On Semi Generalized $\omega\alpha$-Closed Sets in Topological Spaces

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Abstract

The aim of this paper is to introduce the new class of closed sets called semi generalized $\omega\alpha$-closed (briefly $sg\omega\alpha$-closed) sets in topological spaces which is properly lies between the class of semi-closed sets and the class of gs-closed sets. Further we define $sg\omega\alpha$-closure and $sg\omega\alpha$-interior in topological spaces and obtained some of their properties.

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1. Introduction


The aim of this paper is to introduce the new weaker forms of closed sets called $sg\omega\alpha$-closed sets and studied the some of their characterizations and also we define and study the $sg\omega\alpha$-closure and $sg\omega\alpha$-interior and some of their basic properties are investigated.

2. Preliminaries

Throughout this paper, the space $(X, \tau)$ (or simply $X$) always means a topological space on which no separation axioms are assumed unless explicitly stated. For a subset $A$ of a space $(X, \tau)$, then $\text{cl}(A)$, $\text{int}(A)$ and $A^c$ denote the closure of $A$, the interior of $A$ and the compliment of $A$ in $X$ respectively.

Definition 2.1. A subset $A$ of a topological space $X$ is called

(i) regular open [18] if $A = \text{int}(\text{cl}(A))$ and regular closed if $A = \text{cl}(\text{int}(A))$.

(ii) semi-open set [11] if $A \subseteq \text{cl}(\text{int}(A))$ and semi-closed set if $\text{int}(\text{cl}(A)) \subseteq A$.

(iii) pre-open set [16] if $A \subseteq \text{int}(\text{cl}(A))$ and pre-closed set if $\text{cl}(\text{int}(A)) \subseteq A$.

(iv) $\alpha$-open set [17] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ and $\alpha$-closed set if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$.

(v) semi-pre open set [2] (= $\beta$-open [1]) if $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$ and semi-pre closed set [2] (= $\beta$-closed [1]) if $\text{int}(\text{cl}(\text{int}(A))) \subseteq A$.

The intersection of all semi-closed (resp. semi-open) subsets of $(X, \tau)$ containing $A$ is called the semi-closure (resp. semi-kernel) of $A$ and by $\text{scl}(A)$ (resp.$\text{sker}(A)$). Also the intersection of all preclosed (resp. semi-preclosed and $\alpha$-closed) subsets of $(X, \tau)$ containing $A$ is called the pre-closure (resp. semi-preclosure and $\alpha$-closure) of $A$ and is denoted by $\text{pcl}(A)$ (resp. $\text{spcl}(A)$ and $\alpha$-$\text{cl}(A)$).

Definition 2.2. A subset $A$ of a topological space $X$ is called a

(i) generalized closed (briefly g-closed) set [12] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(ii) generalized semi-closed (briefly gs-closed) set [3] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$. 

(iii) $\alpha$-generalized closed (briefly $\alpha g$-closed) set \([14]\) if $\alpha \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(iv) generalized $\alpha$-closed (briefly $g\alpha$-closed) set \([13]\) if $\alpha \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $X$.

(v) generalized pre-closed (briefly gp-closed) set \([15]\) if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(vi) generalized semi-pre-closed (briefly gsp-closed) set \([7]\) if $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

(vii) generalized pre-regular-closed (briefly gpr-closed) set \([8]\) if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular-open in $X$.

(viii) $\omega$-closed \([19]\) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $X$.

(ix) $g^*$-closed set \([20]\) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open set in $X$.

(x) $\alpha$-generalized regular closed (briefly $\alpha gr$-closed) set \([22]\) if $\alpha \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular-open in $X$.

(xi) pre $g^*$-closed (briefly $pg^*$-closed) set \([10]\) if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\omega\alpha$-open in $X$.

(xii) $g^*$-preclosed (briefly $g^*p$-closed) set \([21]\) if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open set in $X$.

(xiii) $\omega\alpha$-closed \([4]\) if $\alpha \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\omega$-open in $X$.

(xiv) generalized $\omega\alpha$-closed (briefly $g\omega\alpha$-closed) set \([5]\) if $\alpha \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\omega\alpha$-open set in $X$.

**Definition 2.3.** A topological space $(X, \tau)$ is said to be semi-normal \([3]\) if for each pair of disjoint semiclosed sets $A$ and $B$ of $X$, there exist disjoint open sets $U$ and $V$ in $X$ such that $A \subseteq U$ and $B \subseteq V$.

3. **$sg\omega\alpha$-Closed Sets in Topological Spaces**

In this section, we introduce semi generalized $\omega\alpha$-closed (briefly $sg\omega\alpha$-closed) sets in topological spaces and obtained some of their properties.

**Definition 3.1.** A subset $A$ of a topological space $(X, \tau)$ is called semi generalized $\omega\alpha$-closed (briefly $sg\omega\alpha$-closed) set if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\omega\alpha$-open in $X$. We denote the set of all $sg\omega\alpha$-closed sets in $(X, \tau)$ by $SG\omega\alpha(X, \tau)$.

**Theorem 3.2.** Every closed set is $sg\omega\alpha$-closed.
Proof. Let \( A \) be a closed and \( G \) be an \( \omega \alpha \)-open set containing \( A \) in \( X \). Since \( A \) is closed, we have \( cl(A) = A \). But \( scl(A) \subseteq cl(A) \) is always true. So that \( scl(A) \subseteq cl(A) \subseteq G \). Therefore \( scl(A) \subseteq G \). Hence \( A \) is \( sg \omega \alpha \)-closed set. The converse of the above theorem need not be true as seen from the following example.

**Example 3.3.** Let \( X = \{ a, b, c \} \) and \( \tau = \{ X, \phi, \{ a \}, \{ b, c \} \} \). Then the set \( \{ a, b \} \) is \( sg \omega \alpha \)-closed but not a closed set in \( X \).

**Theorem 3.4.** Every \( \alpha \)-closed set is \( sg \omega \alpha \)-closed.

*Proof.* Let \( A \) be an \( \alpha \)-closed set and \( G \) be an \( \omega \alpha \)-open set in \( X \) such that \( A \subseteq G \). Since \( A \) is \( \alpha \)-closed, we have \( acl(A) = A \). But \( scl(A) \subseteq acl(A) \) is always true. So that \( scl(A) \subseteq acl(A) \subseteq G \). Therefore \( scl(A) \subseteq G \). Hence \( A \) is \( sg \omega \alpha \)-closed set. The converse of the above theorem need not be true as seen from the following example.

**Example 3.5.** In Example 3.3, the set \( \{ a, b \} \) is \( sg \omega \alpha \)-closed but not an \( \alpha \)-closed set in \( X \).

**Theorem 3.6.** Every semi-closed set is \( sg \omega \alpha \)-closed but not conversely.

*Proof.* Let \( A \) be semi-closed set and \( G \) be an \( \omega \alpha \)-open set in \( X \) such that \( A \subseteq G \). Since \( A \) is semi-closed, we have \( scl(A) = A \subseteq G \). Therefore \( scl(A) \subseteq G \). Hence \( A \) is \( sg \omega \alpha \)-closed set.

**Example 3.7.** In Example 3.3, the set \( \{ a, b \} \) is \( sg \omega \alpha \)-closed but not semi-closed in \( X \).

**Theorem 3.8.** Every \( g \omega \alpha \)-closed set is \( sg \omega \alpha \)-closed.

*Proof.* Let \( A \) be \( g \omega \alpha \)-closed set and \( G \) be an \( \omega \alpha \)-open set in \( X \) such that \( A \subseteq G \). Since \( A \) is \( g \omega \alpha \)-closed, we have \( acl(A) \subseteq G \). But \( scl(A) \subseteq acl(A) \) is always true. So that \( acl(A) \subseteq G \). Therefore \( scl(A) \subseteq G \). Hence \( A \) is \( sg \omega \alpha \)-closed set. The converse of the above theorem need not be true as seen from the following example.

**Example 3.9.** Let \( X = \{ a, b, c \} \) and \( \tau = \{ X, \phi, \{ a \}, \{ b \}, \{ a, b \} \} \). Then the set \( \{ b \} \) is \( sg \omega \alpha \)-closed but not \( g \omega \alpha \)-closed in \( X \).

**Theorem 3.10.** Every \( sg \omega \alpha \)-closed set is gsp-closed.

*Proof.* Let \( A \) be \( sg \omega \alpha \)-closed set and \( G \) be an open set in \( X \) such that \( A \subseteq G \). Since every open set is \( \omega \alpha \)-open set and \( A \) is \( sg \omega \alpha \)-closed, we have \( scl(A) \subseteq G \). But \( spcl(A) \subseteq scl(A) \) is always true. So that \( spcl(A) \subseteq G \). Hence \( A \) is gsp-closed. The converse of the above theorem need not be true as seen from the following example.

**Example 3.11.** Let \( X = \{ a, b, c \} \) and \( \tau = \{ X, \phi, \{ a \} \} \). Then the set \( \{ a, b \} \) is gsp-closed but not \( sg \omega \alpha \)-closed in \( X \).

**Theorem 3.12.** Every \( sg \omega \alpha \)-closed set is gs-closed (resp. wg-closed).

*Proof.* The proof follows from the definitions. The converse of the above theorem need
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not be true as seen from the following example.

**Example 3.13.** In Example 3.11, the set \{a, b\} is gs-closed (resp. wg-closed) but not sg$\omega\alpha$-closed in X.

**Remark 3.14.** The concept of sg$\omega\alpha$-closed set is independent of the concept of sets namely pre-closed, semi-preclosed, g-closed, gp-closed, $\alpha g$-closed, gpr-closed, $\alpha gr$-closed, $g^*$-closed, $g^*p$-closed, $\omega\alpha$-closed sets as seen from the following examples.

**Example 3.15.** In Example 3.11, the set $A = \{a, c\}$ is $\omega\alpha$-closed, gp-closed, g-closed, $\alpha gr$-closed but not sg$\omega\alpha$-closed in X.

**Example 3.16.** Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{b, c\}, \{b, c, d\}, \{a, b, c\}\}$. Then the set $A = \{b\}$ is pre-closed, semi-preclosed but not sg$\omega\alpha$-closed in X.

**Example 3.17.** Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{a, c\}\}$. Then the set $A = \{a, b\}$ is g-closed, $g^*$-closed, $g^*p$-closed but not sg$\omega\alpha$-closed in X.

**Example 3.18.** In Example 3.9, the set $A = \{b\}$ is sg$\omega\alpha$-closed but not pre-closed, semi-preclosed, $\alpha gr$-closed, $g^*$-closed, $g^*p$-closed, gp-closed, $\omega\alpha$-closed in X.

**Remark 3.19.** Union of two sg$\omega\alpha$-closed sets need not be a sg$\omega\alpha$-closed set as seen from the following example.

**Example 3.20.** In Example 3.9, the sets \{a\} and \{b\} are sg$\omega\alpha$-closed sets but their union \{a\} $\cup$ \{b\} = \{a, b\} is not a sg$\omega\alpha$-closed set in X.

**Theorem 3.21.** If a subset $A$ of X is sg$\omega\alpha$-closed, then $scl(A)$-$A$ does not contain any non empty $\omega\alpha$-closed set in $(X, \tau)$.

**Proof.** Suppose that $A$ is sg$\omega\alpha$-closed set and $F$ be a non empty $\omega\alpha$-closed subset of $scl(A)$-$A$. Then $F \subseteq scl(A) \cap (X-F)$. Since $(X - F)$ is $\omega\alpha$-open and $A$ is sg$\omega\alpha$-closed, $scl(A) \subseteq (X - F)$. Therefore $F \subseteq (X - scl(A))$ Then $F \subseteq scl(A) \cap (X - scl(A)) = \emptyset$. That is $F = \emptyset$. Thus $scl(A)$-$A$ does not contain any non-empty $\omega\alpha$-closed set in $(X, \tau)$.

The converse of the above theorem need not be true as seen from the following example.

**Example 3.22.** In Example 3.16, the set $A = \{a, b\}$, then $scl(A)$ - $A = \{c, d\}$ does not contain non empty $\omega\alpha$-closed set. But $A$ is not sg$\omega\alpha$-closed set in $(X, \tau)$.

**Theorem 3.23.** If a subset $A$ of a topological space X is sg$\omega\alpha$-closed such that $A \subseteq B \subseteq scl(A)$, then $B$ is also sg$\omega\alpha$-closed.

**Proof.** Let $U$ be an $\omega\alpha$-open set in X such that $B \subseteq U$, then $A \subseteq U$. Since $A$ is sg$\omega\alpha$-closed, $scl(A) \subseteq U$. By hypothesis $scl(B) \subseteq scl(scl(A)) = scl(A) \subseteq U$. Consequently, $scl(B) \subseteq U$. Therefore $B$ is also sg$\omega\alpha$-closed set in $(X, \tau)$. The converse of the above
Theorem need not be true as seen from the following example.

Example 3.24. In Example 3.3, the set $A = \{a\}$ and $B = \{a, b\}$ such that $A$ and $B$ are $sgωα$-closed sets but $A \subseteq B \nsubseteq scl(A)$.

Theorem 3.25. If $A$ is open and $gs$-closed set, then $A$ is $sgωα$-closed set in $X$.

Proof. Let $A$ be an open and $gs$-closed set in $X$. Let $A \subseteq U$ and let $U$ be $ωα$-open in $X$. Now $A \subseteq A$. By hypothesis $scl(A) \subseteq A$. Thst is $scl(A) \subseteq U$. Thus $A$ is $sgωα$-closed in $X$.

Theorem 3.26. If $A$ is $ωα$-open and $sgωα$-closed set, then $A$ is semi-closed set in $X$.

Proof. Let $A \subseteq A$, where $A$ is $ωα$-open. Then $scl(A) \subseteq A$ as $A$ is $sgωα$-closed in $X$. But $A \subseteq scl(A)$ is always true. Therefore $A = scl(A)$. Hence $A$ is semi-closed in $X$.

Theorem 3.27. For each $x \in X$, either $x$ is $ωα$-closed or $xC$ is $sgωα$-closed in $X$.

Proof. Suppose $\{x\}$ is not $ωα$-closed in $X$, then $xC$ is not $ωα$-open and the only $ωα$-open set containing $xC$ is the space $X$ itself. Therefore $scl(xc) \subseteq X$, and thus $xC$ is $sgωα$-closed in $X$.

Theorem 3.28. If $A$ is a $sgωα$-closed set in $X$ and $A \subseteq Y \subseteq X$, then $A$ is a $sgωα$-closed set relative to $Y$.

Proof. Let $A \subseteq Y \cap G$ where $G$ is an $ωα$-open set in $X$. Then $A \subseteq Y$ and $A \subseteq G$. Since $A$ is $sgωα$-closed set in $X$, so $scl(A) \subseteq G$, which implies that $Y \cap scl(A) \subseteq Y \cap G$. Hence $A$ is $sgωα$-closed set relative to $Y$.

Theorem 3.29. Let $(X, τ)$ be a $s$-normal space and if $Y$ is $sgωα$-closed subset of $X$, then the subspace $Y$ is $s$-normal space.

Proof. If $G_1$ and $G_2$ are disjoint semi-closed sets in $X$ such that $(Y \cap G_1) \cap (Y \cap G_2) = \emptyset$. Then $Y \subseteq (G_1 \cap G_2)^c$ and $(G_1 \cap G_2)^c$ is $ωα$-open and $Y$ is $sgωα$-closed in $X$. Therefore $scl(Y) \subseteq (G_1 \cap G_2)^c$ and hence $(scl(Y) \cap G_1) \cap (scl(Y) \cap G_2) = \emptyset$. Since $X$ is $s$-normal space, there exists disjoint open sets $A$ and $B$ such that $scl(Y) \cap G_1 \subseteq A$ and $scl(Y) \cap G_2 \subseteq B$ such that $Y \cap G_1 \subseteq Y \cap A$ and $Y \cap G_2 \subseteq Y \cap B$. Hence $Y$ is $s$-normal space.

Theorem 3.30. A regular open, $sgωα$-closed is semi-closed and hence clopen.

Proof. Let $A$ be a regular open $sgωα$-closed. Since regular open set is $ωα$-open [4], $scl(A) \subseteq A$. This implies $A$ is semi-closed(regular) open setis (regular) closed. Hence $A$ is clopen.

Theorem 3.31. Let $(X, τ)$ be a topological space. Then if $X$ is hyperconnected if and only if every subset of $X$ is $sgωα$-closed and $X$ is connected.
Proof. Necessity: Let X be hyperconnected. Then by Jankovic [9], the only regular open subsets of X are trivial ones. Hence every subset of X is sgωα-closed. Also every hyperconnected space is trivially connected.

Sufficiently: Let A be a non-void proper regular open subset X. Then A is sgωα-closed, by hypothesis. From Theorem 3.30, it follows that A is clopen, which contradicts the hypothesis, since A is connected. Hence X is hyperconnected. ■

Definition 3.32. [4] The intersection of all ωα-open subsets of (X, τ) containing A is called ωα-kernel of A and is denoted by ωα-ker(A).

Theorem 3.33. A subset A of (X, τ) is sgωα-closed if and only if scl(A) ⊆ ωα-ker(A).

Proof. Suppose that A is sgωα-closed. Then scl(A) ⊆ U, whenever A ⊆ U and U is ωα-open. Let x ∈ scl(A). If x ∉ ωα-ker(A), then there is a ωα-open set U containing A such that x ∉ U. Since U is ωα-open set containing A. We have x ∉ scl(A). This is a contradiction. Hence scl(A) ⊆ ωα-ker(A). Conversely, let scl(A) ⊆ ωα-ker(A). If U is only ωα-open set containing A, then scl(A) ⊆ ωα-ker(A) ⊆ U. Hence A is sgωα-closed. ■

Theorem 3.34. Let A be sgωα-closed in (X, τ), then A is semi-closed if and only if scl(A) - A is ωα-closed.

Proof. Necessity: Suppose A is semi-closed, then scl(A) = A and so scl(A) - A = ∅. Which is ωα-closed.

Sufficiently: Suppose scl(A) - A is ωα-closed. Then scl(A) - A = ∅, Since A is sgωα-closed, that is scl(A) = A or A is semi-closed. ■

Now we introduce the following.

Definition 3.35. A subset A of a topological space (X, τ) is called semi generalized ωα-open (briefly sgωα-open) set if its complement Ac is sgωα-closed set in X.

Theorem 3.36. A subset A of a topological space X is sgωα-open, then F ⊆ sint(A) whenever F is ωα-closed in (X, τ).

Proof. Assume that A is sgωα-open. Then Ac is sgωα-closed. Let F be a ωα-closed set in X contained in A. Then Fc is ωα-open set containing Ac in (X, τ). Since Ac is sgωα-closed, this implies that scl(A) ⊆ Fc. Taking complements on both sides, we have F ⊆ sint(A). ■

Theorem 3.37. A subset A is sgωα-open in (X, τ) then G = X, whenever G is ωα-open and sint(A) ∪ (X - A) ⊆ G.

Proof. Let A be sgωα-open, G be ωα-open and sint(A) ∪ (X-G) ⊆ G. This gives X-
G ⊆ (X-sint(A)) ∩ (X-(X-A)) = (X-sint(A))-(X-A) = scl(X-A)-(X-A). Since X-A is sgoωα-closed and X-G is ωα-closed. Then by Theorem 3.21, it follows that X-G = φ. Therefore X = G. ■

**Theorem 3.38.** Let A and B be any two subsets of X. If sint(A) ⊆ B ⊆ A and A is sgoωα-open in (X, τ) then B is a sgoωα-open set in (X, τ).

**Proof.** By hypothesis, sint(A) ⊆ B ⊆ A, then (X-A) ⊆ (X-B) ⊆ X-sint(A) = scl(X-A). Since A is sgoωα-open set, then (X-A) is sgoωα-closed, By Theorem 3.23, (X-B)is sgoωα-closed set in X. Therefore sgoωα-open in X. ■

4. **sgoωα-Closure and sgoωα-Interior**

In this section, the notion of sgoωα-closure and sgoωα-interior in topological spaces are defined and some of its properties are studied.

**Definition 4.1.** For a subset A of (X, τ), sgoωα-closure of A is denoted by sgoωαcl(A) and is defined as sgoωαcl(A) = ∩ {G: A ⊆ G, G is sgoωα-closed in (X, τ)}.

**Definition 4.2.** For a subset A of (X, τ), sgoωα-interior of A is denoted by sgoωαint(A) and is defined as sgoωαint(A) = ∪ {G: G ⊆ A, G is sgoωα-open in (X, τ)}. That is, sgoωαint(A) is the union of all sgoωα-open sets contained in A.

**Theorem 4.3.** For any x ∈ X, x ∈ sgoωαcl(A) if and only if A ∩ V ≠ φ for every sgoωα-open set V containing x.

**Proof.** Let x ∈ sgoωαcl(A). Suppose there exists a sgoωα-open set V containing x such that V ∩ A = φ. Then A ⊆ X-V, sgoωαcl(A) ⊆ X-V. This implies x ∉ sgoωαcl(A) which is a contradiction. Hence V ∩ A ≠ φ. Conversely Suppose x ∉ sgoωαcl(A), then there exists sgoωα-closed set G containing A such that x ∉ G. Then x ∈ X-G is sgoωα-open. Also (X-G) ∩ A = φ. Which is contradiction to the hypothesis. Hence x ∈ sgoωαcl(A). ■

**Theorem 4.4.** If A ⊆ X then, A ⊆ sgoωαcl(A) ⊆ cl(A).

**Proof.** Since every closed set is sgoωα-closed, the proof follows. ■

**Remark 4.5.** Both containment relations in the Theorem 4.4 may be proper as seen from the following example.

**Example 4.6.** Let X= {a, b, c} and τ = {X, φ, {a, b}}, the set A = {a} then sgoωαcl(A) = {a, c} and cl(A) = X and so A ⊆ sgoωαcl(A) ⊆ cl(A).

**Remark 4.7.** For any subset A of X, int(A) ⊆ sgoωαint(A) ⊆ A.

**Theorem 4.8.** Let A be any subset of a space X, then

(i) sgoωαcl(φ) = φ and sgoωαcl(X) = X.
(ii) $\text{sg}\omega \alpha \text{cl}(A)$ is a $\text{sg}\omega \alpha$-closed set in $X$.

**Proof.** The proof follows from the definition 4.1.

**Theorem 4.9.** Let $A$ and $B$ be any two subsets of a space $X$, then the following properties are true

(i) $A$ is $\text{sg}\omega \alpha$-closed set if and only if $\text{sg}\omega \alpha \text{cl}(A) = A$.

(ii) $A$ is $\text{sg}\omega \alpha$-closed set in $X$, then $\text{sg}\omega \alpha \text{cl}(A)$ is the smallest $\text{sg}\omega \alpha$-closed subset of $X$ containing $A$.

(iii) If $A \subseteq B$, then $\text{sg}\omega \alpha \text{cl}(A) \subseteq \text{sg}\omega \alpha \text{cl}(B)$.

(iv) $\text{sg}\omega \alpha \text{cl}(A \cup B) = \text{sg}\omega \alpha \text{cl}(A) \cup \text{sg}\omega \alpha \text{cl}(B)$.

(v) $\text{sg}\omega \alpha \text{cl}(\text{sg}\omega \alpha \text{cl}(A)) = \text{sg}\omega \alpha \text{cl}(A)$.

**Proof.**

(i) Let $A$ be a $\text{sg}\omega \alpha$-closed set in $X$. Since $A \subseteq A$ and $A \in \{ F \subseteq X : A \subseteq F \text{ and } F \text{ is } \text{sg}\omega \alpha$-closed set $\}$ which implies that $A = \cap \{ F \subseteq X : A \subseteq F \text{ and } F \text{ is } \text{sg}\omega \alpha$-closed set $\} \subseteq A.$ Then $\text{sg}\omega \alpha \text{cl}(A) \subseteq A.$ But $A \subseteq \text{sg}\omega \alpha \text{cl}(A)$ is always true. Hence $A = \text{sg}\omega \alpha \text{cl}(A)$.

Conversely, Suppose $A = \text{sg}\omega \alpha \text{cl}(A)$, then $\text{sg}\omega \alpha \text{cl}(A)$ is $\text{sg}\omega \alpha$-closed set in $X$ which implies that $A$ is $\text{sg}\omega \alpha$-closed set in $X$.

(ii) Suppose $A$ is $\text{sg}\omega \alpha$-closed set in $X$, then $\text{sg}\omega \alpha \text{cl}(A) = \cap \{ F \subseteq X : A \subseteq F \text{ and } F \text{ is } \text{sg}\omega \alpha$-closed set $\}$. Since $A \subseteq A$ and $A$ is $\text{sg}\omega \alpha$-closed set in $X$, From (i), $A = \text{sg}\omega \alpha \text{cl}(A)$ is the smallest $\text{sg}\omega \alpha$-closed set in space $X$ containing $A$.

(iii) We know that $B \subseteq \text{sg}\omega \alpha \text{cl}(B)$, then $A \subseteq B \subseteq \text{sg}\omega \alpha \text{cl}(B)$. So $\text{sg}\omega \alpha \text{cl}(B)$ is the $\text{sg}\omega \alpha$-closed set containing $A$. $\rightarrow (a)$.

But $\text{sg}\omega \alpha \text{cl}(A)$ is the smallest $\text{sg}\omega \alpha$-closed set containing $A$. $\rightarrow (b)$. From (a) and (b) we get, $\text{sg}\omega \alpha \text{cl}(A)$ is smaller than $\text{sg}\omega \alpha \text{cl}(B)$. Therefore $\text{sg}\omega \alpha \text{cl}(A) \subset \text{sg}\omega \alpha \text{cl}(B)$

(iv) We know that $A \subseteq A \cup B$ and $B \subseteq A \cup B$, from (iii), $\text{sg}\omega \alpha \text{cl}(A) \subseteq \text{sg}\omega \alpha \text{cl}(A \cup B)$ and $\text{sg}\omega \alpha \text{cl}(B) \subseteq \text{sg}\omega \alpha \text{cl}(A \cup B)$. Then $\text{sg}\omega \alpha \text{cl}(A) \cup \text{sg}\omega \alpha \text{cl}(B) \subseteq \text{sg}\omega \alpha \text{cl}(A \cup B)$ $\rightarrow (a)$.

Let $x \in X$ be any point such that $x \notin \text{sg}\omega \alpha \text{cl}(A) \cup \text{sg}\omega \alpha \text{cl}(B)$, then there exist $\text{sg}\omega \alpha$-closed sets $G$ and $H$ such that $A \subseteq G$ and $B \subseteq H$, $x \notin G$ and $x \notin H$, it implies that $x \notin G \cup H$ and $A \cup B \subseteq G \cup H$. By the Theorem 3.28, $G \cup H$ is $\text{sg}\omega \alpha$-closed, then $x \notin \text{sg}\omega \alpha \text{cl}(A \cup B)$. Therefore $\text{sg}\omega \alpha \text{cl}(A \cup B) \subseteq \text{sg}\omega \alpha \text{cl}(A) \cup \text{sg}\omega \alpha \text{cl}(B)$ $\rightarrow (b)$. From (a) and (b) we have, $\text{sg}\omega \alpha \text{cl}(A \cup B) = \text{sg}\omega \alpha \text{cl}(A) \cup \text{sg}\omega \alpha \text{cl}(B)$. 


(v) We know that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. From (iii) we get, $sg_{\omega \alpha}cl(A \cap B) \subseteq sg_{\omega \alpha}cl(A)$ and $sg_{\omega \alpha} cl(A \cap B) \subseteq sg_{\omega \alpha}cl(B)$. Then $sg_{\omega \alpha} cl(A \cap B) \subseteq sg_{\omega \alpha}cl(A) \cap sg_{\omega \alpha}cl(B)$.

(vi) From definition 4.1 we have, $sg_{\omega \alpha}cl(A)$ is a $sg_{\omega \alpha}$-closed set in $X$. Let $sg_{\omega \alpha}cl(A) = G$, then $G$ is a $sg_{\omega \alpha}$-closed set in $X$. From (i) we have, $sg_{\omega \alpha} cl(G) = G$. Which implies that $sg_{\omega \alpha}cl(sg_{\omega \alpha}cl(A)) = sg_{\omega \alpha}cl(A)$. The converse of property (ii) and property (v) are need not be true as seen in the following examples.

Example 4.10. In the Example 3.11, the subset $A = \{a, c\}$, then $sg_{\omega \alpha}cl(A) = X$ which is the smallest $sg_{\omega \alpha}$-closed set in $X$ containing $A$ but $A$ is not a $sg_{\omega \alpha}$-closed set in $X$.

Example 4.11. Let $X=\{a, b, c\}$ and $\tau = \{X, \phi, \{a, b\}\}$, the subset $A = \{a\}$ and $B = \{b\}$, then $sg_{\omega \alpha}cl(A) = \{a, c\}$ and $sg_{\omega \alpha}cl(B) = \{b, c\}$ and $sg_{\omega \alpha}cl(A \cap B) = \phi$. Hence $\{c\} = sg_{\omega \alpha}cl(A) \cap sg_{\omega \alpha}cl(B) \not\subseteq sg_{\omega \alpha}cl(A \cap B) = \phi$.

Remark 4.12. The following example shows that for any two subsets $A$ and $B$ of $X$, $A \subseteq B$ implies $sg_{\omega \alpha}cl(A) \neq sg_{\omega \alpha}cl(B)$.

Example 4.13. In the Example 3.17, the subset $A = \{a\}$ and $B = \{a, c\}$, then $A \subseteq B$. Now $sg_{\omega \alpha}cl(A) = \{c\}$ and $sg_{\omega \alpha}cl(B) = X$. Hence $sg_{\omega \alpha}cl(A) \neq sg_{\omega \alpha}cl(B)$.

Remark 4.14. For a subset $A$ of $(X, \tau)$, $sg_{\omega \alpha}cl(A) \neq cl(A)$ as seen from the following example.

Example 4.15. In the Example 3.17, the subset $A = \{c\} \subseteq X, sg_{\omega \alpha}cl(A) = \{c\}$ and $cl(A) = \{b, c\}$. Therefore $sg_{\omega \alpha}cl(A) \neq cl(A)$.

Remark 4.16. For any two subsets $A$ and $B$ of $(X, \tau)$, $sg_{\omega \alpha}cl(A) = sg_{\omega \alpha}cl(B)$ does not imply that $A = B$. This is shown by the following example.

Example 4.17. In the Example 3.11, the subset $A = \{a\}$ and $B = \{a, c\}$, then $sg_{\omega \alpha}cl(A) = sg_{\omega \alpha}cl(B) = X$, but $A \neq B$.

Definition 4.18. Let $(X, \tau)$ be a topological space and let $x \in X$. A subset $N$ of $X$ is said to be $sg_{\omega \alpha}$-neighborhood of point $x \in X$ if there exist a $sg_{\omega \alpha}$-open set $G$ such that $x \in G \subseteq N$.

Definition 4.19. Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$, A subset $N$ of $X$ is said to be $sg_{\omega \alpha}$-neighborhood (briefly $sg_{\omega \alpha}$-nbd) of $A$ is if there exists a $sg_{\omega \alpha}$-open set $G$ such that $A \in G \subseteq N$.

The collection of all $sg_{\omega \alpha}$-neighborhood of $x \in X$ is called $sg_{\omega \alpha}$-neighborhood system of $x$ and shall be denoted by $sg_{\omega \alpha}N(x)$.

Theorem 4.20. A subset $A$ of topological space $X$ is $sg_{\omega \alpha}$-open then it is a $sg_{\omega \alpha}$-neighborhood of each of its points.
Proof. Let a subset $G$ of a topological space $X$ be $sg\omega_\alpha$-open. Then for every $x \in X$, $x \in G \subseteq G$ and therefore $G$ is a $sg\omega_\alpha$-neighborhood of each of its points. The converse of the above theorem need not be true as seen from the following example.

Example 4.21. In the Example 3.11, the subset $A = \{b, c\}$ is $sg\omega_\alpha$-neighborhood of each of its points $b$ and $c$, but $A$ is not $sg\omega_\alpha$-open.

Theorem 4.22. A subset $A$ of topological space $X$ is $sg\omega_\alpha$-closed and $x \in sg\omega_\alpha\text{cl}(A)$ if and only if $N \cap A \neq \phi$ for any $sg\omega_\alpha$-nbd $N$ of $x$ in $X$.

Proof. If $x \notin sg\omega_\alpha\text{cl}(A)$. Then there exist $sg\omega_\alpha$-closed set $F$ of $X$ such that $A \subseteq F$ and $x \notin F$. Thus $x \in (X - F)$ and $(X - F)$ is $sg\omega_\alpha$-open in $X$. But $A \cap (X-F) = \phi$, which is a contradiction. Hence $x \in sg\omega_\alpha\text{cl}(A)$. Conversely, suppose that there exist a $sg\omega_\alpha$-nbd $N$ of a point $x \in X$ such that $N \cap A = \phi$. Then there exist an $sg\omega_\alpha$-open set $F$ of $X$ such that $x \in F \subseteq N$. Therefore we have $F \cap A = \phi$ and $A \in (X-F)$. Then $sg\omega_\alpha\text{cl}(A) \subseteq (X-F)$ and $x \notin sg\omega_\alpha\text{cl}(A)$, which is a contradiction to hypothesis that $x \in sg\omega_\alpha\text{cl}(A)$. Therefore $N \cap A \neq \phi$.

Definition 4.23. Let $A$ be a subset of topological space $X$. Then a point $x \in X$ is said to be a $sg\omega_\alpha$-limit point of $A$ if every $sg\omega_\alpha$-open set of $x$ contains a point of $A$ other than $x$, that is $[G - \{x\}] \cap A \neq \phi$ for every $sg\omega_\alpha$-open set $G$ of $x$.

In topological space $X$, the set of all $sg\omega_\alpha$-limit points of a given subset $A$ of $X$ is called $sg\omega_\alpha$-derived set of $A$ and is denoted by $sg\omega_\alpha d(A)$.

Theorem 4.24. Let $A$ and $B$ be any two subsets of a space $X$, then the following properties are true:

(i) $sg\omega_\alpha d(\phi) = \phi$.

(ii) If $A \subseteq B$, then $sg\omega_\alpha d(A) \subseteq sg\omega_\alpha d(B)$.

(iii) If $x \in sg\omega_\alpha d(A)$, then $x \in sg\omega_\alpha d[A - \{x\}]$.

(iv) $sg\omega_\alpha d(A \cup B) = sg\omega_\alpha d(A) \cup sg\omega_\alpha d(B)$.

(v) $sg\omega_\alpha d(A \cap B) \subseteq sg\omega_\alpha d(A) \cap sg\omega_\alpha d(B)$.

Proof.

(i) Let $x \in X$ and $x \in sg\omega_\alpha d(\phi)$. Then for every $sg\omega_\alpha$-open set $G$ containing $x$, we should have $[G - \{x\}] \cap A \neq \phi$, which is impossible. Therefore $sg\omega_\alpha d(\phi) = \phi$.

(ii) Let $x \in sg\omega_\alpha d(A)$ then $x$ is a limit point of $A$. From definition 4.23, $[G - \{x\}] \cap A \neq \phi$ for every $sg\omega_\alpha$-nbd $G$ containing $x$. Since $A \subseteq B$, then $[G - \{x\}] \cap A \subseteq [G - \{x\}] \cap B$. Therefore $x \in sg\omega_\alpha d(B)$. Hence $sg\omega_\alpha d(A) \subseteq sg\omega_\alpha d(B)$.
(iii) If \( x \in \text{sg}_\omega \alpha \text{d}(A) \), by definition 4.23, every \( \text{sg}_\omega \alpha \text{d} \)-open set \( G \) containing \( x \) contains at least one point other than \( x \) of \( A - \{ x \} \). Hence \( x \) is \( \text{sg}_\omega \alpha \text{d} \)-limit point of \( A - \{ x \} \). Hence \( x \in \text{sg}_\omega \alpha \text{d}[A - \{ x \}] \).

(iv) Since \( A \subseteq A \cup B \) and \( B \subseteq A \cup B \), then from property (ii), \( \text{sg}_\omega \alpha \text{d}(A) \cup \text{sg}_\omega \alpha \text{d}(B) \subseteq \text{sg}_\omega \alpha \text{d}(A \cup B) \).

On the other hand if \( x \notin \text{sg}_\omega \alpha \text{d}(A) \cup \text{sg}_\omega \alpha \text{d}(B) \), then \( x \notin \text{sg}_\omega \alpha \text{d}(A) \) and \( x \notin \text{sg}_\omega \alpha \text{d}(B) \). Therefore there exist \( \text{sg}_\omega \alpha \)-nbds \( G_1 \) and \( G_2 \) of \( x \) such that \( G_1 \cap (A - \{ x \}) = \phi \) and \( G_2 \cap (B - \{ x \}) = \phi \). Since \( G_1 \cap G_2 \) is \( \text{sg}_\omega \alpha \)-nbd of \( x \), then we get \( (G_1 \cap G_2) \cap [A - \{ x \}] = \phi \). Therefore \( x \notin \text{sg}_\omega \alpha \text{d}(A \cup B) \). Therfore \( \text{sg}_\omega \alpha \text{d}(A \cup B) \subseteq \text{sg}_\omega \alpha \text{d}(A) \cup \text{sg}_\omega \alpha \text{d}(B) \).

Therefore from (a) and (b) we get, \( \text{sg}_\omega \alpha \text{d}(A \cup B) = \text{sg}_\omega \alpha \text{d}(A) \cup \text{sg}_\omega \alpha \text{d}(B) \).

(v) Since \( A \cap B \subseteq A \) and \( A \cap B \subseteq B \), then from property (ii) we have \( \text{sg}_\omega \alpha \text{d}(A \cup B) \subseteq \text{sg}_\omega \alpha \text{d}(A) \) and \( \text{sg}_\omega \alpha \text{d}(A \cap B) \subseteq \text{sg}_\omega \alpha \text{d}(B) \). Consequently, \( \text{sg}_\omega \alpha \text{d}(A \cap B) \subseteq \text{sg}_\omega \alpha \text{d}(A) \cap \text{sg}_\omega \alpha \text{d}(B) \).

References


