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Thermodynamics of the black hole in the Grand statistical ensemble

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Abstract

Using the classical thermodynamics, we calculated in this work, the thermodynamic potential of black holes in the presence of a magnetic field. In addition, to this work, we have also investigated the thermodynamics properties of the black hole in the case of the Grand potential interaction as the particle number of the black hole is at all times in incessant fluctuations. It has been shown that for an homogeneous gravitational field and for an external magnetic field in the space-time of the black hole that $C_{\Omega,\Phi,M,\mu}$, $C_{J,Q,B,N}$ lead to a strange Mayer's relation since it is negative.

AMS subject classification:
Keywords:

1. Introduction

In the cosmology theory and the general theory of relativity, the black hole is defined as a region of space-time showing a strong gravitational pull such as no particle or radiation can escape from it. In the framework of this theory, it is checked that a gravitationally collapsing star of mass M will shrink, in short time measured by an observer on the surface about a radius of magnitude, equal to $2GM/c^2$. The boundary of the region from which no escape is possible is called the event horizon. The theoretical foundations of the subject of black hole dynamics were laid by Bekenstein [1, 2], Hawking [3–5], Bardeen et al. [6], and the systematic complete treatment is given by Straumann [7]. The statistical theory of internal (unobservable) configurations and microcanonical ensemble in the black hole framework has been developed and discussed in considerable detail by Hawking [8]. As a consequence to the theory of black hole dynamics, it is possible to develop black hole dynamic potentials and the fundamental relations between the black hole parameters M, J, Q, κ, Ω, and Φ. In this work, we have been developed formula for the specific heats after consideration the magnetic field near the boundary of the black hole: this is done in the framework of the grand canonical statistical ensemble, because the particles number N changes under particles transmutation to energy and transmutation energy to particles with respect to chemical potential μ. Then, we have organized this paper as follows: the section 2 is devoted to derive the thermodynamic potentials for the black hole with its magnetic field. In this section, we have defined for each process, the adequate potential: the internal energy E, the free energy K, the enthaly H and the Gibbs energy F. Considering each potential as a state function, we have developped the corresponding Maxwell equations: ten equations (9, 11–13, 15, 16–20) corresponding to the internal energy E, six (30-35) to the free energy K, ten (44-53) to the enthalpy H and ten (64-73) to the Gibbs energy F (at all 36 Maxwell equations). A conclusion and a discussion are reported in the section 3.

2. Description of the black hole with the grand thermodynamics potentials

2.1. Internal energy E

Since the particles number is not constant (some of them transform to energy), the adequate theory, to investigate the thermodynamic properties of the black hole, is to use the grand thermodynamic potential. The interaction of the black hole with the surrounding fields and matter was widely studied [12-14]. We focused here on the interaction with a constant magnetic field B. The first law of thermodynamics, in differential form, writes as

$$dE = d(Mc^2) = \frac{\kappa c^2}{8\pi G} dA + \Omega dJ + \Phi dQ + MdB + \mu dN \tag{1}$$

where B is a possible external magnetic field surrounding the black hole and μ is the chemical potential (conjugate moment of the particles number N). This formula is very similar to the first law of thermodynamics, with intensive moments (the temperature and the pressure T and P) and extensive moments (the entropy and the volume S and V).

$$dE = TdS - PdV \tag{2}$$

suggesting that one should regard the first equation as a perfect differential from which we can list the following relations

$$\left(\frac{\partial E}{\partial A}\right)_{J,Q,B,N} = \frac{\kappa c^2}{8\pi G} \tag{3}$$

$$\left(\frac{\partial E}{\partial J}\right)_{A,Q,B,N} = \Omega \tag{4}$$

$$\left(\frac{\partial E}{\partial Q}\right)_{A,J,B,N} = \Phi \tag{5}$$

$$\left(\frac{\partial E}{\partial B}\right)_{A,J,Q,N} = M \tag{6}$$

and

$$\left(\frac{\partial E}{\partial N}\right)_{A,J,Q,B} = \mu \tag{7}$$

Because dE is a perfect differential, we have from the equation (1) the following relation

$$\left(\frac{\partial}{\partial J}\left(\frac{\partial E}{\partial A}\right)_{J,Q,B,N}\right)_{A,Q,B,N} = \left(\frac{\partial}{\partial A}\left(\frac{\partial E}{\partial J}\right)_{A,Q,B,N}\right)_{J,Q,B,N} \tag{8}$$

and nine other similar relations deduced from the equation (1). At all, from equation (1) we get ten relations similar to equation (8). We can transform these ten relations, with the help of the equations (3-7) to the following ten equations (9, 11-19) (known as the Maxwell relations in classical thermodynamics): by using eqs (3) and (4) we have from equation (8)

$$\left(\frac{\partial \kappa}{\partial J}\right)_{A,Q,B,N} = \frac{8\pi G}{c^2}\left(\frac{\partial \Omega}{\partial A}\right)_{J,Q,B,N} \tag{9}$$

Similarly we get the second relation (among ten equations) from equation (1)

$$\left(\frac{\partial}{\partial Q}\left(\frac{\partial E}{\partial A}\right)_{J,Q,B,N}\right)_{A,J,B,N} = \left(\frac{\partial}{\partial A}\left(\frac{\partial E}{\partial Q}\right)_{A,J,B,N}\right)_{J,Q,B,N} \tag{10}$$

from which, with the help of eqs (3) and (5), we get the second equation (among ten equations)

$$\left(\frac{\partial \kappa}{\partial Q}\right)_{A,J,B,N} = \frac{8\pi G}{c^2}\left(\frac{\partial \Phi}{\partial A}\right)_{J,Q,B,N} \tag{11}$$

and so on for the remaining eight equations that we list as follow:

$$\left(\frac{\partial\kappa}{\partial B}\right)_{A,Q,J,N} = \frac{8\pi G}{c^2}\left(\frac{\partial M}{\partial A}\right)_{J,Q,B,N} \tag{12}$$

$$\left(\frac{\partial\kappa}{\partial N}\right)_{A,B,J,Q} = \frac{8\pi G}{c^2}\left(\frac{\partial \mu}{\partial A}\right)_{J,Q,B,N} \tag{13}$$

$$\left(\frac{\partial\Omega}{\partial Q}\right)_{A,J,B,N} = \left(\frac{\partial \Phi}{\partial J}\right)_{A,Q,B,N} \tag{14}$$

$$\left(\frac{\partial\Omega}{\partial B}\right)_{A,J,Q,N} = \left(\frac{\partial M}{\partial J}\right)_{A,B,Q,N} \tag{15}$$

$$\left(\frac{\partial\Omega}{\partial N}\right)_{A,J,B,Q} = \left(\frac{\partial \mu}{\partial J}\right)_{A,Q,B,N} \tag{16}$$

$$\left(\frac{\partial\Phi}{\partial B}\right)_{A,J,Q,N} = \left(\frac{\partial M}{\partial Q}\right)_{A,B,J,N} \tag{17}$$

$$\left(\frac{\partial\Phi}{\partial N}\right)_{A,J,B,Q} = \left(\frac{\partial \mu}{\partial Q}\right)_{A,B,J,N} \tag{18}$$

$$\left(\frac{\partial\mu}{\partial B}\right)_{A,J,Q,N} = \left(\frac{\partial M}{\partial N}\right)_{A,Q,J,B} \tag{19}$$

2.2. Free Helmholtz energy K

For any process in which the surface gravity κ is constant and by the zeroth law of the black hole dynamics (which states that for a stationary axi-symmetric black hole in a space-time which is asymptotically flat, it is possible to give a general definition of the surface gravity κ such that κ is constant on the horizon), we can write from eq. (1),

$$\frac{\kappa c^2}{G8\pi}dA = d\left(\frac{\kappa c^2}{G8\pi}\right) \tag{20}$$

Therefore,

$$d\left[E - \frac{\kappa c^2}{G8\pi}A\right] = \Omega dJ + \Phi dQ + MdB + \mu dN \tag{21}$$

or

$$dK = \Omega dJ + \Phi dQ + MdB + \mu dN \tag{22}$$

It is clear that $K = E - \frac{\kappa c^2}{G8\pi}A$ is the additional available energy. As we shall see, the black holes have a negative specific heat. Thus K must obey the condition $K < \frac{1}{4}E$ in order that the black hole be in a state of stable thermal equilibrium. Now from the equation (22) we have,

$$\left(\frac{\partial K}{\partial J}\right)_{Q,B,N} = \Omega \tag{23}$$

$$\left(\frac{\partial K}{\partial Q}\right)_{J,B,N} = \Phi \tag{24}$$

$$\left(\frac{\partial K}{\partial B}\right)_{J,Q,N} = M \tag{25}$$

$$\left(\frac{\partial K}{\partial N}\right)_{J,Q,B} = \mu \tag{26}$$

Since dK is a perfect differential, and using last four equations, we have the following six relations

$$\left(\frac{\partial \Omega}{\partial Q}\right)_{J,B,N} = \left(\frac{\partial \Phi}{\partial J}\right)_{Q,B,N} \tag{27}$$

$$\left(\frac{\partial \Omega}{\partial B}\right)_{J,Q,N} = \left(\frac{\partial M}{\partial J}\right)_{Q,B,N} \tag{28}$$

$$\left(\frac{\partial \Omega}{\partial N}\right)_{J,Q,B} = \left(\frac{\partial \mu}{\partial J}\right)_{Q,B,N} \tag{29}$$

$$\left(\frac{\partial \Phi}{\partial B}\right)_{J,Q,N} = \left(\frac{\partial M}{\partial Q}\right)_{J,B,N} \tag{30}$$

$$\left(\frac{\partial \Phi}{\partial N}\right)_{J,Q,B} = \left(\frac{\partial \mu}{\partial Q}\right)_{J,B,N} \tag{31}$$

$$\left(\frac{\partial \mu}{\partial B}\right)_{Q,J,N} = \left(\frac{\partial M}{\partial N}\right)_{Q,B,J} \tag{32}$$

2.3. The enthalpy energy H

As we have said in the introduction, the black hole have an incessant fluctuations in the particles number. This means that the black hole devors the matter and emit the thermal radiation at all time. In other words, the black hole has an enthalpy H given by

$$H = E - \Omega J + \Phi Q + MB + \mu N \tag{33}$$

Therefore

$$\begin{aligned} dH = dE - \Omega dJ + \Phi dQ + MdB - Jd\Omega \\ + Qd\Phi + BdM + \mu dN + Nd\mu \end{aligned} \tag{34}$$

or using eq (1) we have

$$dH = \frac{\kappa c^2}{8\pi G} dA - Jd\Omega - Qd\Phi - BdM - Nd\mu \tag{35}$$

Since dH is a perfect differential we can write immediately

$$\left(\frac{\partial H}{\partial A}\right)_{\Omega,\Phi,M,\mu} = \frac{\kappa c^2}{8\pi G} \tag{36}$$

$$\left(\frac{\partial H}{\partial \Omega}\right)_{A,\Phi,M,\mu} = -J \tag{37}$$

$$\left(\frac{\partial H}{\partial \Phi}\right)_{A,\Omega,M,\mu} = -Q \tag{38}$$

$$\left(\frac{\partial H}{\partial M}\right)_{A,\Omega,\Phi,\mu} = -B \tag{39}$$

$$\left(\frac{\partial H}{\partial \mu}\right)_{A,\Omega,\Phi,M} = -N \tag{40}$$

Because dH is a perfect differential, and by using the last five relations from what we can deduce the following ten Maxwell equations

$$\left(\frac{\partial \kappa}{\partial \Omega}\right)_{A,\Phi,M,\mu} = \frac{8\pi G}{c^2}\left(\frac{\partial J}{\partial A}\right)_{\Omega,\Phi,M,\mu} \tag{41}$$

$$\left(\frac{\partial \kappa}{\partial \Phi}\right)_{A,\Omega,M,\mu} = \frac{8\pi G}{c^2}\left(\frac{\partial Q}{\partial A}\right)_{\Omega,\Phi,M,\mu} \tag{42}$$

$$\left(\frac{\partial \kappa}{\partial M}\right)_{A,\Omega,\Phi,\mu} = \frac{8\pi G}{c^2}\left(\frac{\partial B}{\partial A}\right)_{\Omega,\Phi,M,\mu} \tag{43}$$

$$\left(\frac{\partial \kappa}{\partial \mu}\right)_{A,\Omega,\Phi,M} = \frac{8\pi G}{c^2}\left(\frac{\partial N}{\partial A}\right)_{\Omega,\Phi,M,\mu} \tag{44}$$

$$\left(\frac{\partial J}{\partial \Phi}\right)_{A,\Omega,M,\mu} = \left(\frac{\partial Q}{\partial \Omega}\right)_{A,\Phi,M,\mu} \tag{45}$$

$$\left(\frac{\partial J}{\partial M}\right)_{A,\Omega,\Phi,\mu} = \left(\frac{\partial B}{\partial \Omega}\right)_{A,\Phi,M,\mu} \tag{46}$$

$$\left(\frac{\partial J}{\partial N}\right)_{A,\Omega,\Phi,M} = \left(\frac{\partial \mu}{\partial \Omega}\right)_{A,\Phi,M,N} \tag{47}$$

$$\left(\frac{\partial \Omega}{\partial N}\right)_{A,\Phi,M,\mu} = \left(\frac{\partial \mu}{\partial J}\right)_{A,\Omega,\Phi,M} \tag{48}$$

$$\left(\frac{\partial \Omega}{\partial B}\right)_{A,\Phi,M,\mu} = \left(\frac{\partial M}{\partial J}\right)_{A,\Omega,\Phi,\mu} \tag{49}$$

$$\left(\frac{\partial \mu}{\partial B}\right)_{A,\Omega,\Phi,M} = \left(\frac{\partial M}{\partial N}\right)_{A,\Omega,\Phi,\mu} \tag{50}$$

2.4. The free enthalpy energy F (Gibbs energy)

We can't consider eqs (43-45) as a fundamental relations, because we haven't any relation between (the charge Q, the magnetic field B, the chemical potential μ) with the angular frequency Ω, like angular momentum J under normal situation does not relate to (Φ, M, N), unless someone proves in the future, relations between these quantities which were till now unknown because the unobservable nature of the internal configurations of the black hole and the quantum nature of singularity of the black holes. By using the equation (32) with the constancy of the quantities: the potential Φ of the area of the event horizon A, the angular frequency Ω, the surface gravity κ, the magnetic moment M and the chemical potential μ, we can write

$$d\left[H - \frac{\kappa c^2}{8\pi G}A\right] = 0 \tag{51}$$

We define an auxiliary free energy F as

$$F = H - \frac{\kappa c^2}{8\pi G}A \tag{52}$$

therefore

$$dF = 0 \tag{53}$$

$$dH = dE - \Omega dJ - \Phi dQ - MdB - \mu dN + Jd\Omega + Qd\Phi + BdM - Nd\mu \tag{54}$$

Substituting the expression of dE in the last equation we find

$$dF = \frac{\kappa c^2}{8\pi G}dA - Jd\Omega - Qd\Phi - BdM - Nd\mu \tag{55}$$

hence we get

$$\left(\frac{\partial F}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu} = \frac{Ac^2}{8\pi G} \tag{56}$$

$$\left(\frac{\partial F}{\partial \Omega}\right)_{A,\Phi,M,\mu} = -J \tag{57}$$

$$\left(\frac{\partial F}{\partial \Phi}\right)_{A,\Omega,M,\mu} = -Q \tag{58}$$

$$\left(\frac{\partial F}{\partial M}\right)_{A,\Omega,\Phi,\mu} = -B \tag{59}$$

$$\left(\frac{\partial F}{\partial \mu}\right)_{A,\Omega,\Phi,M} = -N \tag{60}$$

Because dF is a perfect differential, and after using the last five equations, we get the following ten relations

$$\left(\frac{\partial A}{\partial \Omega}\right)_{\kappa,\Phi,M,\mu} = \frac{8\pi G}{c^2}\left(\frac{\partial J}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu} \tag{61}$$

$$\left(\frac{\partial A}{\partial \Phi}\right)_{\kappa,\Omega,M,\mu} = \frac{8\pi G}{c^2}\left(\frac{\partial Q}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu} \tag{62}$$

$$\left(\frac{\partial A}{\partial M}\right)_{\kappa,\Omega,\Phi,\mu} = \frac{8\pi G}{c^2}\left(\frac{\partial B}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu} \tag{63}$$

$$\left(\frac{\partial A}{\partial \mu}\right)_{\kappa,\Omega,\Phi,M} = \frac{8\pi G}{c^2}\left(\frac{\partial N}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu} \tag{64}$$

$$\left(\frac{\partial J}{\partial \Phi}\right)_{\kappa,\Omega,M,\mu} = \left(\frac{\partial Q}{\partial \Omega}\right)_{\kappa,\Phi,M,\mu} \tag{65}$$

$$\left(\frac{\partial J}{\partial M}\right)_{\kappa,\Omega,\Phi,\mu} = \left(\frac{\partial B}{\partial \Omega}\right)_{\kappa,\Phi,M,\mu} \tag{66}$$

$$\left(\frac{\partial J}{\partial \mu}\right)_{\kappa,\Omega,\Phi,M} = \left(\frac{\partial N}{\partial \Omega}\right)_{\kappa,\Phi,M,\mu} \tag{67}$$

$$\left(\frac{\partial Q}{\partial M}\right)_{\kappa,\Omega,\Phi,\mu} = \left(\frac{\partial B}{\partial \Phi}\right)_{\kappa,\Omega,M,\mu} \tag{68}$$

$$\left(\frac{\partial Q}{\partial \mu}\right)_{\kappa,\Omega,\Phi,M} = \left(\frac{\partial N}{\partial \Phi}\right)_{\kappa,\Omega,M,\mu} \tag{69}$$

$$\left(\frac{\partial B}{\partial \mu}\right)_{\kappa,\Omega,\Phi,M} = \left(\frac{\partial N}{\partial M}\right)_{\kappa,\Phi,\Omega,\mu} \tag{70}$$

The fundamental black hole dynamic relations are given by eqs (9) to (19) and (27) to, (32) and (61-70). Comparing eq. (3) and eq. (36); we find an auxiliary relation

$$\left(\frac{\partial E}{\partial A}\right)_{J,Q,B,N} = \left(\frac{\partial H}{\partial A}\right)_{\Omega,\Phi,M,\mu}. \tag{71}$$

To deduce an equation involving $(\kappa.c2/(8\pi.G))dA$ term, we use the differential of the area of the event horizon A as the following

$$dA = \left(\frac{\partial A}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu} d\kappa + \left(\frac{\partial A}{\partial \Omega}\right)_{\kappa,\Phi,M,\mu} d\Omega + \left(\frac{\partial A}{\partial \Phi}\right)_{\kappa,\Omega,M,\mu} d\Phi + \left(\frac{\partial A}{\partial M}\right)_{\kappa,\Omega,\Phi,\mu} dM + \left(\frac{\partial A}{\partial \mu}\right)_{\kappa,\Omega,\Phi,M} d\mu \tag{72}$$

we have

$$dA \geq 0 \tag{73}$$

or equivalently

$$\frac{c^2 dA}{8\pi G} = \frac{c^2}{8\pi G}\left(\frac{\partial A}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu} d\kappa + \frac{c^2}{8\pi G}\left(\frac{\partial A}{\partial \Omega}\right)_{\kappa,\Phi,M,\mu} d\Omega$$
$$+ \frac{c^2}{8\pi G}\left(\frac{\partial A}{\partial \Phi}\right)_{\kappa,\Omega,M,\mu} d\Phi + \tag{74}$$
$$\frac{c^2}{8\pi G}\left(\frac{\partial A}{\partial M}\right)_{\kappa,\Omega,\Phi,\mu} dM + \frac{c^2}{8\pi G}\left(\frac{\partial A}{\partial \mu}\right)_{k,\Omega,\Phi,M} d\mu \tag{75}$$

Now we define the specific heat of the black hole at constant Ω, Φ, M and μ as

$$C_{\Omega,\Phi,M,\mu} = \frac{c^2}{8\pi G}\left(\frac{\partial A}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu} d\kappa \tag{76}$$

and using eqs (61) and (62) we find

$$\frac{c^2}{8\pi G} dA = C_{\Omega,\Phi,M,\mu} + \kappa \left[\left(\frac{\partial J}{\partial \kappa}\right)_{\Phi,M,\mu} d\Omega + \left(\frac{\partial Q}{\partial \kappa}\right)_{\Omega,M,\mu} d\Phi \right.$$
$$\left. + \left(\frac{\partial B}{\partial \kappa}\right)_{\Omega,\Phi,\mu} dM + \left(\frac{\partial N}{\partial \kappa}\right)_{\Omega,\Phi,M} d\mu \right] \tag{77}$$

Similarly, we can write

$$\frac{c^2 dA}{8\pi G} = C_{\Omega,\Phi,M,\mu} + \kappa \left[\frac{c^2}{8\pi G}\left(\frac{\partial A}{\partial \kappa}\right)_{J,Q,B,N} d\kappa + \frac{c^2}{8\pi G}\left(\frac{\partial A}{\partial \Omega}\right)_{\kappa,Q,B,N} d\Omega \right]$$
$$+\kappa \left[\frac{c^2}{8\pi G}\left(\frac{\partial A}{\partial \Phi}\right)_{\kappa,J,B,N} d\Phi + \frac{c^2}{8\pi G}\left(\frac{\partial A}{\partial M}\right)_{\kappa,J,Q,N} dM \right.$$
$$\left. + \frac{c^2}{8\pi G}\left(\frac{\partial A}{\partial \mu}\right)_{\kappa,J,Q,B} d\mu \right] \tag{78}$$

We define the specific heat at constant J, Q, B and N as

$$C_{J,Q,B,N} = \frac{c^2}{8\pi G}\left(\frac{\partial A}{\partial \kappa}\right)_{J,Q,B,N} d\kappa. \tag{79}$$

About the fundamental relations of the additional available energy, we have,

$$\frac{c^2}{8\pi G}dA = C_{J,Q,B,N} + \kappa\left[\left(\frac{\partial J}{\partial\kappa}\right)_{A,Q,B,N} d\Omega + \left(\frac{\partial Q}{\partial\kappa}\right)_{A,J,B,N} d\Phi + d\mu\right] + \kappa\left[\left(\frac{\partial B}{\partial\kappa}\right)_{A,J,Q,N} dM + \left(\frac{\partial N}{\partial\kappa}\right)\right] \tag{80}$$

The specific heats $C_{\Omega,\Phi,M,\mu}$ and $C_{J,Q,B,N}$ are negatives. The energy $E = mc^2$ of the black hole is in principle the sum of the additional available energy (or the first free energy function as we have called it) K and energy that is not available for work $(\kappa c^2/(8\pi G))dA$. Then K is the energy available for work in time-reversible processes (white holes) observing constancy of surface gravity κ. As it is known, the area of the event horizon A always tends to increase. Then the bound energy of the black hole always increases. From this, it results that, the additional available energy K for work, decreases. This decreasing in K makes the black holes to approach to a stable thermal equilibrium state.

$$dK = dE - d\left(\frac{\kappa c^2 A}{8\pi G}\right) = dE - \frac{\kappa c^2}{8\pi G}dA - \frac{Ac^2}{8\pi G}d\kappa \tag{81}$$

Using eq. (1), we have

$$dK = -\frac{Ac^2}{8\pi G}d\kappa + \Omega dJ + \Phi dQ + MdB + \mu dN \tag{82}$$

The surface gravity is constant $d\kappa = 0$ and we have eq. (22)

$$dA = \Omega dJ + \Phi dQ + MdB + \mu dN \tag{83}$$

that is to say

$$K_2 - K_1 = \int_1^2 \Omega dJ \int_1^2 \Phi dQ + \int_1^2 MdB + \int_1^2 \mu dN \tag{84}$$

that is, the variation of the extra available energy of a black hole. During a time-reversible process, the measurement of the constancy of the surface gravity equals to the work performed on the black hole. In other words, all the work in such a process is done at the cost of the additional energy available from the black hole. So a black hole that undergoes a reversible isothermal process in time does a work to the detriment of its extra available energy. Clearly we are faced with a hypothetical situation unless the black hole is stationary (or at least quasi-stationary) and axisymmetric throughout any reversible process in time. Since the temperature of a thermodynamic system as a function of the energy E of the system and the entropy, are related by $T = dS/dE$, therefore, one can define the temperature of the black hole to be:

$$\frac{1}{T_H} = \left(\frac{\partial S_H}{\partial E}\right)_{J,Q,B,N} \tag{85}$$

The second law of black hole dynamics is equivalent as in any classical process, that the heat cannot travel from cooler system to a warm one: the area of the event horizon does not decreases nor does the black holes entropy. The second law of black hole mechanics can, however, be violated if the quantum effect is taken into account, namely that the area of the event horizon can be reduced via Hawking radiation. Note that the proof of this, depends on the cosmic censorship conjecture [9] which states that in the black holes universe enshroud singularities so that no information about singularities can reach an outside.

Now we can define the area of event horizon A of a black hole as the rate of change of the additional available energy with the surface gravity at a constant angular momentum J, a charge Q, a magnetic field B and particles number N, i.e,

$$A = -\frac{8\pi G}{c^2}\left(\frac{\partial K}{\partial \kappa}\right)_{J,Q,B,N} \tag{86}$$

and we have from eqs (21) and (22),

$$E = K + \frac{8\pi G}{c^2}A \tag{87}$$

$$E = K - \left(\kappa\frac{\partial K}{\partial \kappa}\right)_{J,Q,B,N} \tag{88}$$

At the end, we need an useful approximation that holds for the case of an homogeneous gravitational field in the space-time of the black holes. In this approximation the area A is a function of the surface gravity, the angular momentum, the charge, the magnetic field and the particles number:

$$\begin{aligned}
\frac{c^2}{8\pi G}\left(\frac{\partial A}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu} &= \frac{c^2}{8\pi G}\left(\frac{\partial A}{\partial \kappa}\right)_{J,Q,B,N} \\
&+\kappa\left[\left(\frac{\partial J}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu}\left(\frac{\partial \Omega}{\partial \kappa}\right)_{J,Q,B,N} d\Omega\right] \\
&+\kappa\left[\left(\frac{\partial Q}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu}\left(\frac{\partial \Phi}{\partial \kappa}\right)_{J,Q,B,N} d\Phi\right. \\
&+\left.\left(\frac{\partial B}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu}\left(\frac{\partial M}{\partial \kappa}\right)_{J,Q,B,N} dM\right] \\
&+\kappa\left[\left(\frac{\partial N}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu}\left(\frac{\partial \mu}{\partial \kappa}\right)_{J,Q,B,N} d\mu\right]
\end{aligned} \tag{89}$$

$$C_{\Omega,\Phi,M,\mu} - C_{J,Q,B,N} = \kappa \left[\left(\frac{\partial J}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu} \left(\frac{\partial \Omega}{\partial \kappa}\right)_{J,Q,B,N} + \left(\frac{\partial Q}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu} \left(\frac{\partial \Phi}{\partial \kappa}\right)_{J,Q,B,N} \right] + \left(\frac{\partial B}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu} \left(\frac{\partial M}{\partial \kappa}\right)_{J,Q,B,N} + \left(\frac{\partial N}{\partial \kappa}\right)_{\Omega,\Phi,M,\mu} \left(\frac{\partial \mu}{\partial \kappa}\right)_{J,Q,B,N} \tag{90}$$

3. Conclusion

In this work, we have considered the thermodynamics of the black hole in the Grand canonical ensemble with a possible a fluctuating particles number. We have established the relations between the black hole parameters in the presence of a constant external magnetic field present in the surrounding black hole. These calculations show that, in the presence of an the external magnetic field, the sign of the difference between the specific heats $C_{\Omega,\Phi,M,\mu}$ and $C_{J,Q,B,N}$ (Mayer's relation) changes. This feature leads to a new property of the black holes radiation.

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