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p-Compactness and C-p.Compactness

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Abstract

The aim of this paper is to introduce new types of compactness called p-Compactness and C-p.compactness of topological spaces using the concept of p-open sets. Also we characterized p-Compactness and C-p.compactness using nets and filters and studied some of its properties. We identified spaces in which p-Compactness is equivalent to compactness.

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1. Introduction

In [3] Gierz introduced prime elements in a continuous lattice. Later many authors [4], [6], [8] studied about the prime elements in many different contexts. Motivated by those definitions of prime element; in [12] we introduced a new collection of open sets called p-open sets in topological spaces and studied some of the basic notions in topology using the concept of p-open sets. Also in [12] we apply the concept of p-open set to generalised closed sets which was introduced by Norman Levine [7] in 1970. Generalised closed sets introduced by Norman Levine plays a significant role in topology and many research work has been done in this area. In [12] we studied some new separation axioms using p-open sets and generalised p-closed sets and identified the spaces in which all those separation axioms are equivalent.

In present work, we defined new types of compactness called p-Compactness and C-p.compactness using the concept of p-open sets. We introduce p-continuity, p-homeomorphism, p-topological property etc and proved that p-Compactness is a p-topological property. Also we characterize p-Compactness using nets and filters and characterize C-p.compactness using regular p-open sets and also using nets and filters. We introduce regular p-open sets and regular p-closed sets for this purpose. We obtained the condition for the equivalence of p-Compactness and compactness of any arbitrary topological space.

2. Preliminaries

Definition 2.1. [12] Let (X, T) be any arbitrary topological space. The open sets in T forms a complete lattice with smallest element 0 and largest element 1; where $0 = \phi$ and $1 = X$. We define an open set G$\neq$ 1 in T to be *prime open set* if $H \cap K \subseteq G \Rightarrow H \subseteq G$ or $K \subseteq G$; where H, K are open sets in T such that $H \cap K \neq \phi$. Clearly 0 and 1 are prime in T. Prime open sets are denoted by *p-open* sets. Complements of p-open sets are called *p-closed sets*.

Definition 2.2. [8] Any space is said to be a sober space if the only prime elements in it is of the form $X - \overline{\{x\}}$ where $x \in X$.

Theorem 2.3. [12] Let (X, T) be a hausdorff space and $x \in X$ then the only p-open sets are $X - \{x\}$.

Definition 2.4. [12] Let (X, T) be a topological space and let $A \subseteq X$, then the *p-closure* of A with respect to T is defined as the minimal p-closed super set of A in X and is denoted as p-$cl(A)$.

Proposition 2.5. [12] Let (X, T) be a topological space, then for every p-open set there always exists a unique p-closed set containing A.

Definition 2.6. [12] Let (X, T) be a topological space and let $A \subseteq X$, then the *p-interior* of A with respect to T is defined as the maximal p-open subset of A in X and is denoted as p-int(A).

Proposition 2.7. **[12]** Let (X, T) be a topological space, then for every p-closed set there always exists a unique p-open set contained in A.

Theorem 2.8. **[12]** Let (X, T) be a topological space and $Y \subseteq X$. U p-open in X implies $U \cap Y$ p-open in Y.

3. p-Continuity and p-Compactness In Topological Spaces

For any arbitrary topological space compactness and discreteness occurs simultaneously only when the underlying set is finite. Introduction of p-Compactness gives a variety of infinite p-Compact discrete topological spaces; in fact in $\Sigma(X)$ the largest element is always p-Compact where $\Sigma(X)$ denotes the collection of all topologies on X.

Definition 3.1. Let $(X, T), (Y, T^{'})$ be two topological spaces and let $f : (X, T) \to (Y, T^{'})$ be a mapping between this two topological spaces. f is called *p-continuous* if the inverse image of p-open sets in $T^{'}$ are p-open in T.

Remark 3.2. Let $(X, T), (Y, T^{'})$ be two topological spaces and let $f : (X, T) \to (Y, T^{'})$ be a continuous mapping between this two topological spaces. Then f need not be p-continuous; for example let $X = \{a, b, c\}$ and let $f : (X, D) \to (Y, T)$ be the identity mapping such that D is the discrete topology on X and $T = \{X, \phi, \{a\}, \{a, b\}\}$ then f is continuous but not p-continuous.

Remark 3.3. Let $(X, T), (Y, T^{'})$ be two topological spaces and let $f : (X, T) \to (Y, T^{'})$ be a p-continuous mapping between this two topological spaces. Then f need not be continuous; for example Let $X_1 = R$ with cofinite topology and $X_2 = R$ with discrete topology. Identity function $f : X_1 \to X_2$ is p-continuous but not continuous.

Definition 3.4. Let $(X, T), (Y, T^{'})$ be two topological spaces and $f : (X, T) \to (Y, T^{'})$ be a mapping. f is said to be a *p-homeomorphism* if f is one-one, onto and both f, f^{-1} are p-continuous.

Theorem 3.5. Homeomorphism implies p-homeomorphism.

Proof. Let $(X, T), (Y, T^{'})$ be two topological spaces. Let $f : (X, T) \to (Y, T^{'})$ be a homeomorphism between this two topological spaces. Let G be a prime open set in T' to prove that $f^{-1}(G)$ is prime in T. Always $f^{-1}(G)$ is open in T since f is continuous. On contradiction assume that $f^{-1}(G)$ is not prime in T then there exists two open subsets $f^{-1}(H), f^{-1}(K)$ of X such that $f^{-1}(G) \subset f^{-1}(H)$, $f^{-1}(G) \subset f^{-1}(K)$ and $f^{-1}(H) \cap f^{-1}(K) \subseteq f^{-1}(G)$
$\Rightarrow f(f^{-1}(G)) \subset f(f^{-1}(H)), f(f^{-1}(G)) \subset f(f^{-1}(K))$ and $f(f^{-1}(H) \cap f^{-1}(K)) \subseteq f(f^{-1}(G))$
$\Rightarrow G \subset H, G \subset K$ and $H \cap K \subset G$, where H and K are open subsets of Y since f is an open continuous one one mapping. Hence G is not prime in T which is not possible and

therefore inverse image of p-open sets are p-open that is f is p-continuous. Similarly we can prove that f^{-1} is also continuous. Hence f is a p-homeomorphism. ■

Remark 3.6. Converse of above theorem is not true and example in Remark 3.3 illustrates it.

Definition 3.7. A property P is said to be a *p-topological property*, if whenever a space X has that property P then any space p-homeomorphic to that space also has the same property P.

Definition 3.8. Let (X, T) be a topological space and let $A \subset X$ then the collection $\{Pi : i \in I\}$ of prime open sets in T is said to be a *p-open cover* of A if $A \subseteq \bigcup_i P_i$.

Definition 3.9. Let (X, T) be a topological space, (X, T) is said to be *p-Compact* if every p-open cover has a finite sub cover.

Example 3.10. Any sober space is p-Compact.

Theorem 3.11. Any hausdorff space is p-Compact.

Proof. Proof is trivial since all hausdorff spaces are sober and any sober space happens to be p-Compact. ■

Theorem 3.12. Every compact space is p-Compact.

Proof. Since any p-open cover is an open cover, existence of finite sub cover for p-open cover obviously follows from the compactness of the space. ■

Remark 3.13. The converse of above theorem is not true; for example Let X be any infinite set with discrete topology. Then X is p-Compact but not compact.

Proposition 3.14. Let (X, T) be a topological space and $A \subset X$. Then A is p-Compact with respect to T if and only if it is p-Compact with respect to T_A; where T_A is the relative topology on A with respect to T.

Proof. For necessity we assume that A is p-Compact with respect to T to prove that A is p-Compact with respect to T_A. Let $\{G_i : i \in I\}$ be a p-open cover of A where each $G_i \in T_A$. Now by Theorem 2.8 each G_i is of the form $H_i \cap A$ such that H_i is p-open in X but by our assumption H_i has a finite sub collection which covers A; that is

$$A \subseteq \bigcup_i G_i \subseteq \bigcup_i (H_i \cap A) \subseteq \bigcup_i H_i$$

$$\Rightarrow A \subseteq H_{i_1} \cup H_{i_2} \cup \ldots\ldots \cup H_{i_n}$$

$$\Rightarrow A \subseteq (H_{i_1} \cup H_{i_2} \cup \ldots\ldots \cup H_{i_n}) \cap A = G_{i_1} \cup G_{i_2} \cup \ldots\ldots \cup G_{i_n}$$

Now in order to prove the sufficiency part let H_i be a p-open cover of A by p-open sets in T

Then $A \subseteq \bigcup_i H_i$

$\Rightarrow A \subseteq A \cap (\bigcup_i H_i) = \bigcup_i G_i$

where each $G_i = A \cap H_i$ and is p-open in T_A again by Theorem 2.8 which implies
$A \subseteq G_{i_1} \cup G_{i_2} ..... \cup G_{i_n}$
$\subseteq A \cap (H_{i_1} \cup H_{i_2} \cup ...... \cup H_{i_n})$
$\subseteq H_{i_1} \cup H_{i_2} \cup ...... \cup H_{i_n}$. Hence A is p-Compact with respect to T. ■

Theorem 3.15. Every p-closed subset of a p-Compact space is p-Compact.

Proof. Let Y be a p-closed subset of a p-Compact space X we have to prove that Y is p-Compact. Let G be a p-open covering of Y by sets p-open in X then $H = G \cup \{X - Y\}$ is a p-open covering of X and since X is p-Compact H has a finite sub cover in particular G has a finite sub cover which covers Y. Hence Y is p-Compact. ■

Theorem 3.16. The p-continuous image of a p-Compact space is p-Compact.

Proof. Let $f : X \to Y$ be p-continuous and let X be p-Compact. Let G be a p-open covering of set $f(X)$ by sets p-open in Y. The collection $\{f^{-1}(A)/A \in G\}$ is a collection of sets covering X; these sets are p-open in X because f is p-continuous. Hence finitely many of them say $f^{-1}(A_1)$, $f^{-1}(A_2)$, ...$f^{-1}(A_n)$ cover X. Then trivially the sets $A_1, A_2........A_n$ cover $f(X)$. ■

Theorem 3.17. p-Compactness is a p-topological property.

Proof. Let (X, T), $(Y, T^{'})$ be two topological spaces and $f : (X, T) \to (Y, T^{'})$ be a p-homeomorphism between them. Let X be p-Compact we have to prove that Y is also p-Compact. Since p-continuous image of a p-Compact space is p-Compact and f is onto; we obtain $f(X) = Y$ as a p-Compact space. Hence p-Compactness is a p-topological property. ■

4. Characterizations of p-Compactness

Theorem 4.1. The following statements are equivalent

1. X is p-Compact.

2. For every class $\{F_i\}$ of p-closed subsets of X, $\cap_i Fi = \phi$ implies $\{F_i\}$ contains a finite subclass $\{F_{i_1}, ...F_{i_m}\}$ with $\{F_{i_1} \cap .... \cap F_{i_m}\} = \phi$.

Proof. Case 1: To prove that (1) $\Rightarrow$ (2)
Suppose $\cap_i Fi = \phi$, then by De-Morgan's law $X = \phi^c = (\cap_i F_i)^c = \cup_i (F_i)^c$, so $\{(F_i)^c\}$ is a p-open cover of X, but p-Compactness of X implies existence of $(F_{i_1})^c$, ...$(F_{i_m})^c$ such that $X = (F_{i_1})^c \cup .... \cup (F_{i_m})^c$, then again by applying De-Morgans law

$\phi = X^c = \{F_{i_1} \cap .... \cap F_{i_m}\}$. Hence (1) $\Longrightarrow$ (2).

Case 2: To prove that (2) $\Longrightarrow$ (1)
Let $\{G_i\}$ be a p-open cover of X, that is $X = \cup_i G_i$. By De-Morgan's law $\phi = X^c = (\cup_i G_i)^c = \cap_i (G_i)^c$. Since each G_i is p-open, $\{G_i^c\}$ is a class of p-closed sets and has an empty intersection. Hence there exists $(G_{i_1})^c, (G_{i_2})^c, ...(G_{i_m})^c \in \{(G_i)^c\}$ such that $(G_{i_1})^c \cap (G_{i_2})^c \cap .... \cap (G_{i_m})^c = \phi$. Thus by De-Morgan's law $X = G_{i_1} \cup G_{i_2} \cup ..... \cup G_{i_m}$. Accordingly X is p-Compact and so (2) $\Rightarrow$ (1). ∎

Theorem 4.2. A topological space X is p-Compact if and only if every class $\{F_i\}$ of p-closed subsets of X which satisfies the finite intersection property has itself a non-empty intersection.

Proof. Using the preceding theorem, it suffices to show that the following statements are equivalent,

1. $F_{i_1} \cap .... \cap F_{i_m} \neq \phi; \forall i_1, ..., i_m \Rightarrow \cap_i F_i \neq \phi$.
2. $\cap_i F_i = \phi \Rightarrow \exists i_1, i_2 ....i_m$ such that $F_{i_1} \cap .... \cap F_{i_m} = \phi$.

But these statements are contra positives and therefore proof is trivial. ∎

Definition 4.3. Let (X, T) be a topological space and let $S : D \to X$ be a net. A point $x \in X$ is said to be a *p-cluster point/p-limit point* if for every p-open set U containing 'x' and $m \in D$, there exists an $n \in D$ such that $n \geq m$ and $S(n) \cap U \neq \phi$. Also S is said to *p-converges* to a point $x \in X$ if for every p-open set U containing 'x', there exists an $m \in D$ such that for all $n \geq m, n \in D$; $S(n) \cap U \neq \phi$.

Definition 4.4. Let (X, T) be a topological space and F be a filter on X. A point $x \in X$ is said to be a *p-cluster point/p-limit point* of F if every p-open set containing 'x' intersects every member of F. Also F is said to *p-converges* to a point $x \in X$ if every p-open set containing 'x' is a member of F.

Remark 4.5. Every limit point is a p-limit point but converse need not be true; for example let N the set of all natural numbers be the directed set and let (N, D) be the given discrete topological space. Then $I : N \to N$ be the identity mapping, that defines a net on N. Clearly any point of N is a p-limit point but not a limit point of the net I.

Theorem 4.6. Let (X, T) be a topological space then the following statements are equivalent

1. X is p-Compact.
2. Every net/filter in X has a p-cluster point.

Proof. Case 1: To prove that (1) $\Longrightarrow$ (2)
Let $S : D \to X$ be a net in X, we have to prove that S has a p-cluster point. On contradiction we assume that S has no p-cluster point. Then by definition of p-cluster point, for

each $x \in X$ there exist a p-open set N_x such that $x \in N_x$ and $m_x \in D$ such that for every $n \in D, n \geq m_x \Rightarrow S(n) \in X - N_x$. Let $X = \bigcup \{N_x / x \in X\}$. Since X is p-Compact there exists $x_1, x_2, \ldots, x_k$ such that $X = \bigcup \left\{N_{x_i} / i = 1, 2, \ldots, k\right\}$. Let the corresponding elements in D be $m_{x_1}, m_{x_2}, \ldots, m_{x_k}$. Since D is a directed set there exists $n \in D$ such that $n \geq m_{x_i}$ for $i = 1, 2, \ldots, k$. But then $S(n) \in \bigcap \left\{X - N_{x_i} / i = 1, 2, \ldots, k\right\} = \phi$; which is a contradiction. Hence our assumption is wrong and S has at least one p-cluster point in X. Then since S is arbitrary, (1) $\Rightarrow$ (2).

Case 2: To prove that (2) $\Rightarrow$ (1)
We assume that every net in X has a p-cluster point in X. To prove that X is p-Compact. Let $\mathbf{C}$ be a family of p-closed sets in X having finite intersection property. Let $\mathbf{D}$ be the family of all finite intersection of members of $\mathbf{C}$. We make $\mathbf{D}$ a directed set by defining for $D, E \in \mathbf{D}$; $D \geq E \Rightarrow D \subset E$; Clearly $D \neq \phi$. Now define a net $S : D \to X$ by $S(D) =$ any point in D. By our assumption S has a p-cluster point say $x \in X$. We claim that $x \in \bigcap \{C/C \in \mathbf{C}\}$. Then $\bigcap \{C/C \in \mathbf{C}\}$ is non-empty. Hence X is p-Compact. On contradiction we assume that $x \notin \bigcap \{C/C \in \mathbf{C}\}$ which implies there exists $C \in \mathbf{C}$ such that $x \notin C$. Then $X - C$ is a p-open set containing 'x' and $C \in \mathbf{D}$. Since 'x' is a p-cluster point, by definition there exists $D \in \mathbf{D}$ such that $D \geq C$ and $S(D) \in X - C$; which implies $X - C \subset X - D$ and $S(D) \in X - D$ which is a contradiction to our definition of net. Hence $x \in \bigcap \{C/C \in \mathbf{C}\}$ and X is p-Compact. ■

Theorem 4.7. A topological space is p-Compact iff every ultra filter/ultra net in it is p-convergent.

Proof. Necessary part is trivial by **Theorem 4.6.**, conversely let (X, T) be a topological space and assume that every ultra filter in it is p-convergent. To show that X is p-Compact it is enough to prove that every filter on X has a p-cluster point. Suppose F is a filter on X then there exists an ultra filter G containing F. By our assumption G p-converges to a point say 'x' on X which implies 'x' is a p-cluster point of G. So every p-open set containing 'x' meets every member of G and in particular every member of F since $F \subset G$. Hence 'x' is a p-cluster point of F. Thus every filter on X has a p-cluster point which implies p-Compactness of X. ■

Analyzing **Theorem 4.2**, **Theorem 4.6** and **Theorem 4.7** we characterize p-Compactness of any arbitrary topological space according to the following result.

Theorem 4.8. Let (X, T) be any topological space. The following statements are equivalent

1. X is p-Compact.

2. Each family of p-closed subsets of X with the finite intersection property has non-empty intersection.

3. Every net/filter in X has a p-cluster point.

4. Every ultra filter in X p-converges.

Theorem 4.9. Let (X, T) be a topological space. Then p-Compactness of X implies compactness of X if and only if there exists no net S in X such that S has a p-cluster point but has no cluster points.

Proof. Necessary Condition. Assume that X p-Compact implies X is compact. Compactness of X implies that every net in X has a cluster point which in turn implies that there exists no net S in X such that S has a p-cluster point but has no cluster points in X.

Sufficiency Condition. Assume that there exists no net S in X such that S has a p-cluster point but has no cluster points and X is p-compact. To prove that X is compact. p-Compactness of X implies every net in X has a p-cluster point and by our assumption above every net has a cluster point hence X is compact. ■

5. C-p. compactness and Its Characterizations

Definition 5.1. Let (X, T) be a topological space. A set $U \subseteq X$ is said to be *regular prime open (regular p-open)* if p-int(p-$cl(U)) = U$ and U is called regular p-closed set if it is the complement of a regular p-open set.

Example 5.2. Consider $X = \{a, b, c, d\}$, $T = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ then (X, T) is a topological space and $\{a\}$ is a regular p-open set.

Remark 5.3. In a discrete space all subsets are regular open but all of them are not regular p-open hence regular open sets need not be regular p-open.

Theorem 5.4. p-interior of a set $A \subset X$ is regular p-open, if it is p-closed.

Proof. In order to prove that p-int(A) is regular p-open it is enough to prove that p-int(A)=p-int(p-$cl(p - int(A))$). Since A is p-closed p-$cl(p - int(A))$ is A itself. Hence the result. ■

Corollary 5.5. For any set $U \subseteq X$, p-int(p-$cl(U)$) is a regular p-open set always.

Proof. Proof is trivial by last theorem since p-$cl(U)$ is always p-closed. ■

Proposition 5.6. Let (X, T) be a topological space, then for every p-open set there always exists a unique regular p-open set containing A.

Proof. Proof is trivial by proposition 2.5 and proposition 2.7. ■

Definition 5.7. A space X is said to be *C-p.compact* if for each p-closed $A \subset X$ and each p-open cover $\{U_\alpha / \alpha \in \Delta\}$ of A, there exists a finite sub collection $\{U_{\alpha_i} / i = 1, 2..n\}$

such that $A \subseteq \bigcup p\text{-}cl(U_{\alpha_i})/i = 1, 2, \ldots, n$.

Remark 5.8. Compactness implies p-compactness implies C-p.compactness.

Remark 5.9. C-p.compactness need not implies C-compactness; in order to prove this we consider an example due to S.Sakai [10]. Let $X = \{(a, b) : n, m \in N\}$ such that $a = 1/n, b = 1/m$ or $a = 1/n, b = 0$ or $a = 0, b = 0$ where N stands for the set of all positive integers. Also let $\{N_i/i \in N\}$ be the partition of N to infinitely many disjoint classes. Define subsets of X as follows:
$H_{ik} = \{\{(1/i, 0)\} \cup \{(1/i, 1/m)/m \geq k\}\} \cup \{(1/n, 1/m)/n \geq k, m \in N_i\}$,
$L_k = \{(0, 0)\} \cup \{(1/n, 1/m)/n > k, m \notin N_i, 1 \leq i \leq k\}$.
Let T be the topology on X generated by $\{\{(1/n, 1/m)\}/n, m \in N\} \cup \{H_{ik}/i, k \in N\} \cup \{L_k/k \in N\}$. Then (X, T) is a c-compact hausdorff space. Now let $Y = \{y_0, y_1, y_2 ....\}$ be a one point compactification of a countable discrete space $\{y_1, y_2 ....\}$. Consider $X \times Y$ S.Sakai in [10] proved that this is a non c-compact space, but this space is hausdorff since both X and Y are hausdorff and hence it happens to be a p-Compact and thus C-p.compact space.

Theorem 5.10. A space (X, T) is C-p.compact if and only if for each p-closed $A \subseteq X$ and regular p-open cover (Cover in which all elements are regular p-open) $\{U_\alpha/\alpha \in \Delta\}$ there exists a finite sub collection $\{U_{\alpha_i} : i = 1, 2...n\}$ such that $A \subseteq \bigcup p\text{-}cl(U_{\alpha_i})/i = 1, 2.....n$.

Proof. Necessary Part: If X is C-p.compact all p-open covers satisfies the given condition we have to prove that each regular p-open cover $\{U_\alpha/\alpha \in \Delta\}$ satisfies the given condition but since each U_α is regular p-open; p-int(p-$cl(U_\alpha)) = U_\alpha$ for every $\alpha \in \Delta$ that is each U_α is p-open and since each p-open cover satisfies the given condition $\{U_\alpha/\alpha \in \Delta\}$ also satisfies the given condition. Hence necessary part is trivial.

Sufficiency Part: Assume that for each p-closed $A \subseteq X$ and regular p-open cover $\{V_\alpha/\alpha \in \Delta\}$ there exists a finite sub collection $\{V_{\alpha_i} : i = 1, 2...n\}$ such that $A \subseteq \bigcup p\text{-}cl(V_{\alpha_i})/i = 1, 2.....n$. Let $\{U_\alpha/\alpha \in \Delta\}$ be any p-open cover of A. Then $\{p - int(p - cl(U_\alpha))/\alpha \in \Delta\}$ is a regular p-open cover of A. So there exists a finite sub collection $\{p - int(p - cl(U_{\alpha_i})) : i = 1, 2....n\}$ such that $A \subseteq \bigcup p\text{-}cl(p - int(p\text{-}cl(U_{\alpha_i})))/i = 1, 2.....n$. But p-$cl(p\text{-}int(p\text{-}cl(U_{\alpha_i}))) = p\text{-}cl(U_{\alpha_i})$ hence $A \subseteq \bigcup p\text{-}cl(U_{\alpha_i})$; $i = 1, 2.....n$; that is A is C-p.compact. ∎

Definition 5.11. Let (X, T) be a topological space and let $X \neq \phi$, $\mathbf{F}$ a filter on X. Then $\mathbf{F}$ is said to be *p-closure convergent* to a point $a \in X$ if for every p-open set V in X containing 'a'; p-$cl(V) \in \mathbf{F}$ and $a \in X$ is said to be a *p-closure limit point* of $\mathbf{F}$ if every p-open set V containing 'a' is such that p-$cl(V) \cap A \neq \phi$ for every $A \in \mathbf{F}$.

Remark 5.12. p-convergence implies p-closure convergence but converse is not true.

Similarly for p-closure limit point.

Example 5.13. Let $(R, T = \{R, \phi, Q, Q'\})$ be a topological space and a filter be defined on it as $F = U(x)/x \in Q'$ that is principal filter generated by 'x'. Then any rational point is a p-closure convergent point but not a p-convergent point.

Definition 5.14. Let (X, T) be a topological space and let $S : D \to X$ be a net on X, then S *p-closure converges* to a point $x \in X$ if for every p-open set U containing 'x' there exists $m \in D$ such that for every $n \geq m$; $S(n) \cap p\text{-}cl(U) \neq \phi$ and 'x' is said to be a *p-closure limit point* of S if for every p-open set U containing 'x' and $m \in D$; there exists $n \in D$ such that $n \geq m$ and $S(n) \cap p\text{-}cl(U) \neq \phi$.

Now we characterize C-p.compactness as follows

Theorem 5.15. Let (X, T) be a topological space then the following conditions are equivalent:

1. X is C-p.compact.
2. For each p-closed $A \subseteq X$ and regular p-open cover (Cover in which all elements are regular p-open) $\{U_\alpha/\alpha \in \Delta\}$ there exists a finite sub collection $\{U_{\alpha_i} : i = 1, 2...n\}$ such that $A \subseteq \bigcup p\text{-}cl(U_{\alpha_i})/i = 1, 2.....n$.
3. For each p-closed $A \subseteq X$ and each collection of non-empty regular p-closed sets $\{F_\alpha/\alpha \in \Delta\}$ such that $(\bigcap F_\alpha) \cap A = \phi$ there exists a finite sub collection $\{F_{\alpha_i}/i = 1, 2......n\}$ such that $(\bigcap p\text{-}int(F_{\alpha_i})) \cap A = \phi$; $i = 1, 2.......n$.
4. For each p-closed $A \subseteq X$ and each filter with base $F = \{A_\alpha/\alpha \in \Delta\}$ in A, there exists an $a \in A$ such that the filter has 'a' as a p-closure cluster point.

Proof.

- (1) $\Leftrightarrow$ (2) already proved.
- To prove (1) $\Rightarrow$ (3).

  Assume that X is C-p.compact. Let A be a p-closed set and each collection of non-empty regular p-closed sets $\{F_\alpha/\alpha \in \Delta\}$ such that $(\bigcap F_\alpha) \cap A = \phi$. To prove that $(\bigcap p\text{-}int(F_{\alpha_i})) \cap A = \phi$ for i=1,2...n. Consider $\mathbf{U} = \{U_\alpha/U_\alpha = X - F_\alpha\}$ then each U_α is a regular p-open set and $A \subseteq \bigcup (X - F_\alpha)/\alpha \in \Delta$. Since X is C-p.compact $A \subseteq \bigcup p\text{-}cl(U_{\alpha i})/i = 1, 2.....n$. Now consider $\bigcap p\text{-}int(F_{\alpha_i}) = X - \bigcup p\text{-}cl(U_{\alpha_i}) \subseteq X - A$ where $i = 1, 2.......n$ and hence $(\bigcap p\text{-}int(F_{\alpha_i})) \cap A = \phi$.

- To prove (3) $\Rightarrow$ (2)

  Let $\{U_\alpha/\alpha \in \Delta\}$ be a regular p-open cover of A, then $A \subseteq \bigcup U_\alpha$ implies $A \cap (\bigcap(X - U_\alpha)) = \phi$ but by condition (3)
  $(\bigcap p\text{-}int(X - U_{\alpha_i})) \cap A = \phi; i = 1, 2...n$ which implies
  $A \subseteq X - (\bigcap p\text{-}int(X - U_{\alpha_i})); i = 1, 2.....n$
  $=\bigcup(X - p\text{-}int(X - U_{\alpha_i})); i = 1, 2..........n$
  $=\bigcup p\text{-}cl(U_{\alpha_i}); i = 1, 2......n$. Hence condition (2) is satisfied.

- To prove condition (1) $\Rightarrow$ (4)

  Suppose there exists a filter **F** on A with filter base $\mathbf{F} = \{A_\alpha/\alpha \in \Delta\}$ in A such that **F** has no p-closure limit point. Then for each $a \in A$ there exists a p-open set $U(a)$ and some $A_{\alpha(a)} \in \mathbf{F}$ such that $A_{\alpha(a)} \cap p\text{-}cl(U(a)) = \phi$. But for every $a \in A$, there exists a p-open set $U(a)$ satisfying the above condition hence $\{U(a)/a \in A\}$ forms a p-open cover of A and C-p.compactness of X implies $A \subseteq \bigcup p\text{-}cl(U(a_i)); i = 1, 2............n$. For each $U(a_i); i = 1, 2......n$ there exists $A_{\alpha(a_i)} \in \mathbf{F}$ which implies $\bigcap A_{\alpha(a_i)}; i = 1, 2.......n$ belongs to the filter and there exists A_{α_0} such that $A_{\alpha_0} \subseteq \bigcap A_{\alpha(a_i)}$; $i = 1, 2.......n$; but $A_{\alpha_0} \neq \phi$ and $A_{\alpha_0} \subseteq A \subseteq \bigcup p\text{-}cl(U(a_i)); i = 1, 2.....n$ that in turn implies $A_{\alpha(a_j)} \cap p\text{-}cl(U(a_j)) \neq \phi$ which is a contradiction. Hence the result.

- To prove (4) $\Rightarrow$ (3)

  In order to prove condition (3) it is enough to prove its contra positive statement. Suppose there exists a p-closed set $A \subseteq X$ and a collection of regular p-closed sets $\{F_\alpha/\alpha \in \Delta\}$ such that each finite sub collection $\{F_{\alpha_i}/i = 1, 2.......n\}$ has the property that $(\bigcap(p\text{-}int(F_{\alpha_i}))) \cap A \neq \phi : i = 1, 2....n$ but $(\bigcap F_\alpha) \cap A = \phi$. Then the sets $\{(p - int(F_\alpha)) \cap A/\alpha \in \Delta\}$ together with all finite intersections of the form $(\bigcap(p\text{-}int(F_{\alpha_i}))) \cap A : i = 1, 2.......n$ will form a filter base for a filter **F** on A ; then by our assumption **F** has a p-closure limit point say $a \in A$. Then for any p-open set $U(a)$ containing 'a' and each $p\text{-}int(F_\alpha)$; $p\text{-}cl(U(a)) \cap (p\text{-}int(F_\alpha) \cap A) \neq \phi$. But $F_\alpha \cap A \neq \phi$ for every $\alpha \in \Delta$ and $(\bigcap F_\alpha) \cap A = \phi$ together implies there exists $\alpha_0 \in \Delta$ such that $a \notin F_{\alpha_0}$. Therefore $a \notin p\text{-}int(F_{\alpha_0})$ implies $a \in X - F_{\alpha_0} \subset p\text{-}cl(X - F_{\alpha_0}) \subset X - p\text{-}int(F_{\alpha_0})$ which implies $p\text{-}cl(X - F_{\alpha_0}) \cap p\text{-}int(F_{\alpha_0}) = \phi$ which implies 'a' is not a p-closure limit point of F which is a contradiction. Hence $(\bigcap F_\alpha) \cap A \neq \phi$. Hence proving the contra positive statement of (3). ■

Remark 5.16. Let X be a topological space and 'x' in X is said to be a p-closure cluster point of the filter F on X if and only if $'x'$ is a p-closure cluster point for the net associated with the filter and conversely. Hence the above equivalent condition for C-p.compactness holds for any net on any p-closed subset of the corresponding topological space.

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