gx_{2n-1}) \\ &\quad + d(Sx_{2n-1}, Fx_{2n})\}, \\ &\leq \alpha d(y_{2n-1}, y_{2n}) + \beta \max\{d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} \\ &\quad + \gamma \max\{d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n+1})\}, \\ &\leq \alpha d(y_{2n-1}, y_{2n}) + \beta \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} \\ &\quad + \gamma \{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\}, \\ q d(Tx_{2n}, Sx_{2n+1}) &\leq (\alpha + \gamma) d(y_{2n-1}, y_{2n}) + \beta \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} \\ &\quad + \gamma d(y_{2n}, y_{2n+1}). \end{aligned} \tag{3.2}$$

If we suppose that $d(y_{2n-1}, y_{2n}) \leq d(y_{2n}, y_{2n+1})$ then we obtain

$$q d(Tx_{2n}, Sx_{2n+1}) \leq (\alpha + \beta + 2\gamma) d(y_{2n}, y_{2n+1})$$

which is a contradiction. Therefore from (3.2) we obtain

$$q d(Tx_{2n}, Sx_{2n+1}) \leq (\alpha + \beta + \gamma) d(y_{2n}, y_{2n-1}) + \gamma d(y_{2n}, y_{2n+1})$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \leq \left(\frac{\alpha + \beta + \gamma}{q - \gamma}\right) d(Sx_{2n-1}, Tx_{2n}). \tag{3.3}$$

Similarly if $(Sx_{2n-1}, Tx_{2n}) \in Q_o \times P_o$, then

$$d(Sx_{2n-1}, Tx_{2n}) \leq \left(\frac{\alpha + \beta + \gamma}{q - \gamma}\right) d(Sx_{2n-1}, Tx_{2n-2}). \tag{3.4}$$

Case 2. If $(Tx_{2n}, Sx_{2n+1}) \in P_o \times Q_1$ then

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}),$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \leq d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1}),$$

and hence

$$q d(Tx_{2n}, Sx_{2n+1}) \leq q d(y_{2n}, y_{2n+1}) \leq H(Fx_{2n}, Gx_{2n-1}).$$

Now, proceeding as in Case 1, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq \left(\frac{\alpha + \beta + \gamma}{q - \gamma} \right) d(Sx_{2n-1}, Tx_{2n}).$$

If $(Sx_{2n-1}, Tx_{2n}) \in Q_1 \times P_o$ then as earlier, we also obtain

$$d(Sx_{2n-1}, Tx_{2n}) \leq \left(\frac{\alpha + \beta + \gamma}{q - \gamma} \right) d(Sx_{2n-1}, Tx_{2n-2}).$$

Case 3. If $(Tx_{2n}, Sx_{2n+1}) \in P_1 \times Q_o$ then $Sx_{2n-1} = y_{2n-1}$. As in Case 1, we get

$$\begin{aligned} q d(Tx_{2n}, Sx_{2n+1}) &= q d(Tx_{2n}, y_{2n+1}) \leq q \{d(Tx_{2n}, y_{2n}) + d(y_{2n}, y_{2n+1})\}, \\ &\leq q d(Tx_{2n}, y_{2n}) + q d(y_{2n}, y_{2n+1}) \leq q d(Tx_{2n}, y_{2n}) \\ &\quad + H(Fx_{2n}, Gx_{2n-1}), \\ &\leq q d(Tx_{2n}, y_{2n}) + \alpha d(Tx_{2n}, Sx_{2n-1}) \\ &\quad + \beta \max\{d(Tx_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} \\ &\quad + \gamma \max\{d(Tx_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n}), d(Tx_{2n}, y_{2n}) \\ &\quad + d(Sx_{2n-1}, Sx_{2n+1})\}. \end{aligned}$$

Since $\alpha < q$ and $d(Tx_{2n}, y_{2n}) + d(Tx_{2n}, Sx_{2n-1}) = d(Sx_{2n-1}, y_{2n})$ we obtain

$$q d(Tx_{2n}, y_{2n}) + \alpha d(Tx_{2n}, Sx_{2n-1}) \leq q d(Sx_{2n-1}, y_{2n}).$$

Also, by the triangle inequality, we obtain

$$\begin{aligned} d(Tx_{2n}, y_{2n}) + d(Sx_{2n-1}, Sx_{2n+1}) &\leq d(Tx_{2n}, y_{2n}) + d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, Sx_{2n+1}) \\ &\leq d(Sx_{2n-1}, y_{2n}) + d(Tx_{2n}, Sx_{2n+1}). \end{aligned}$$

Therefore

$$\begin{aligned} q d(Tx_{2n}, Sx_{2n+1}) &\leq q d(Sx_{2n-1}, y_{2n}) + \beta \max\{d(Tx_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\} \\ &\quad + \gamma \{d(Sx_{2n-1}, y_{2n}) + d(Tx_{2n}, y_{2n+1})\} \end{aligned}$$

If $d(Tx_{2n}, y_{2n+1}) \geq d(y_{2n-1}, y_{2n})$ then we obtain

$$d(Tx_{2n}, Sx_{2n+1}) \leq \left(\frac{q + \gamma}{q - \beta - \gamma} \right) d(Sx_{2n-1}, y_{2n}).$$

Otherwise, if $d(Tx_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n})$ then

$$d(Tx_{2n}, Sx_{2n+1}) \leq \left(\frac{q + \beta + \gamma}{q - \gamma} \right) d(Sx_{2n-1}, y_{2n}) \leq \left(\frac{q + \gamma}{q - \beta - \gamma} \right) d(Sx_{2n-1}, y_{2n}).$$

Now, proceeding as earlier, we also obtain

$$d(Sx_{2n-1}, y_{2n}) \leq \left(\frac{\alpha + \beta + \gamma}{q - \gamma} \right) d(Sx_{2n-1}, Tx_{2n-2}).$$

Therefore combining above inequalities, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq kd(Sx_{2n-1}, Tx_{2n-2}), \text{ where } k = \left(\frac{q + \gamma}{q - \beta - \gamma} \right) \left(\frac{\alpha + \beta + \gamma}{q - \gamma} \right).$$

Thus in all the cases, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq k \max \{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n-2}, Sx_{2n-1})\}, \tag{3.5}$$

whereas

$$d(Sx_{2n+1}, Tx_{2n+2}) \leq k \max \{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1})\}. \tag{3.6}$$

Now on the lines of Assad and Kirk [4], it can be shown by induction that for $n \geq 1$, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq k^n \delta \text{ and } d(Sx_{2n+1}, Tx_{2n+2}) \leq k^{n+\frac{1}{2}} \delta, \text{ whereas } \delta = k^{\frac{-1}{2}} \max \{d(Tx_0, Sx_1), d(Sx_1, Tx_2)\}.$$

Thus the sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, \dots, Sx_{2n-1}, Tx_{2n}, Sx_{2n+1}, \dots\}$ is Cauchy and converges to some point z . It follows that $\{Tx_{2n}\} \rightarrow z$ as $n \rightarrow \infty$ that is $y_{2n} \rightarrow z$ as $n \rightarrow \infty$. For each y_{2n} denoted by Y_{2n} one of the subsets $\{Gx_{2n-1}\}$ which contains y_{2n} and also y_{2n+1} denoted by Y_{2n+1} one of the subsets $\{Fx_{2n}\}$ which contains y_{2n+1} . Then

$$H(Y_{2n}, Y_{2n+1}) \leq \left(\frac{\alpha + \beta + \gamma}{q - \gamma} \right) d(Sx_{2n-1}, Tx_{2n}) \text{ (See (3.3)), or}$$

$$H(Y_{2n+1}, Y_{2n+2}) \leq k \max \{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1})\} \text{ See (3.6).}$$

Since the sequence is Cauchy it follows that $\{Y_n\}$ is Cauchy in the complete metric space $(CB(X), H)$. Thus $\lim_{n \rightarrow \infty} Y_n = Y$ for some $Y \in CB(X)$. Now we have

$$D(z, Y) \leq d(z, y_n) + H(Y_n, Y) \rightarrow 0 \text{ as } n \rightarrow \infty$$

and hence $z \in Y$. By the construction of the sequence there exists at least one subsequence $\{Tx_{2n_k}\}$ or $\{Sx_{2n_k+1}\}$ which is contained in P_\circ or Q_\circ respectively. Consequently the subsequence $\{Sx_{2n_k+1}\}$ which is contained in Q_\circ for each $k \in N$, converges to z . Since $Sx_{2n+1} = y_{2n+1}$, $\{SSx_{2n_k+1}\}$ and $\{GSx_{2n_k-1}\}$ are well defined.

$$\text{Set } L_k = d(Sx_{2n_k+1}, GSx_{2n_k-1}) \text{ and } R_k = H(Gx_{2n_k-1}, GSx_{2n_k-1}).$$

Therefore

$$\begin{aligned} R_k &\leq H(Y_{2n_k}, Y_{2n_k+1}) + H(Fx_{2n_k}, GSx_{2n_k-1}) \\ &\leq H(Y_{2n_k}, Y_{2n_k+1}) + \alpha d(Tx_{2n_k}, SSx_{2n_k-1}) + \beta \max\{d(Tx_{2n_k}, Fx_{2n_k}), \\ &d(SSx_{2n_k-1}, GSx_{2n_k-1})\} + \gamma \max\{d(Tx_{2n_k}, Fx_{2n_k}) + d(SSx_{2n_k-1}, GSx_{2n_k-1}), \\ &d(Tx_{2n_k}, GSx_{2n_k-1}) + d(Fx_{2n_k}, SSx_{2n_k-1})\} \\ &\leq H(Y_{2n_k}, Y_{2n_k+1}) + \alpha d(y_{2n_k}, SSx_{2n_k-1}) + \beta \max\{d(y_{2n_k}, y_{2n_k+1}), \\ &d(SSx_{2n_k-1}, GSx_{2n_k-1})\} + \gamma \max\{d(y_{2n_k}, y_{2n_k+1}) + d(SSx_{2n_k-1}, GSx_{2n_k-1}), \\ &d(y_{2n_k}, GSx_{2n_k-1}) + d(y_{2n_k+1}, SSx_{2n_k-1})\} \\ &\leq H(Y_{2n_k}, Y_{2n_k+1}) + \alpha [d(GSx_{2n_k-1}, SSx_{2n_k-1}) + R_k] + \beta [d(SSx_{2n_k-1}, GSx_{2n_k-1}) + R_k] \\ &\quad + \gamma [d(Sx_{2n_k+1}, SSx_{2n_k-1}) + R_k]. \end{aligned}$$

Hence

$$\begin{aligned} R_k &\leq H(Y_{2n_k}, Y_{2n_k+1}) + (\alpha + \beta) [d(SSx_{2n_k-1}, GSx_{2n_k-1}) + R_k] \\ &\quad + \gamma [d(Sx_{2n_k+1}, SSx_{2n_k-1}) + R_k] \end{aligned} \tag{3.7}$$

Since $Y_n \rightarrow Y$, $Sx_{2n_k+1} \rightarrow z$ and as S is continuous, it follows that the real sequence $\{R_k\}$ is bounded. Thus $\limsup_{n \rightarrow \infty} R_k$ exists. Since G and S are weakly compatible and $Sx_{2n_k+1} = y_{2n_k+1} \in K$, $y_{2n_k+1} \in GK \cap K$ and $\lim_{n \rightarrow \infty} d(Sx_{2n_k+1}, y_{2n_k}) = 0$. Using Definition 2.3, one gets

$$\limsup_{n \rightarrow \infty} L_k \leq \limsup_{n \rightarrow \infty} R_k \tag{3.8}$$

$$\limsup_{n \rightarrow \infty} d(SSx_{2n_k+1}, Sx_{2n_k-1}) \leq \limsup_{n \rightarrow \infty} R_k \tag{3.9}$$

Denoting $\limsup_{n \rightarrow \infty} R_k$ as R and taking the upper limit in (3.7), by (3.8) and (3.9), we get $R \leq (\alpha + \beta + \gamma)R$. Hence $R = 0$. Then from (3.9), $d(Sz, z) = 0$. Thus $Sz = z$. Similarly, using the foregoing arguments, we obtain $Tz = z$. In order to show that $z \in Gz$, consider

$$\begin{aligned} q d(y_{2n_k+1}, Gz) &\leq H(Fx_{2n_k}, Gz) \\ &\leq \alpha d(Tx_{2n_k}, Sz) + \beta \max\{d(Tx_{2n_k}, Fx_{2n_k}), d(Sz, Gz)\} \end{aligned}$$

$$\begin{aligned}
 & +\gamma \max\{d(Tx_{2n_k}, Fx_{2n_k}) + d(Sz, Gz), d(Tx_{2n_k}, Gz) + d(Sz, Fx_{2n_k})\}, \\
 & \leq \beta d(z, Gz) + \gamma d(z, Gz),
 \end{aligned}$$

implying thereby $z \in Gz$. Next consider

$$\begin{aligned}
 q d(Fz, z) & \leq H(Fz, Gz) \leq \alpha d(Tz, Sz) + \beta \max\{d(Tz, Fz), d(Sz, Gz)\} \\
 & + \gamma \max\{d(Tz, Fz) + d(Sz, Gz), d(Tz, Gz) + d(Sz, Fz)\}, \\
 & \leq \beta d(z, Fz) + \gamma d(z, Fz),
 \end{aligned}$$

implying thereby $z \in Fz$. Thus we obtained $z = Tz = Sz \in Fz \cap Gz$. This completes the proof. ■

Remark 3.2. By setting $F = G, S = T = I_K$ and $\beta = 0 = \gamma$ in Theorem 3.1, we deduce a theorem due to Assad and Kirk [4].

Remark 3.3. By setting $S = T = I_K$ in Theorem 3.1, we deduce a theorem due to Ćirić and Ume [8].

Remark 3.4. By setting $S = T = I_K$ in Theorem 3.1 we deduce a partially generalized form of a result due to Ćirić and Ume [7].

Remark 3.5. By setting $F = G$ and $S = T = I_K$ in Theorem 3.1, we deduce a theorem due to Rhoades [25].

Theorem 3.6. [25] Let (X, d) be a complete metrically convex metric space and K a nonempty closed subset of X . Let $F : K \rightarrow CB(X)$ satisfying the contraction condition:

$$H(Fx, Fy) \leq \alpha d(x, y) + \beta \max\{d(x, Fx), d(y, Fy)\} + \gamma \{d(x, Fy) + d(y, Fx)\} \tag{3.10}$$

for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ such that

$$\left(\frac{1 + \alpha + \gamma}{1 - \beta - \gamma}\right) \left(\frac{\alpha + \beta + \gamma}{1 - \gamma}\right) < 1.$$

If $Fx \subseteq K$ for each $x \in \delta K$, then there exists $z \in K$ such that $z \in Fz$.

Proof. From the condition of Rhoades [25], one can write

$$\begin{aligned}
 \left(\frac{1 + \alpha + \gamma}{1 - \beta - \gamma}\right) \left(\frac{\alpha + \beta + \gamma}{1 - \gamma}\right) & = \frac{\alpha + \beta + \gamma + \alpha^2 + \alpha\beta + \alpha\gamma + \alpha\gamma + \beta\gamma + \gamma^2}{1 - \gamma - \beta + \beta\gamma - \gamma + \gamma^2} \\
 & = \frac{\alpha + 2\beta + 3\gamma + \alpha\gamma + \alpha^2 + \alpha\beta + \alpha\gamma + \beta\gamma + \gamma^2 - \beta - 2\gamma}{1 + \gamma(\beta + \gamma) - \beta - 2\gamma} \\
 & = \frac{\alpha + 2\beta + 3\gamma + \alpha\gamma + \alpha(\alpha + \beta + \gamma) + (\beta + \gamma)\gamma - \beta - 2\gamma}{1 + \gamma(\beta + \gamma) - \beta - 2\gamma}
 \end{aligned}$$

Rhoades condition implies that $\alpha + 2\beta + 3\gamma + \alpha\gamma + \alpha(\alpha + \beta + \gamma) < 1$. Thus, we can say that Theorem 3.1 is a generalization of the theorem of Rhoades. Notice that condition (3.1) for $\beta = 0$ and $\gamma = 0$ reduces to $\alpha < 1$. ■

Remark 3.7. Theorem 3.1 is a generalization and extension of Theorem 2.1 due to Ćirić and Ume [8] and generalization of theorem due to Assad and Kirk [4], Assad [5], Khan [23], Itoh [17] and Rhoades [25]. The present technique of proof gives a simplification of the corresponding proofs given by Ćirić and Ume [8], Itoh [17], Khan [23] and Rhoades [25].

In the next theorem we utilize the closedness of TK and SK (or FK and GK) to relax the continuity requirements besides minimizing the commutativity requirements to merely coincidence points.

Theorem 3.8. Let (X, d) be a complete metrically convex metric space and K a nonempty closed subset of X . Let $F, G : K \rightarrow CB(X)$ and $S, T : K \rightarrow X$ satisfying (3.1), (ix) and (x). Suppose that

(xvii) TK and SK (or FK and GK) are closed subspaces of X . Then

(xviii) the pair (G, S) as well as (F, T) has a point of coincidence.

Moreover, (F, T) has a common fixed point if T is quasi-coincidentally commuting and occasionally coincidentally idempotent w.r.t F , whereas (G, S) has a common fixed point provided S is quasi-coincidentally commuting and occasionally coincidentally idempotent w.r.t G .

Proof. Proceeding as in Theorem 3.1, we assume that there exists a subsequence $\{Tx_{2n_k}\}$ which is contained in P_\circ and TK as well as SK are closed subspaces of X . Since $\{Tx_{2n_k}\}$ is Cauchy in TK , it converges to a point $u \in TK$. Let $v \in T^{-1}u$, then $Tv = u$. Since $\{Sx_{2n_k+1}\}$ is a subsequence of Cauchy sequence, $\{Sx_{2n_k+1}\}$ converges to u as well. Using (3.1.1) we can write

$$\begin{aligned} q d(Fv, Tx_{2n_k}) &\leq H(Fv, Gx_{2n_k-1}) \\ &\leq \alpha d(Tv, Sx_{2n_k-1}) + \beta \max\{d(Sx_{2n_k-1}, Gx_{2n_k-1}), d(Tv, Fv)\} \\ &\quad + \gamma \max\{d(Sx_{2n_k-1}, Gx_{2n_k-1}) + d(Tv, Fv), d(Tv, Gx_{2n_k-1}) + d(Sx_{2n_k-1}, Fv)\}, \end{aligned}$$

which on letting $k \rightarrow \infty$, reduces to

$$\begin{aligned} q d(Fv, u) &\leq \beta \max\{d(u, Fv), 0\} + \gamma \max\{d(Fv, u), d(Fv, u)\}, \\ &\leq (\beta + \gamma) d(u, Fv), \end{aligned}$$

yielding thereby $u \in Fv$, which implies that $u = Tv \in Fv$ as Fv is closed.

Since Cauchy sequence $\{Tx_{2n_k}\}$ converges to $u \in K$ and $u \in Fv, u \in FK \cap K \subseteq SK$, there exists $w \in K$ such that $Sw = u$. Again using (3.1) we get

$$\begin{aligned} q d(Sw, Gw) &= q d(Tv, Gw) \leq H(Fv, Gw) \\ &\leq \alpha d(Tv, Sw) + \beta \max\{d(Tv, Fv), d(Sw, Gw)\} \\ &\quad + \gamma \max\{d(Tv, Fv) + d(Sw, Gw), d(Tv, Gw) + d(Sw, Fv)\}, \\ &\leq (\alpha + \beta + \gamma) d(Sw, Gw), \end{aligned}$$

implying thereby $Sw \in Gw$, that is w is a coincidence point of (S, G) .

In case FK and GK are closed subspaces, then $u \in FK \cap K \subseteq SK$ or $GK \cap K \subseteq TK$. The analogous arguments establish the desired conclusions. If we assume that there exists a subsequence $\{Sx_{2n_k+1}\}$ contained in Q_o with TK as well as SK are closed subspaces of X , then noting that $\{Sx_{2n_k+1}\}$ is Cauchy in SK , the foregoing arguments establish that $Tz \in Fz$ and $Sw \in Gw$.

Since v is a coincidence point of (F, T) therefore using quasi-coincidentally commuting property of (F, T) and occasionally coincidentally idempotent property of T w.r.t F we have

$$Tv \in Fv \text{ and } u = Tv \Rightarrow Tu = TTv = Tv = u.$$

Therefore $u = Tu = TTv \in TFv \subset FTv = Fu$, which shows that u is the common fixed point of (F, T) . Similarly using the quasi-coincidentally commuting property of (G, S) and occasionally coincidentally idempotent property of S w.r.t G we can show that (G, S) has a common fixed point as well. This completes the proof. ■

Finally, we prove a theorem when closedness of K is replaced by compactness of K .

Theorem 3.9. Let (X, d) be a complete metrically convex metric space and K a nonempty compact subset of X . Let $F, G : K \rightarrow CB(X)$ and $T : K \rightarrow X$ satisfying:

(xix) $\delta K \subseteq TK, (FK \cup GK) \cap K \subseteq TK$

(xx) $Tx \in \delta K \Rightarrow Fx \cup Gx \subseteq K$, with

$$\begin{aligned} H(Fx, Gy) &< M(x, y), \text{ when } M(x, y) > 0, \text{ for all } x, y \in K \text{ where} \\ M(x, y) &= \alpha d(Tx, Ty) + \beta \max\{d(Tx, Fx), d(Ty, Gy)\} \\ &\quad + \gamma \max\{d(Tx, Fx) + d(Ty, Gy), d(Tx, Gy) + d(Ty, Fx)\} \end{aligned} \quad (3.11)$$

for all $x, y \in X$ with $x \neq y$, where α, β and γ are non-negative reals with $\alpha + 2\beta + 3\gamma + \alpha\gamma \leq q \leq 1$. If T is weakly compatible with F and G then F, G and T have a common coincidence. Moreover F, G and T have a common fixed point provided Tz remains fixed under T .

Proof. We assert that $M(x, y) = 0$ for some $x, y \in K$. Otherwise $M(x, y) \neq 0$, for any $x, y \in K$ implies that

$$f(x, y) = \frac{H(Fx, Gy)}{M(x, y)}$$

is continuous and satisfies $f(x, y) < 1$ for all $(x, y) \in K \times K$. Since $K \times K$ is compact, there exists $(u, v) \in K \times K$ such that $f(x, y) \leq f(u, v) = c < 1$ for $x, y \in K$, which in turn yields $H(Fx, Gy) \leq cM(x, y)$ for $x, y \in K$ and $0 < c < 1$. Therefore using (3.11) we obtain

$$\max \left\{ \frac{\alpha + \beta + \gamma}{q - \gamma}, \frac{\alpha + \gamma}{q - \beta - \gamma} \right\} < 1.$$

Now by Theorem 3.1 (with restriction $S = T$), we get $Tz \in Fz \cap Gz$ for some $z \in K$ and one concludes $M(z, z) = 0$, contradicting the facts that $M(x, y) > 0$. Therefore $M(x, y) = 0$ for some $x, y \in K$ which implies $Tx \in Fx$ and $Tx = Ty \in Gy$. If $M(x, x) = 0$ then $Tx \in Gx$ and if $M(x, x) \neq 0$ then using (3.11) we infer that $d(Tx, Gx) \leq 0$ yielding thereby $Tx \in Gx$. Similarly in either of the cases $M(y, y) = 0$ or $M(y, y) > 0$ we conclude that $Ty \in Fy$. Thus we have shown that F, G and T have a common point of coincidence.

By setting $T = I_K$ in Theorem 3.9, we deduce the following corollary for a pair of maps. ■

Corollary 3.10. Let (X, d) be a complete metrically convex metric space and K be a nonempty compact subset of X . Let $F, G : K \rightarrow CB(X)$ satisfying:

(xxi) $x \in \delta K \Rightarrow Fx, Gx \subset K$,

$$H(Fx, Gy) < \alpha d(x, y) + \beta \max\{d(x, Fx), d(y, Gy)\} + \gamma \max\{d(x, Fx) + d(y, Gy), d(x, Gy) + d(y, Fx)\}$$

for all $x, y \in X$ with $x \neq y$, where α, β and γ are non-negative reals with $\alpha + 2\beta + 3\gamma + \alpha\gamma \leq q \leq 1$. Then $z \in Fz$ and $z \in Gz$.

Remark 3.11. Similar corollaries can also be derived for a single map by setting $F = G$ and $T = I_K$ in Theorem 3.9.

4. An illustrative example

Now, we furnish an example which demonstrates the validity of the hypotheses of Theorem 3.1 besides establishing the genuineness of our extension over some other relevant results of the existing literature.

Example 4.1. Consider $X = \Re$ be the set of reals equipped with natural distance and $K = \left\{ \frac{-1}{3} \right\} \cup [0, 1]$. Define $F, G : K \rightarrow CB(X)$ and $S, T : K \rightarrow X$ by

$$Fx = \begin{cases} \left[\frac{-x}{2}, 0 \right], & \text{if } 0 \leq x < 1 \\ \{0\}, & \text{if } x \in \left\{ \frac{-1}{3}, 1 \right\} \end{cases} \quad Tx = \begin{cases} \frac{-x}{3}, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = \left\{ \frac{-1}{3}, 1 \right\} \end{cases}$$

$$Gx = \begin{cases} \left[\frac{-x}{4}, 0 \right], & \text{if } 0 \leq x < 1 \\ \{0\}, & \text{if } x \in \left\{ \frac{-1}{3}, 1 \right\} \end{cases} \quad \text{and } Sx = \begin{cases} \frac{-x}{4}, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = \left\{ \frac{-1}{3}, 1 \right\} \end{cases}$$

Since δK (the boundary of K) = $\left\{ \frac{-1}{3}, 0, 1 \right\}$. Clearly $\delta K \subset TK \cap SK$.

Further $FK \cap K = \left(\frac{-1}{2}, 0 \right] \cap K \subset TK$ and $GK \cap K = \left(\frac{-1}{4}, 0 \right] \cap K \subset SK$.

Also

$$T\left(\frac{-1}{3}\right) = 1 \in \delta K \Rightarrow F\left(\frac{-1}{3}\right) = \{0\} \subseteq K, \quad T(0) = 0 \in \delta K \Rightarrow F(0) = \{0\} \subseteq K$$

$$T(1) = 1 \in \delta K \Rightarrow F(1) = \{0\} \subseteq K, \quad S\left(\frac{-1}{3}\right) = 1 \in \delta K \Rightarrow G\left(\frac{-1}{3}\right) = \{0\} \subseteq K,$$

$$S(0) = 0 \in \delta K \Rightarrow G(0) = \{0\} \subseteq K \text{ and } S(1) = 1 \in \delta K \Rightarrow G(1) = \{0\} \subseteq K.$$

Similarly, by a routine calculation we can show that (F, G) is a generalized (S, T) contraction pair of K into $CB(X)$ with $\alpha = \frac{1}{3}$ and $\beta = \gamma = \frac{1}{7}$. and also '0' is a point of coincidence as $T0 = F0$ and $S0 = G0$. Thus F, G, S and T have a point of common coincidence, whereas both the pairs (F, T) and (G, S) are weakly compatible pairs and the pair (T, S) is continuous. Thus all the conditions of the Theorem 3.1 are satisfied and note that '0' is the unique common fixed point of F, G, S and T .

5. An application

Now we state and prove the following theorem in Banach spaces as an application of our main Theorem 3.1 for single valued nonself mappings. While proving this result the notion of demiclosed and starshaped subset are utilized.

Definition 5.1. ([9]) Let K be a non empty subset of a normed linear space X . A mapping $T : K \rightarrow X$ is said to be demiclosed if $\{x_n\} \subset K, x_n \rightarrow x$ and $Tx_n \rightarrow y \in X$ then $Tx = y$.

Definition 5.2. ([15]) Let K be a non empty subset of a normed linear space X is said to be starshaped if there exists atleast one point $p \in K$ such that for each $x \in K$ and $t \in [0, 1], (1 - t)p + tx \in K$.

Definition 5.3. ([18]) A pair (F, T) of nonself mappings defined on a nonempty subset K of a set X is said to be weakly compatible if $Fx = Tx$ for some $x \in K$ with $Fx, Tx \in K \Rightarrow FTx = TFx$.

Theorem 5.4. Let K a nonempty weakly compact starshaped subset of a Banach space X . Let (F, G) a generalized (S, T) nonexpansive mappings of K into X satisfying:

$$(xxii) \delta K \subseteq SK \cap TK, FK \cap K \subseteq SK, GK \cap K \subseteq TK,$$

$$(xxiii) Tx \in \delta K \Rightarrow Fx \in K, Sx \in \delta K \Rightarrow Gx \in K,$$

(xxiv) (F, T) and (G, S) are weakly compatibles pairs, with

$$d(Fx, Gy) < \alpha d(Tx, Ty) + \beta \max\{d(Tx, Fx), d(Ty, Gy)\} + \gamma \max\{d(Tx, Fx) + d(Ty, Gy), d(Tx, Gy) + d(Ty, Fx)\} \quad (5.1)$$

for all $x, y \in X$ with $x \neq y$, where α, β and γ are non-negative reals with $\alpha + 2\beta + 3\gamma + \alpha\gamma \leq 1$. Moreover, if $(I - F)$ and $(I - G)$ are demiclosed, then the mappings F, G, S and T have a common fixed point $z \in K$ provided S and T are continuous.

Proof. Let us choose $p \in K$ such that $(1 - t)p + tx \in K$ for all $x \in K$ and all $t \in (0, 1)$. Let us take $k_n = 1 - \frac{1}{n}, n = 2, 3, 4, \dots$ and define $F_n, G_n : K \rightarrow X$ by $F_n(x) = (1 - k_n)p + k_n(Fx)$ and $G_n(x) = (1 - k_n)p + k_n(Gx)$ for all $x \in K$. It is easy to verify that (F_n, G_n) is a generalized (S, T) contractive mapping of K into X and F_n, G_n satisfy conditions (xxii), (xxiii) and (xxiv). Since weak topology is Hausdorff and K is weakly compact we can conclude that K is weakly closed and hence strongly closed. Now by Theorem 3.1 (for single valued setting) for each $n \geq 2, F_n, G_n, S$ and T have a unique common fixed point, say $z_n \in K$. Now it follows that z_n has a weakly convergent subsequence and we can assume that z_n itself converges to $z \in K$ weakly. Since weakly convergent sequences are norm bounded, therefore we can find a constant $M > 0$ such that $\|z_n\| < M$ for all $n \geq 2$. Now for every $n \geq 2$, we can have

$$\begin{aligned} \|(I - F)z_n\| &= \|z_n - [k_n^{-1}\{Fz_n - (1 - k_n)p\}]\| \\ &= \|z_n - [k_n^{-1}Fz_n - k_n^{-1}(1 - k_n)p]\| \\ &= \|(1 - k_n^{-1})z_n - (1 - k_n^{-1})p\| \\ &= \|(1 - k_n^{-1})(z_n - p)\| \\ &\leq |(1 - k_n^{-1})|(\|z_n\| + \|p\|) \\ &\leq |(1 - k_n^{-1})|(M + \|p\|) \end{aligned}$$

since $\|z_n\| \leq M$. Since $k_n^{-1} \rightarrow 1$ as $n \rightarrow \infty$ we can have $(I - F)z_n \rightarrow 0 \in K$. Also $z_n \rightarrow z \in K$ and $(I - F)$ is demiclosed, it follows that $(I - F)z = 0$ giving thereby $Fz = z$. Similarly using the demiclosedness of $(I - G)$ we can show that $Gz = z$. Since for each $n \geq 2, Tz_n = z_n$ and $Sz_n = z_n$, therefore taking the limit as $n \rightarrow \infty$, we obtain $Tz = Sz = z$ as T and S are continuous. This completes the proof. ■

By setting $S = I_K$ in Theorem 5.1 we deduce the following corollary for three maps.

Corollary 5.5. Let K be a nonempty weakly compact starshaped subset of Banach space X . Let (F, G) be a generalized T non-expansive mappings of K into X satisfying:

$$(xxv) \quad Tx \in \delta K \Rightarrow Fx, Gx \in K,$$

$$(xxvi) \quad \delta K \subseteq TK, (FK \cup GK) \cap K \subseteq TK$$

(xxvii) F and T are weakly compatible pairs.

Moreover, if $(I - F)$ is demiclosed, then the mappings F , G and T have a common fixed point $z \in K$ provided T is continuous.

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