Generalization of Uniqueness of Meromorphic Functions Concerning Differential Polynomials

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Abstract

In this paper, we study the uniqueness of meromorphic functions concerning differential polynomials sharing a small function \( a(z) \). The results generalize and improve the corresponding results obtained by Jianming Qi and Lei Qiao [7].

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1. INTRODUCTION AND MAIN RESULTS

Let \( \mathbb{C} \) denote the complex plane. A meromorphic function will mean meromorphic in \( \mathbb{C} \). We shall use the standard notations and terms in the Nevanlinna value distribution theory such as \( T(r, f), N(r, f), \overline{N}(r, f), m(r, f) \) (see [4, 5, 11, 14]). The notation \( S(r, f) \) is defined to be any quantity satisfying \( S(r, f) = o(T(r, f)) \) as \( r \to \infty \) outside of a possible exceptional set of finite linear measure. A meromorphic function \( a(z) \) is called a small function with respect to \( f(z) \), provided that \( T(r, a) = S(r, f) \).
We denote by $E_k(a,f)$ the set of zeros of $f - a$ with multiplicities at most $k$, where each zero is counted according to its multiplicity. We denote by $\overline{E}_k(a,f)$ the set of zeros of $f - a$ with multiplicities at most $k$, where each zero is counted only once. In addition, we denote by $N_k(r, \frac{1}{f-a})$ the counting function with respect to the set $E_k(a,f)$ ($\overline{E}_k(a,f)$). We denote by $N(k,r,1\overline{f-a})$ the counting function of $\alpha$-points of $f$ (counted with proper multiplicities) whose multiplicities are not less than $k$, we denote by $\overline{N}(k,r,1\overline{f-a})$ the corresponding reduced counting function (ignoring multiplicity). Set 

$$N_k(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}(2,r, \frac{1}{f-a}) + \cdots + \overline{N}(k,r, \frac{1}{f-a}).$$

We denote by $N_{11}(r, 1\overline{f})$ the counting function for common simple 1-points of both $f(z)$ and $g(z)$ where multiplicity is not counted. $\overline{N}_L(r, v_0^f > v_0^g)$ is the counting function for 1-points of both $f$ and $g$ about which $f$ has larger multiplicity than $g$, with multiplicity being not counted. Similarly, we have the notation $\overline{N}_L(r, v_0^g > v_0^f)$.

Let $f(z)$ be a nonconstant meromorphic function in the complex plane $C$, $k$ be a positive integer and $a(z)$ is a small function of $f$, where a zero point with multiplicity $m$ is counted $m$ times in the set. If there zero points are only counted once, then we denote the set by $\overline{E}(a(z), f)$. Define, $E_k(a(z), f) = \{z|f(z) - a(z) = 0\}$, where each zero of $f(z) - a(z)$ with multiplicity $m$ is counted $m$ times where $m \leq k$. We say $f$ and $g$ share a CM (IM), if $E(a,f) = E(a,g) (\overline{E}(a,f) = \overline{E}(a,g))$. Similarly, we define that $f$ and $g$ share a small function $a(z)$ (≠ 0, ∞) CM (IM), if $E(a(z),f) = E(a(z),g) (\overline{E}(a(z),f) = \overline{E}(a(z),g))$. If $E_k(a(z), f) = E_k(a(z), g)$, then we say that $f - a(z)$ and $g - a(z)$ have the same zeros with multiplicities at most $k$.

In 1997, Yang and Hua Studied meromorphic functions sharing one value. They proved the following result

**Theorem A [13].** Let $f$ and $g$ be two nonconstant meromorphic functions, and $n(\geq 11)$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1CM, then either $f = c_1 e^{cz}$, $g = ...$
where \(c_1, c_2\) and \(c\) are three constants, satisfying \((c_1c_2)^{n+1}c^2 = -1\), or \(f = tg\) for a constant \(t\) such that \(t^{n+1} = 1\).

Later, Fang and Qiu investigated meromorphic functions sharing fixed points, which is an improvement of Theorem A

**Theorem B [3].** Let \(f\) and \(g\) be two nonconstant meromorphic (entire) functions, and \(n \geq 11\) be a positive integer. If \(f^n f'\) and \(g^n g'\) share \(z\) CM, then either \(f = c_1 e^{cz^2}, g = c_2 e^{-cz^2}\), where \(c_1, c_2\) and \(c\) are three constants, satisfying \(4(c_1c_2)^{n+1}c^2 = -1\), or \(f = tg\) for a constant \(t\) such that \(t^{n+1} = 1\).

Corresponding to transcendental meromorphic functions, Wang and Gao further extended the above results as follows

**Theorem C [8].** Let \(f\) and \(g\) be two transcendental meromorphic functions, and let \(a(z) \neq 0\) be a common small function with respect to them, and let \(n \geq 11\) be a positive integer. If \(f^n f'\) and \(g^n g'\) share \(a(z)\) CM, then either \(f^n f' g^n g' \equiv a^2(z)\), or \(f = tg\) for a constant \(t\) such that \(t^{n+1} = 1\).

Corresponding to nonconstant meromorphic functions, in 2015, Jianming Qi and Lei Qiao [7], obtained the following results

**Theorem D.** Let \(f\) and \(g\) be two nonconstant meromorphic functions and let \(n \geq 11\) and \(k \geq 3\), be two positive integers. If \(E_k(a(z), f^n f') = E_k(a(z), g^n g')\) then either \(f^n f' g^n g' \equiv a^2(z)\), or \(f = tg\) for a constant \(t\) such that \(t^{n+1} = 1\), where \(a(z)\) is a meromorphic function such that \(a(z) \neq 0\) and \(a(z)\) is a small function of \(f\) and \(g\).

**Theorem E.** Let \(f\) and \(g\) be two nonconstant meromorphic functions and \(n \geq 13\) be a positive integer. If \(E_2(a(z), f^n f') = E_2(a(z), g^n g')\) then either \(f^n f' g^n g' \equiv a^2(z)\), or \(f = tg\) for a constant \(t\) such that \(t^{n+1} = 1\), where \(a(z)\) is a meromorphic function such that \(a(z) \neq 0\) and \(a(z)\) is a small function of \(f\) and \(g\).

**Theorem F.** Let \(f\) and \(g\) be two nonconstant meromorphic functions and \(n \geq 19\) be a positive integer. If \(E_3(a(z), f^n f') = E_3(a(z), g^n g')\) then either \(f^n f' g^n g' \equiv a^2(z)\), or \(f = tg\) for a constant \(t\) such that \(t^{n+1} = 1\), where \(a(z)\) is a meromorphic function such that \(a(z) \neq 0\) and \(a(z)\) is a small function of \(f\) and \(g\).

**Theorem G.** Let \(f\) and \(g\) be two nonconstant meromorphic functions and \(n \geq 23\) be a positive integer. If \(E_4(a(z), f^n f') = E_4(a(z), g^n g')\) then either \(f^n f' g^n g' \equiv a^2(z)\), or \(f = tg\) for a constant \(t\) such that \(t^{n+1} = 1\), where \(a(z)\) is a meromorphic function such that \(a(z) \neq 0\) and \(a(z)\) is a small function of \(f\) and \(g\).
This paper is motivated by the following question

**Question.** What will happen in Theorems D - G if $f^n f'$ and $g^n g'$ is replaced by $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ respectively.

We will concentrate our attention to the above question and provide an affirmative answer in this direction.

We now state our main results.

**Theorem 1.** Let $f$ and $g$ be two nonconstant meromorphic functions and let $n (> 3k + m + 8)$ and $k (\geq 3)$ be two positive integers. If $E_k(a(z), [f^n P(f)]^{(k)}) = E_k(a(z), [g^n P(g)]^{(k)})$, then

(i) when $P(w) \equiv c_0, f(z) \equiv tg(z)$ for a constant $t$ such that $t^n = 1$.

(ii) when $P(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0$, where $a_0 \neq 0, a_1, ..., a_{m-1}, a_m \neq 0$, then either $f \equiv tg$ for a constant $t$ such that $t^d = 1$, where $d = \gcd(n + m, ..., n + m - i, ..., n), (a_{m-i} \neq 0)$, for some $i = 0, 1, ..., m$, or $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(w_1, w_2) = w_1^n (a_m w_1^m + a_{m-1} w_1^{m-1} + \cdots + a_0) - w_2^n (a_m w_2^m + a_{m-1} w_2^{m-1} + \cdots + a_0)$.

(iii) $[f^n P(f)]^k [g^n P(g)]^k \equiv a^2(z)$.

**Theorem 2.** Let $f$ and $g$ be two nonconstant meromorphic functions and $n \left(> 4k + \frac{3m}{2} + 9\right)$ be a positive integer. If $E_2(a(z), [f^n P(f)]^{(k)}) = E_2(a(z), [g^n P(g)]^{(k)})$, then the conclusion of theorem 1 holds.

**Theorem 3.** Let $f$ and $g$ be two nonconstant meromorphic functions and $n (> 7k + 3m + 12)$ be a positive integer. If $E_1(a(z), [f^n P(f)]^{(k)}) = E_1(a(z), [g^n P(g)]^{(k)})$, then the conclusion of theorem 1 holds.

**Theorem 4.** Let $f$ and $g$ be two nonconstant meromorphic functions and $n (> 9k + 4m + 14)$ be a positive integer. If $E_1(a(z), [f^n P(f)]^{(k)}) = E_1(a(z), [g^n P(g)]^{(k)})$, then the conclusion of theorem 1 holds.

2. Some Lemmas

In this section, we present some lemmas which will be needed in the sequel.

**Lemma 1 ([14]).** Let $f$ be a non constant meromorphic function and $k$ be a positive integer, then

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\bar{N}(r, f) + S(r, f).$$
Lemma 2 ([10]). Let $f$ be a non-constant meromorphic function and $P(f) = a_nf^n + a_{n-1}f^{n-1} + \cdots + a_1f + a_0$, where $a_0, a_1, ..., a_n$ are constants and $a_n \neq 0$. Then
\[ T(r, P(f)) = nT(r, f) + S(r, f). \]

Lemma 3 ([9]). Let $f, g$ be two nonconstant meromorphic functions, $k, n > 2k + 1$ be two positive integers. If $[f^n]^{(k)} = [g^n]^{(k)}$, then $f = tg$ for a constant $t$ such that $t^n = 1$.

Lemma 4 ([9]). Let $f$ and $g$ be two nonconstant meromorphic functions. Let $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$ or $P(w) \equiv c_0$, where $a_0 \neq 0, a_1, ..., a_{m-i}, a_m \neq 0, c_0 \neq 0$ are complex constants, and $n > 0, k > 0$, and $m \geq 0$ be three integers with $n > 2k + m + 1$. If $[f^nP(f)]^{(k)} = [g^nP(g)]^{(k)}$, then $f^nP(f) = g^nP(g)$.

Lemma 5. Let $f$ and $g$ be two nonconstant meromorphic functions, $F = \frac{[f^nP(f)]^{(k)}}{a(z)}$ and $G = \frac{[g^nP(g)]^{(k)}}{a(z)}$, where $n \geq 2k + m + 1$ is a positive integer. If $F \equiv G$, then

(i) when $P(w) \equiv c_0$, $f \equiv tg$ for a constant $t$ such that $t^n = 1$.

(ii) when $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$, where $a_0 \neq 0, a_1, ..., a_{m-i}, a_m \neq 0$, then either $f \equiv tg$ for a constant $t$ such that $t^d = 1$, where $d = \text{GCD}(n + m, ..., n + m - i, ..., n)$, $(a_{m-i} \neq 0)$, for some $i = 0, 1, ..., m$, or $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where
\[ R(w_1, w_2) = w_1^n(a_mw_1^m + a_{m-1}w_1^{m-1} + \cdots + a_0) - w_2^n(a_mw_2^m + a_{m-1}w_2^{m-1} + \cdots + a_0). \]

Proof. If $F \equiv G$, then by Lemma 4, we have
\[ f^nP(f) \equiv g^nP(g). \]  

(i) when $P(w) \equiv c_0$, then from (2.1), we get $f(z) \equiv tg(z)$ for a constant $t$ such that $t^n = 1$.

(ii) when $P(w) = a_mw^m + a_{m-1}w^{m-1} + \cdots + a_1w + a_0$, then from (2.1), we get
\[ f^n(a_mf^m + a_{m-1}f^{m-1} + \cdots + a_1f + a_0) \equiv g^n(a_mg^m + a_{m-1}g^{m-1} + \cdots + a_1g + a_0). \]  

Let $h = \frac{f}{g}$ if $h$ is a constant, then substituting $f = gh$ into (2.2), we deduce
\[ a_m g^{n+m}(h^{n+m} - 1) + a_{m-1} g^{n+m-1}(h^{n+m-1} - 1) + \ldots + a_0 g(h^n - 1) \equiv 0, \]

(2.3)

which implies \( h^d = 1 \), where \( d = \text{GCD}(n + m, n + m - i, \ldots, n) \), \( a_{m-i} \neq 0 \) for some \( i = 0, 1, \ldots, m \). Thus \( f(z) \equiv tg(z) \) for a constant \( t \) such that \( t^d = 1 \). If \( h \) is not a constant, then we know by (2.2) that \( f \) and \( g \) satisfy the algebraic equation

\[ R(f, g) \equiv 0, \]

where

\[ R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1} w_1^{m-1} + \ldots + a_0) - w_2^n(a_m w_2^m + a_{m-1} w_2^{m-1} + \ldots + a_0). \]

So Lemma 5 is proved.

**Lemma 6** [2]. Let \( f \) and \( g \) be two meromorphic functions, and let \( k \) be a positive integer. If \( E_k(1, f) = E_k(1, g) \), then one of the following occurs:

(i) \( T(r, f) + T(r, g) \leq N_2(r, f) + N_2\left(r, \frac{1}{f}\right) + N_2(r, g) + N_2\left(r, \frac{1}{g}\right) \]

\[ + N\left(r, \frac{1}{f-1}\right) + \tilde{N}\left(r, \frac{1}{g-1}\right) - N_{11}\left(r, \frac{1}{f-1}\right) + N_{k+1}\left(r, \frac{1}{f-1}\right) \]

\[ + \tilde{N}_{k+1}\left(r, \frac{1}{g-1}\right) + S(r, f). \]

(ii) \( f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)} \), where \( a \neq 0, b \) are two constants.

**Lemma 7** [1]. Let \( f \) and \( g \) be two meromorphic functions. If \( \overline{E}(1, f) = \overline{E}(1, g) \), then one of the following occurs:

(i) \( T(r, f) + T(r, g) \leq 2 \left[ N_2(r, f) + N_2\left(r, \frac{1}{f}\right) + N_2(r, g) + N_2\left(r, \frac{1}{g}\right) \right] \]

\[ + 3\overline{N}_L\left(r, v_0 f > v_0 g \right) + 3\overline{N}_L\left(r, v_0 g > v_0 f \right) \]

\[ + S(r, f) + S(r, g). \]

(ii) \( f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)} \), where \( a \neq 0, b \) are two constants.

**Lemma 8.** Let \( f \) and \( g \) be two meromorphic functions, \( n(> 3k + m + 3) \) be a positive integer and let \( F \) and \( G \) be defined as in Lemma 5. If

\[ F = \frac{(b+1)G+(a-b-1)}{bg+(a-b)} \]

(2.4)

where \( a(\neq 0), b \) are two constants, then

(i) when \( P(w) \equiv c_0, f \equiv tg \) for a constant \( t \) such that \( t^{n+1} = 1 \).

(ii) when \( P(w) = a_m w^m + a_{m-1} w^{m-1} + \ldots + a_1 w + a_0, \) where \( a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0 \), then either \( f \equiv tg \) for a constant \( t \) such that \( t^d = 1 \), where
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\[ d = \text{GCD}(n + m, \ldots, n + m - i, \ldots, n), (a_{m-i} \neq 0), \] for some \( i = 0, 1, \ldots, m, \) or \( f \) and \( g \) satisfy the algebraic equation \( R(f, g) \equiv 0, \) where \( R(w_1, w_2) = w_1^n(a_m w_1^m + a_{m-1} w_1^{m-1} + \cdots + a_0) - w_2^n(a_m w_2^m + a_{m-1} w_2^{m-1} + \cdots + a_0). \)

(iii) \([f^n P(f)]^{(k)} [g^n P(g)]^{(k)} \equiv a^2(z).\]

**Proof.** By Lemma 2, we get

\[
T(r, F) = T(r, [f^n P(f)]^{k})
\]

\[
\leq (n + m)T(r, f) + k\bar{N}(r, f) + S(r, f)
\]

\[
\leq (n + m + k)T(r, f) + S(r, f).
\]

(2.5)

On the other hand, we have

\[
(n + m)T(r, f) \leq T(r, f^n P(f)) + S(r, f)
\]

\[
\leq N(r, f^n P(f)) + m(r, f^n P(f)) + S(r, f)
\]

\[
\leq N(r, [f^n P(f)]^{(k)}) - k\bar{N}(r, f) + m\left(r, \frac{f^n P(f)}{[f^n P(f)]^{(k)}}\right)
\]

\[
+ m(r, [f^n P(f)]^{(k)}) + S(r, f)
\]

\[
(n + m)T(r, f) \leq T(r, F) - k\bar{N}(r, f) + S(r, f).
\]

So

\[
T(r, F) \geq (n + m + k)T(r, f) + S(r, f). \quad (2.6)
\]

Thus, by Eqs. (2.5) and (2.6), and \( n > 3k + m + 3, \) we get \( S(r, F) = S(r, f). \)

Similarly, we get

\[
T(r, G) \geq (n + m + k)T(r, g) + S(r, g), \quad (2.7)
\]

and \( S(r, G) = S(r, g). \)

Without loss of generality, we suppose that there exists a set \( I \) with infinite measure such that \( T(r, f) \leq T(r, g), r \in I. \) Next, we consider three cases.

**Case 1.** \( b \neq -1, 0. \) If \( a - b - 1 \neq 0, \) then by Eq. (2.4) we have

\[
\bar{N}\left(r, \frac{1}{G + \frac{a-b-1}{b+1}}\right) = \bar{N}\left(r, \frac{1}{F}\right).
\]

By Nevanlinna second fundamental theorem and Lemma 1 we get
\[ T(r, G) \leq \bar{N}(r, G) + \bar{N} \left( r, \frac{1}{G} \right) + \bar{N} \left( r, \frac{1}{G + \frac{a-b-1}{b+1}} \right) + S(r, G) \]

\[ = \bar{N}(r, G) + \bar{N} \left( r, \frac{1}{G} \right) + \bar{N} \left( r, \frac{1}{F} \right) + S(r, g) \]

\[ \leq \bar{N}(r, g) + \bar{N} \left( r, \frac{1}{[g^np(g)]^{(k)}} \right) + \bar{N} \left( r, \frac{1}{[f^n p(f)]^{(k)}} \right) + S(r, g) \]

\[ \leq \bar{N}(r, g) + (k + 1)\bar{N} \left( r, \frac{1}{g} \right) + m\bar{N} \left( r, \frac{1}{g} \right) + k\bar{N}(r, g) \]

\[ + (k + 1)\bar{N} \left( r, \frac{1}{f} \right) + m\bar{N} \left( r, \frac{1}{f} \right) + k\bar{N}(r, f) + S(r, g) \]

\[ \leq (4k + 2m + 3)T(r, g) + S(r, g). \]

Hence, by \( n > 3k + m + 3 \) and Eq. (2.7), we have \( T(r, g) \leq S(r, g), r \in I. \) This is impossible.

If \( a - b - 1 = 0 \), then by Eq. (2.4) we have

\[ \bar{N} \left( r, \frac{1}{G + \frac{1}{b}} \right) = \bar{N}(r, F). \]

By Nevanlinna second fundamental theorem and Lemma 1 we get

\[ T(r, G) \leq \bar{N}(r, G) + \bar{N} \left( r, \frac{1}{G} \right) + \bar{N} \left( r, \frac{1}{G + \frac{1}{b}} \right) + S(r, G) \]

\[ = \bar{N}(r, G) + \bar{N} \left( r, \frac{1}{G} \right) + \bar{N}(r, F) + S(r, g) \]

\[ \leq \bar{N}(r, g) + \bar{N} \left( r, \frac{1}{[g^np(g)]^{(k)}} \right) + \bar{N}(r, f) + S(r, g) \]

\[ \leq \bar{N}(r, g) + (k + 1)\bar{N} \left( r, \frac{1}{g} \right) + m\bar{N} \left( r, \frac{1}{g} \right) + k\bar{N}(r, g) + \bar{N}(r, f) + S(r, g) \]

\[ \leq (2k + m + 3)T(r, g) + S(r, g). \]

Thus by \( n > 3k + m + 3 \) and Eq. (2.7), we know \( T(r, g) \leq S(r, g), r \in I, \) which is a contradiction.
Case 2. Assume that $b = -1$. Then by Eq. (2.4) we have $F = \frac{a}{a+1-G}$.

If $a + 1 \neq 0$, then $\bar{N}(r, \frac{1}{G-a-1}) = \bar{N}(r, F)$. Proceeding as the proof of case 1, we get a contradiction.

If $a + 1 = 0$, then $FG \equiv 1$, that is,

$$[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \equiv a^2(z).$$

Case 3. Assume that $b = 0$. Then by Eq. (2.4) we have $F = \frac{G+a-1}{a}$.

If $a - 1 \neq 0$, then $\bar{N}(r, \frac{1}{G+a-1}) = \bar{N}\left(r, \frac{1}{f}\right)$. Proceeding as the proof of case 1, we get a contradiction.

If $a - 1 = 0$, then $F \equiv G$. By Lemma 5, we obtain the remaining (i) and (ii) conclusions of this Lemma.

3. Proof of the Theorems

$F$ and $G$ be defined as in Lemma 5.

Proof of Theorem 1. Since $k \geq 3$, we have

$$\bar{N}\left(r, \frac{1}{F-1}\right) + \bar{N}\left(r, \frac{1}{G-1}\right) - N_{11}\left(r, \frac{1}{F-1}\right) + N_{k+1}\left(r, \frac{1}{G-1}\right)$$

$$\leq \frac{1}{2} N\left(r, \frac{1}{F-1}\right) + \frac{1}{2} N\left(r, \frac{1}{G-1}\right)$$

$$\leq \frac{1}{2} T(r, F) + \frac{1}{2} T(r, G) + S(r, f) + S(r, g).$$

Then (i) in Lemma 6 becomes

$$T(r, F) + T(r, G)$$

$$\leq 2 \left\{ N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right) + N_2(r, G) \right\} + S(r, f) + S(r, g).$$

(3.2)

By logarithmic derivatives and the first main theorem of Nevanlinna, we have

$$N_2(r, F) + N_2\left(r, \frac{1}{F}\right) \leq N_2\left(r, \frac{[f^n P(f)]^{(k)}}{a(z)}\right) + N_2\left(r, \frac{a(z)}{[f^n P(f)]^{(k)}}\right)$$

$$\leq (k + 2)\bar{N}\left(r, \frac{1}{F}\right) + m N\left(r, \frac{1}{F}\right) + k \bar{N}(r, f) + 2 \bar{N}(r, f).$$

(3.3)
Similarly, we get
\[ N_2(r,G) + N_2 \left( \frac{1}{G} \right) \leq (k + 2)N \left( \frac{1}{f} \right) + m N \left( \frac{1}{g} \right) + kN(r,f) + 2N(r,g). \]  
(3.4)

Suppose that Eq. (3.2) holds. By Lemma 1, Eqs. (3.3) and (3.4), we have
\[
T(r,F) + T(r,G) \leq 2 \left[ (k + 2)N \left( \frac{1}{f} \right) + m N \left( \frac{1}{f} \right) + kN(r,f) + 2N(r,f) \right]
\]
\[
+ 2 \left[ (k + 2)N \left( \frac{1}{g} \right) + m N \left( \frac{1}{g} \right) + kN(r,g) + 2N(r,g) \right]
\]
\[
+ S(r,f) + S(r,g)
\]
\[
\leq (4k + 2m + 8)T(r,f) + (4k + 2m + 8)T(r,g)
\]
\[
+ S(r,f) + S(r,g).
\]

By \( n > 3k + m + 8 \), Eqs. (2.6) and (2.7), we obtain a contradiction. Thus by Lemma 6, \( F = \frac{(b+1)G+(a-b-1)}{bG+(a-b)} \), where \( a \neq 0 \), \( b \) are two constants. Then by Lemma 8, we can prove Theorem 1.

**Proof of Theorem 2.** Obviously, we have
\[
N \left( \frac{1}{F-1} \right) + N \left( \frac{1}{G-1} \right) - N_{11} \left( \frac{1}{F-1} \right) + \frac{1}{2}N_{(3)} \left( \frac{1}{F-1} \right) + \frac{1}{2}N_{(3)} \left( \frac{1}{G-1} \right)
\]
\[
\leq \frac{1}{2}N \left( \frac{1}{F-1} \right) + \frac{1}{2}N \left( \frac{1}{G-1} \right)
\]
\[
\leq \frac{1}{2}T(r,F) + \frac{1}{2}T(r,G) + S(r,f) + S(r,g).
\]  
(3.5)

Then (i) in Lemma 6 becomes
\[
T(r,F) + T(r,G) \leq 2 \left[ N_2(r,F) + N_2 \left( \frac{1}{F} \right) + N_2(r,G) + N_2 \left( \frac{1}{G} \right) \right] + \tilde{N}_{(3)} \left( \frac{1}{F-1} \right)
\]
\[
+ \tilde{N}_{(3)} \left( \frac{1}{G-1} \right) + S(r,f) + S(r,g).
\]  
(3.6)
Consider
\[ \tilde{N}_3\left( r, \frac{1}{F-1} \right) \leq \frac{1}{2} N\left( r, \frac{F}{F} \right) \]
\[ \leq \frac{1}{2} N\left( r, \frac{1}{F} \right) + S(r, f) \]
\[ \leq \frac{1}{2} \tilde{N}(r, F) + \frac{1}{2} \tilde{N}\left( r, \frac{1}{F} \right) + s(r, f) \]
\[ \leq \frac{1}{2} \left[ N_1\left( r, \frac{1}{f^n P(f)} \right) + \tilde{N}(r, f) \right] + S(r, f) \]
\[ \leq \frac{1}{2} (k + 1) \tilde{N}\left( r, \frac{1}{f} \right) + mN\left( r, \frac{1}{f} \right) + k\tilde{N}(r, f) + \tilde{N}(r, f) + S(r, f) \]
\[ \leq \frac{[2k+m+2]}{2} T(r, f) + S(r, f). \quad (3.7) \]

Similarly, we get
\[ \tilde{N}_3\left( r, \frac{1}{G-1} \right) \leq \left[ \frac{2k+m+2}{2} \right] T(r, g) + S(r, g). \quad (3.8) \]

Suppose that Eq. (3.6) holds. Combining Eqs. (3.3), (3.4), (3.7), and (3.8), we have
\[ T(r, F) + T(r, G) \leq 2 \left[ N_2\left( r, \frac{1}{f^n P(f)} \right) + N_2\left( r, \left[ f^n P(f) \right]^{(k)} \right) + \right. \]
\[ + \left. 2 \left[ N_2\left( r, \frac{1}{g^n P(g)} \right) + N_2\left( r, \left[ g^n P(g) \right]^{(k)} \right) \right] \right. \]
\[ + \left. \left[ \frac{2k+m+2}{2} \right] T(r, f) + \left[ \frac{2k+m+2}{2} \right] T(r, g) \right. \]
\[ + \left. S(r, f) \right. \]
\[ + \left. S(r, g) \right. \]
\[ (n + m + k)[T(r, f) + T(r, g)] \leq \left( \frac{10k + 5m + 18}{2} \right)[T(r, f) + T(r, g)] \]
\[ + S(r, f) + S(r, g) \]
\[ \left( n - \frac{(8k+3m+18)}{2} \right)[T(r, f) + T(r, g)] \leq S(r, f) + S(r, g). \quad (3.9) \]

By \( n > \frac{8k+3m+18}{2} \), Eqs. (2.6), (2.7) and (3.9), we get a contradiction. Thus, by Lemma 6, \( F = \frac{(b+1)G+(a-b-1)}{bG+(a-b)} \), where \( a \neq 0, b \) are two constants. Then by Lemma 8, we can prove Theorem 2.

**Proof of Theorem 3.** Similarly as in proof of Theorem 2, we have
\[ \bar{N}(r, \frac{1}{F-1}) + \bar{N}(r, \frac{1}{G-1}) - N_{11}(r, \frac{1}{F-1}) \]
\[ \leq \frac{1}{2} N(r, \frac{1}{F-1}) + \frac{1}{2} N(r, \frac{1}{G-1}) \]
\[ \leq \frac{1}{2} T(r, F) + \frac{1}{2} T(r, G) + S(r, f) + S(r, g). \]

(3.10)

Then (i) in Lemma 6 becomes
\[ T(r, F) + T(r, G) \]
\[ \leq 2 \left\{ N_2(r, F) + N_2 \left( r, \frac{1}{F} \right) + N_2(r, G) + N_2 \left( r, \frac{1}{G} \right) + \bar{N}_2 \left( r, \frac{1}{F-1} \right) \right\} + S(r, f) + S(r, g). \]

(3.11)

Consider
\[ \bar{N}_2 \left( r, \frac{1}{F-1} \right) \leq N \left( r, \frac{F}{F'} \right) \]
\[ \leq N \left( r, \frac{F}{F} \right) + S(r, f) \]
\[ \leq \bar{N}(r, F) + \bar{N} \left( r, \frac{1}{F} \right) + s(r, f) \]
\[ \leq \bar{N}(r, f) + (k + 1) \bar{N} \left( r, \frac{1}{f} \right) + mN \left( r, \frac{1}{f} \right) + k\bar{N}(r, f) \]
\[ + S(r, f) \]
\[ \leq (2k + m + 2) T(r, f) + S(r, f). \]

(3.12)

Similarly, we get
\[ \bar{N}_2 \left( r, \frac{1}{G-1} \right) \leq (2k + m + 2) T(r, g) + S(r, g). \]

(3.13)

Suppose that Eq. (3.11) holds. Combining Eqs. (3.3), (3.4), (3.12), and (3.13), we have
\[ T(r, F) + T(r, G) \leq 2 \left[ N_2 \left( r, \frac{1}{[f^nP(f)]^{(k)}} \right) + N_2(r, [f^nP(f)]^{(k)}) \right] \]
\[ + 2 \left[ N_2 \left( r, \frac{1}{[g^nP(g)]^{(k)}} \right) + N_2(r, [g^nP(g)]^{(k)}) \right] \]


\[ + [4k + 2m + 4]T(r, f) + [4k + 2m + 4]T(r, g) \]
\[ + S(r, f) + S(r, g). \]

\[(n + m + k)[T(r, f) + T(r, g)] \leq (8k + 4m + 12)[T(r, f) + T(r, g)] \]
\[ + S(r, f) + S(r, g) \]
\[(n - 7k - 3m - 12)[T(r, f) + T(r, g)] \leq S(r, f) + S(r, g). \]

(3.14)

From Eqs. (2.6), (2.7), (3.14), and \( n > 7k + 3m + 12 \), we get a contradiction. Thus, by Lemma 6, \( F = \frac{(b+1)G+(a-b-1)}{bG+(a-b)} \), where \( a \neq 0 \), \( b \) are two constants. Then by Lemma 8, we can prove Theorem 3.

**Proof of Theorem 4.** Obviously, we get

\[ \tilde{N}_L \left( r, \frac{1}{F-1} \right) \leq N \left( r, \frac{F}{F} \right) \leq N \left( r, \frac{F}{F} \right) + S(r, f) \]
\[ \leq \tilde{N}(r, F) + \tilde{N} \left( r, \frac{1}{F} \right) + s(r, f) \]
\[ \leq (2k + m + 2)T(r, f) + S(r, f). \]

(3.15)

Similarly, we have

\[ \tilde{N}_L \left( r, \frac{1}{G-1} \right) \leq (2k + m + 2)T(r, g) + S(r, f). \]

(3.16)

Suppose that \( F \) and \( G \) satisfied (i) in Lemma 7, we get

\[ T(r, F) + T(r, G) \]
\[ \leq 2 \left\{ N_2(r, F) + N_2 \left( r, \frac{1}{F} \right) + N_2(r, G) + N_2 \left( r, \frac{1}{G} \right) \right\} + 3\tilde{N}_L \left( r, \frac{1}{F-1} \right) \]
\[ + 3\tilde{N}_L \left( r, \frac{1}{G-1} \right) + S(r, f) + S(r, g). \]

(3.17)

Combining Eqs. (3.3), (3.4), (3.15), (3.16), and (3.17), we have

\[ T(r, F) + T(r, G) \leq (4k + 2m + 8)[T(r, f) + T(r, g)] + 3[2k + m + 2]T(r, f) + 3[2k + m + 2]T(r, g) + S(r, f) + S(r, g) \]
\[ \leq (4k + 2m + 8)[T(r, f) + T(r, g)] + (6k + 3m + 6) \]
\[ [T(r, f) + T(r, g)] + S(r, f) + S(r, g) \]
\[(n + m + k)[T(r, f) + T(r, g)] \leq (10k + 5m + 14)[T(r, f) + T(r, g)] \]
(n - 9k - 4m - 14)\[T(r,f) + T(r,g)\] \leq S(r,f) + S(r,g). \quad (3.18)

From Eqs. (2.6), (2.7), (3.18), and \(n > 9k + 4m + 14\), we get a contradiction. Thus, by Lemma 7, 

\[F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)}\]

is a solution, where \(a(\neq 0), b\) are two constants. Then by Lemma 8, we can prove Theorem 4.

REFERENCES