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Some characterizations of Inversely Open and Inversely Closed maps via Ideals

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Abstract

We use the theory of local functions via ideals defined on a topological space to characterize inversely open and inversely closed maps between ideal spaces.

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1. Introduction and Preliminaries

In order to obtain some interesting and new characterization of open, closed and continuous mappings, Noorie and Bala in [7], used sets defined by fibers of the given map $f : X \to Y$. More specifically, any map $f : X \to Y$, apart from inducing the usual image and inverse image maps between power sets also induces another map $f^{\#} : \wp(X) \to \wp(X)$ given by $f^{\#}(E) = \{y \in Y : f^{-1}(y) \subseteq E\}$[1] for any subset E of X. The second author used this concept in [8] to obtain new new characterizations of inversely open and inversely closed maps. At the same time, theory of ideals has been widely used by various authors, for instance ([3], [4], [10],[11]) to give generalizations of the results in topology using local functions with respect to $*$-topology induced on the space via an ideal (*where an ideal $\mathcal{I}$ on a topological space (X, τ) is a collection of subsets of X which is closed under finite unions and closed downwards meaning that every subset of a member of $\mathcal{I}$ is in $\mathcal{I}$ i.e. if $B \in \mathcal{I}$ then $\wp(B) \subseteq \mathcal{I}$, where $\wp(B)$ means collection of all subsets of A*).

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In Section 2 of this paper, we use the theory of local functions with respect to an ideal and $f^{\#}$ operator to study inversely open and inversely closed maps with respect to an ideal and obtain new characterizations of these concepts [Theorem 2.3 and 2.10 below]. We study the important case of at most singleton fibers i.e. injective maps, to obtain new characterizations of inversely-$\mathcal{I}$-open maps in this special case [Corollary 2.5 and Theorem 2.4 below]. Inversely-$\mathcal{I}$-closed maps are also characterized [Theorem 2.11 below]. In Section 3, examples are given to illustrate the results and provide counterexamples which show the necessity of some of the conditions assumed in the results of Section 2.

We shall make use of the following definitions and results in this paper:

Definition 1.1. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be codense [2]if $\tau \cap \mathcal{I} = \emptyset$, $\mathcal{I}$-dense[2] if $A^* = X$ and $*$-dense in itself [3] if $A \subseteq A^*$.

Definition 1.2. [5] A mapping $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be pointwise-$\mathcal{I}$-continuous if the inverse image of every open set in Y is $\tau^*(\mathcal{I})$-open. Equivalently, $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is pointwise-$\mathcal{I}$-continuous if and only if $f : (X, \tau^*(\mathcal{I})) \to (Y, \sigma)$ is continuous.

Definition 1.3. [8] A map $f : (X, \tau) \to (Y, \sigma)$ is said to be inversely open if $int(f(A)) \subseteq f(int(A))$ for any subset A of X.

Definition 1.4. [8] A map $f : (X, \tau) \to (Y, \sigma)$ is said to be inversely closed if for any subset A of X, A is closed in X whenever $f(A)$ is closed in Y.

Theorem 1.5. [9] Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a bijective map, then f is pointwise-$\mathcal{I}$-continuous if and only if $f(A^*) \subseteq (f(A))^*$ for every subset A of X.

Lemma 1.6. [7] For any sets X and Y, let $f : X \to Y$ be any map and E be any subset of X. Then:

(a) $f^{\#}(E^C) = (f(E))^C$ and so $f^{\#}(E) = (f(E^C))^C$ and $f(E) = (f^{\#}(E^C))^C$.

(b) $f^{\#}(X) = Y$.

Lemma 1.7. [4] Let $(X, \tau, \mathcal{I})$ be an ideal space and A and B be subsets of X, then the following holds:

(a) $A^* = cl(A^*) \subseteq cl(A)$ and so A^* is a closed subset of $cl(A)$.

(b) A is $\tau^*(\mathcal{I})$-closed if and only if $A^* \subseteq A$.

(c) $A^*(\mathcal{I}, \tau) = A^*(\mathcal{I}, \tau^*(\mathcal{I}))$.

(d) $\tau \cap \mathcal{I} = \emptyset$ if and only if $X = X^*$.

2. Inversely Open And Inversely Closed Maps With Respect To An Ideal

We begin by introducing the following definitions and results:

Definition 2.1. A mapping $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be inversely open with respect to an ideal(written inversely-$\mathcal{I}$-open) if for any subset A of X, $int(f(A)) \subseteq f(int^*(A))$.

Remark 2.2. Every inversely open map is inversely-$\mathcal{I}$-open map, since $\emptyset \in \mathcal{I}$, but converse is not true.

The following theorem gives various characterizations of inversely-$\mathcal{I}$-open maps and is a generalization of Theorem 2.3 of [8].

Theorem 2.3. For any map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following conditions are equivalent:

(a) f is inversely-$\mathcal{I}$-open i.e. for each subset A of X, $int(f(A)) \subseteq f(int^*(A))$.

(b) $f^{\#}(cl^*(A)) \subseteq cl(f^{\#}(A))$.

(c) if V is an open subset of Y and $V \subseteq f(X)$, then $S = \bigcup_{y\in V}\{x \in X : x \in f^{-1}(y)\}$ is $\tau^*(\mathcal{I})$-open in X.

(d) for any subset A of X, A is $\tau^*(\mathcal{I})$-open in X, whenever $f(A)$ is open subset of Y.

(e) for any subset A of X, A is $\tau^*(\mathcal{I})$-closed in X, whenever $f^{\#}(A)$ is closed subset of Y.

Proof. The proof is similar to that of Theorem 2.3 of [8] and hence is omitted. ■

We now show that in case of injective map inversely-$\mathcal{I}$-open map is equivalent to $f^{\#}(A^*) \subseteq (f^{\#}(A))^*$ for every subset A of X. In fact we prove a slightly stronger result in the following:

Theorem 2.4. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ with $\mathcal{J} = f(\mathcal{I})$ be any map.Then the following holds.

(a) If $f^{\#}(A^*) \subseteq (f^{\#}(A))^*$ for every subset A of X and f is injective then for any subset A of X, A is τ^*-open in X, whenever $f(A)$ is σ^*-open in Y and hence in particular, f is inversely-$\mathcal{I}$-open.

(b) If f is inversely-$\mathcal{I}$-open then $f^{\#}(A^*) \subseteq (f^{\#}(A))^*$ for every subset A of X.

Proof.

(a) Let A be any subset of X such that $f(A)$ is σ^*-open in Y. So $(f(A))^C$ is σ^*-closed in Y and so $((f(A))^C)^* \subseteq (f(A))^C$ using Lemma 1.7(b). Therefore, by Lemma 1.6(a), we have $(f^{\#}(A^C))^* \subseteq f^{\#}(A^C)$. So, $f^{\#}((A^C)^*) \subseteq (f^{\#}(A^C))^* \subseteq f^{\#}(A^C)$. So $f^{-1}(f^{\#}((A^C)^*)) \subseteq f^{-1}(f^{\#}(A^C)) \subseteq A^C$. Since for injective map, $A \subseteq f^{-1}(f^{\#}(A))$ for every subset A of X, so $(A^C)^* \subseteq A^C$. Therefore, A^C is τ^*-closed in X and so A is τ^*-open in X. Hence f is inversely-$\mathcal{I}$-open, since $\sigma \subseteq \sigma^*$.

(b) Let $y \notin (f^{\#}(A))^*$. So there exists an open nhd. W of y in Y such that $W \cap f^{\#}(A) \in f(\mathcal{I})$. Since $(f(X))^C \subseteq f^{\#}(A)$ for any subset A of X, so $W \cap (f(X))^C \in f(\mathcal{I})$ implies that $W \cap (f(X))^C = f(I)$ for some $I \in \mathcal{I}$ and so $W \cap (f(X))^C = \emptyset$, so $W \subseteq f(X)$. Now consider the subset O of X given by $O = \{x \in f^{-1}(z) \cap I : z \in W \cap f(I)\} \bigcup \{x \in f^{-1}(z) - A : z \in W - f(I)\}$. So O contains atleast one point say x of the fiber $f^{-1}(y)$ and so by using the equivalence of (a) and (c) of Theorem 2.3, we have O is τ^*-open in X. Now it can be easily checked that $O \cap A = I$. Therefore, there exists τ^*- nhd. O of $x \in f^{-1}(y)$ such that $O \cap A \in \mathcal{I}$, so $x \notin A^*$ and so $y \notin f^{\#}(A^*)$. Hence $f^{\#}(A^*) \subseteq (f^{\#}(A))^*$. ■

Corollary 2.5. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be injective map. Then the following are equivalent:

(a) f is inversely-$\mathcal{I}$-open.

(b) $f^{\#}(A^*) \subseteq (f^{\#}(A))^*$ for every subset A of X.

(c) for any subset A of X, A is τ^*-open in X, whenever $f(A)$ is σ^*-open in Y.

The following corollary characterizes # images of $\mathcal{I}$-dense and $*$-dense in itself subsets of X under inversely-$\mathcal{I}$-open maps.

Corollary 2.6. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be Inversely-$\mathcal{I}$-open map, then the following holds.

(a) The # image of every $\mathcal{I}$-dense subset of X is $\mathcal{J}$-dense subset of Y.

(b) The # image of every $*$ - dense in itself subset of X is $*$ -dense in itself subset of Y.

Proof.

(a) Let A be $\mathcal{I}$-dense subset of X, so $A^* = X$. Therefore, using Theorem 2.4(b), it follows that $f^{\#}(X) \subseteq (f^{\#}(A))^*$. Since, by Lemma 1.6(b), $f^{\#}(X) = Y$, so $(f^{\#}(A))^* = Y$. Hence $f^{\#}(A)$ is $\mathcal{J}$-dense subset of Y.

(b) follows from Theorem 2.4(b). ■

The following theorem shows that under inversely-$\mathcal{I}$-open maps if an ideal $\mathcal{I}$ is codense then its image ideal $f(\mathcal{I})$ is also codense.

Theorem 2.7. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a inversely-$\mathcal{I}$-open map and $\tau \cap \mathcal{I} = \emptyset$, then $\sigma \cap \mathcal{J} = \emptyset$.

Proof. For the subset $A = X$, Theorem 2.4(b) implies that $f^{\#}(X^*) \subseteq (f^{\#}(X))^*$. Now using Lemma 1.7(d), $\tau \cap \mathcal{I} = \emptyset$ implies that $f^{\#}(X) \subseteq (f^{\#}(X))^*$. So by using Lemma 1.6(b), we have $Y \subseteq Y^*$. Therefore $Y = Y^*$. Hence by Lemma 1.7(d), $\sigma \cap \mathcal{J} = \emptyset$. ■

For our next results we introduce inversely closed maps with respect to an ideal.

Definition 2.8. A mapping $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be inversely closed with respect to an ideal (written inversely-$\mathcal{I}$-closed) if for any subset A of X, A is $\tau^*(\mathcal{I})$-closed in X, whenever $f(A)$ is closed subset of Y.

Remark 2.9. Every inversely closed map is inversely-$\mathcal{I}$-closed map, since $\emptyset \in \mathcal{I}$, but converse is not true.

The following theorem gives different characterizations of inversely-$\mathcal{I}$-closed maps and is a generalization of Theorem 2.9 of [8].

Theorem 2.10. For any map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, the following conditions are equivalent:

(a) f is inversely-$\mathcal{I}$-closed i.e. for each subset A of X, A is $\tau^*(\mathcal{I})$-closed in X, whenever $f(A)$ is closed subset of Y.

(b) for any subset A of X, A is $\tau^*(\mathcal{I})$-open in X, whenever $f^{\#}(A)$ is open subset of Y.

(c) if F is a closed subset of Y and $F \subseteq f(X)$, then $S = \bigcup_{y \in F} \{x \in X : x \in f^{-1}(y)\}$ is $\tau^*(\mathcal{I})$-closed in X.

Proof. The proof is similar to that of Theorem 2.9 of [8] and hence is omitted. ■

The following theorem characterizes inversely-$\mathcal{I}$-closed maps in terms of inclusion relation involving f and the local function.

Theorem 2.11. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be any map, then the following holds:

(a) If $f(A^*) \subseteq (f(A))^*$ for every subset A of X and f is injective, then A is τ^*-closed in X, whenever $f(A)$ is σ^*-closed in Y and hence in particular, f is inversely-$\mathcal{I}$-closed.

(b) If f is inversely-$\mathcal{I}$-closed bijective map, then $f(A^*) \subseteq (f(A))^*$ for every subset A of X.

Proof.

(a) Let $f(A)$ be σ^*-closed in Y, then $(f(A))^* \subseteq f(A)$ using Lemma 1.7(b). So $f(A^*) \subseteq (f(A))^* \subseteq f(A)$ and so $f^{-1}(f(A^*)) \subseteq f^{-1}(f(A))$. Therefore, $A^* \subseteq A$, since f is injective and so A is τ^*-closed in X using Lemma 1.7(b). Hence f is inversely-$\mathcal{I}$-closed, since $\sigma \subseteq \sigma^*$.

(b) Let A be any subset of X and $y \in Y$ be any element such that $y \notin (f(A))^*$, then there exists open nhd. W of y in Y such that $W \cap f(A) \in f(\mathcal{I})$. So $W = f(U)$ for some subset U of X, since f is bijective. Therefore, $f^\#(U) \cap f(A) \in f(\mathcal{I})$, since for bijective maps $f^\#(A) = f(A)$ for every subset A of X. So, $U \cap A \in \mathcal{I}$ and U is τ^*-open nhd. of $f^{-1}(y)$ in X, using Theorem 2.10. Thus, $y \notin f(A^*(\mathcal{I}, \tau^*)$ and hence $y \notin f(A^*))$ using Lemma 1.7(c). ■

The following Corollary shows the equivalence of various conditions for bijective maps.

Corollary 2.12. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be bijective map. Then the following are equivalent:

(a) f is inversely-$\mathcal{I}$-open.

(b) f is inversely-$\mathcal{I}$-closed.

(c) for any subset A of X, A is τ^*-closed in X, whenever $f(A)$ is σ^*-closed in Y.

(d) $f(A^*) \subseteq (f(A))^*$ for every subset A of X.

(e) f is pointwise-$\mathcal{I}$-continuous.

Proof. The proof follows from Theorem 1.5 and the fact that for bijective maps, $f^\#(A) = f(A)$ for every subset A of X. ■

The following theorem characterizes images of $\mathcal{I}$-dense and $*$-dense in itself subsets of X under inversely-$\mathcal{I}$-closed bijective maps.

Corollary 2.13. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be Inversely-$\mathcal{I}$-closed bijective map , then the following holds.

(a) The image of every $\mathcal{I}$-dense subset of X is $\mathcal{J}$-dense subset of Y.

(b) The image of every $*$-dense in itself subset of X is $*$-dense in itself subset of Y.

Proof. Proof follows from Corollary 2.12 and is similar to that of Corollary 2.6 and hence is omitted. ■

The following theorem shows that under inversely-$\mathcal{I}$-closed bijective maps if an ideal $\mathcal{I}$ is codense then its image ideal $f(\mathcal{I})$ is also codense.

Theorem 2.14. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a inversely-$\mathcal{I}$-closed bijective map and $\tau \cap \mathcal{I} = \emptyset$, then $\sigma \cap \mathcal{J} = \emptyset$.

Proof. Proof follows from Corollary 2.12 and is similar to that of Theorem 2.7 and hence is omitted. ■

3. Examples

The following example shows that inversely-$\mathcal{I}$-open (inversely-$\mathcal{I}$-closed) map need not be inversely open (inversely closed).

Example 3.1. Let $X = \{a, b\}$, $\tau = \{\emptyset, \{a\}, X\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$. So $\tau^* = \wp(X)$ and $Y = \{0, 1\}$, $\sigma = \{\emptyset, \{0\}, Y\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ by $f(a) = 1$, $f(b) = 0$. Then f is inversely-$\mathcal{I}$-open(Inversely-$\mathcal{I}$-closed) but not inversely open(inversely closed). Since $f(\{b\}) = \{0\}(f(\{a\}) = \{1\})$ is open(closed) in Y but $\{b\}(\{a\})$ is not open(closed) in X.

The following Example shows that the injection condition in Theorem 2.4(a) cannot be dropped.

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$, $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, $Y = \{0, 1\}$ and $\sigma = \{\emptyset, \{0\}, Y\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = 0$, $f(b) = 0$ and $f(c) = 1$ with $\mathcal{J} = f(\mathcal{I}) = \{\emptyset, \{0\}\}$. Then $\tau^*(\mathcal{I}) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. Now $\{a\}^* = \{b\}^* = \{a, b\}^* = \emptyset$ and $\{c\}^* = \{b, c\}^* = \{a, c\}^* = X^* = \{b, c\}$. Also $f^{\#}\{c\} = f^{\#}\{b, c\} = f^{\#}\{a, c\} = \{1\}$ and $(\{1\})^* = Y^* = \{1\}$. Hence $f^{\#}(A^*) \subseteq (f^{\#}(A))^*$ for every subset A of X. But f is not inversely-$\mathcal{I}$-open, since $f\{b\} = \{0\}$ is σ-open in Y but $\{b\}$ is not $\tau^*(\mathcal{I})$-open in X.

The following Example shows that the injection condition in Theorem 2.11(a) cannot be dropped.

Example 3.3. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$, $Y = \{0, 1\}$ and $\sigma = \{\emptyset, \{0\}, Y\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = 0$, $f(b) = 1$ and $f(c) = 1$ with $\mathcal{J} = f(\mathcal{I}) = \{\emptyset, \{0\}\}$. Then $\tau^*(\mathcal{I}) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. Now $\{a\}^* = \emptyset$, $\{b\}^* = \{b\}$ and $\{c\}^* = \{b, c\}$. Also $(f\{b\})^* = (f\{c\})^* = (f\{b, c\})^* = \{1\}^* = \{1\}$. Hence $f(A^*) \subseteq (f(A))^*$ for every subset A of X. But f is not inversely-$\mathcal{I}$-closed, since $f\{c\} = \{1\}$ is closed in Y but $\{c\}$ is not $\tau^*(\mathcal{I})$-closed in X.

The following Example shows that if f is not bijective, then Theorem 2.11(b) need not be hold. Also for any subset A of X, whenever $f(A)$ is σ^*-closed in Y and f is inversely-$\mathcal{I}$-closed, A may not be τ^*-closed in X.

Example 3.4. Let $X = \{a, b\}$, $\tau = \{\emptyset, \{a\}, X\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$. So $\tau = \tau^*$. And $Y = \{0, 1, 2\}$, $\sigma = \{\emptyset, \{0\}, \{0, 1\}, Y\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = 0$, $f(b) = 0$. So $\mathcal{J} = f(\mathcal{I}) = \{0\}$ and $\sigma^* = \{\emptyset, \{0\}, \{1\}, \{0, 1\}, \{1, 2\}, Y\}$. Then $f(X)$ has no closed subsets and therefore, f is vacuously inversely-$\mathcal{I}$-closed. Now for the subset $A = \{a\}$, $A^* = \{a, b\}$. So $f(A^*) = \{0\}$, but $(f(A))^* = \{0\}^* = \emptyset$. Therefore, $f(A^*) \nsubseteq (f(A))^*$. Also $f(\{a\}) = \{0\}$ is σ^*-closed in Y, but $\{a\}$ is not τ^*-closed in X.

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