Remark 4.2: It is a graph with $\frac{(n-2)\,l+4}{2}$ vertices and $\frac{n+l\,(n-2)+2}{2}$ edges if n is even and $\frac{(n-3)\,l+6}{2}$ vertices and $\frac{n+l\,(n-3)+3}{2}$ edges if n is odd.

Parallel P_l chord coloring procedure

***Procedure* 4.1:** The parallel P_l chords joining vertices v_i and v_{n-i} of the cycle C_n are colored in the following manner.

The chord joining vertices v_i and v_{n-i}, $i = 1, 2, 3, ..., \alpha$ is a path on l vertices. Let the internal vertices of path P_l be labeled as $v_{i,\,1}, v_{i,\,2}, v_{i,\,3}, \ldots, v_{i,\,(l-2)}$. In the parallel P_l chords, the right most edge $(v_{i,\,1}, v_i)$ is colored first, followed by the coloring of the leftmost edge $(v_{n-i}, v_{i,\,(l-2)})$. Next the second rightmost edge $(v_{i,\,2}, v_{i,\,1})$ is colored, followed by the coloring of the second leftmost edge $(v_{i,\,(l-2)}, v_{i,\,(l-3)})$. This manner of coloring is continued till all the edges of the chord are colored.

We now give an algorithm for skew edge coloring of a cycle C_n with parallel P_l chords, $3 \le l \le 10$.

Algorithm 4.2

Input: A cycle C_n with parallel P_l chords.

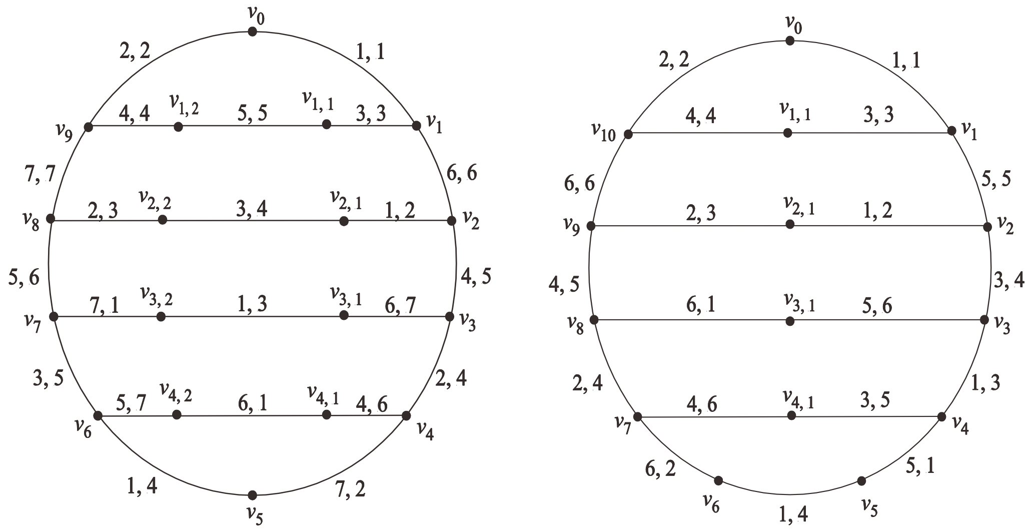
Step 1: Find the smallest positive integer '*k*' satisfying $\binom{k+1}{2} \ge m$, where m denotes the number of edges of the given graph.

Step 2: Consider the set of colors {1, 2, 3,…, *k*} so that $\binom{k+1}{2}$ unordered pairs of colors are available for skew edge coloring of the given graph.

Step 3: Consider the vertex v_0. The edge (v_0, v_1) is colored first, followed by the coloring of the edge (v_0, v_{n-1}). Next the P_l chord (v_1, v_{n-1}) is colored as per the procedure 4.1. In this manner, for each $i = 1, 2, 3, ..., \alpha$, $j = n - i$, the edges (v_i, v_{i+1}), (v_j, v_{j-1}) and the P_l chord (v_{i+1}, v_{j-1}) are colored taken in order. Assignment of colors in this step is done using the ***Algorithms A or B*** of step 3 of ***Algorithm 4.1*** depending upon the nature of '*k*' which may be odd or even.

Output: Skew edge coloring of the cycle C_n with parallel P_l chords.

Remark 4.3: The above algorithm holds good for all cycles C_n with parallel P_l chords where $l = 2i+1$ and $l = 2i+2$, for $n \geq 2i+8$, $i = 1, 2, 3, 4$. See Figure 5.



(a) (b)

Figure 5: (a) A cycle C_{10} with parallel P_4 chords and $s(G) = 7$.

(b) A cycle C_{11} with parallel P_3 chords and $s(G) = 6$.

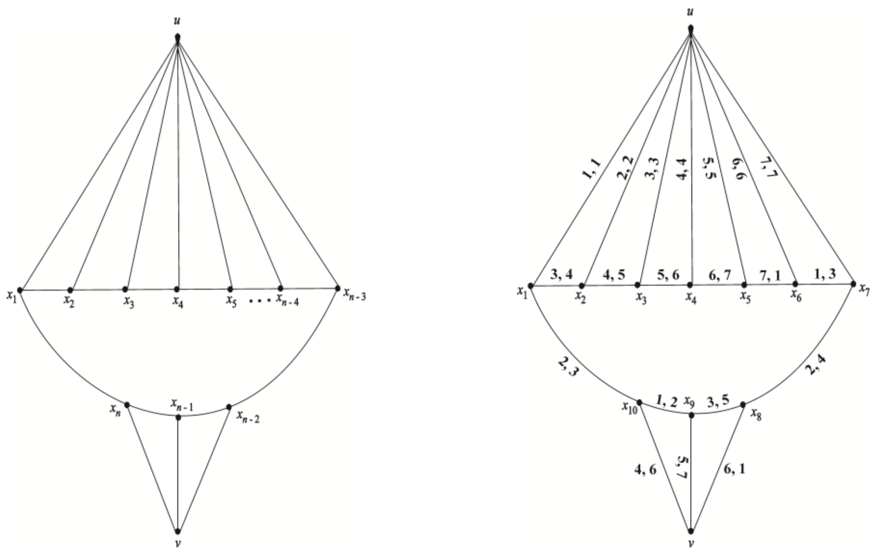
Theorem 4.3: Let G be a cycle C_n with parallel P_l chords, $3 \leq l \leq 10$ and n is even. Then $s(G) = \left\lceil \frac{-1+\sqrt{4n-4l(2-n)+9}}{2} \right\rceil$.

Theorem 4.4: Let G be a cycle C_n with parallel P_l chords, $3 \leq l \leq 10$ and n is odd. Then $s(G) = \left\lceil \frac{-1+\sqrt{4n-4l(3-n)+13}}{2} \right\rceil$.

(c) Double Headed Circular Fan

In this section, we obtain the skew chromatic index of double headed circular fan $DF(n)$.

Definition 4.4: [8] Let $C: x_1x_2x_3\ldots x_nx_1$ be a cycle on *n* vertices. For $x_i \in V(C)$, the graph obtained by adding edges $(x_i, u), i = 1, 2, ..., n-3$ and $(x_i, v), i = n-2, n-1, n$ to *C* is called a double headed circular fan. It is denoted by *DF*(*n*). The new edges are called spokes of *DF*(*n*). See Figure 6.



(a) (b)

Figure 6: (a) *DF*(n), (b) skew edge colored *DF*(10)

Now, we give an algorithm for skew edge coloring of *DF*(*n*) for $n \geq 9$.

***Algorithm 4.3*:**

Input: A double headed circular fan *DF*(*n*), $n \geq 9$.

Step 1: Find *deg*(*u*). Let it be '*k*' (say).

Step 2: Consider the list of colors {1, 2, 3, …, *k*} so that $\binom{k+1}{2}$ pairs are available for skew edge coloring of *DF*(*n*).

Step 3: Assign the pairs of colors of the form (*i*, *i*) to the spokes (x_i, *u*), *i* = 1, 2, 3, …, *n* – 3.

Step 4: Color the edges of the cycle $C_n: x_1x_2x_3\ldots x_nx_1$.

The first *k* –1 edges of the cycle $C_n: x_{n-1}x_nx_1x_2x_3\ldots x_{n-3}x_{n-2}x_{n-1}$ are assigned the colors of the form (*i*, *i* +1), *i* = 1, 2, 3,…, *k* – 1. The next edge is assigned the color (*k*, 1). The remaining $n - k$ edges of the cycle C_n are assigned the colors of the form $(i, i + 2)$, i = 1, 2, 3.

Step 5:

Case (i) If $k = 6$

The spokes (x_i, v), $i = n, n - 1, n - 2$ are assigned the next consecutive colors taken in order of the form (4, 6), (5, 1), (6, 2) respectively.

Case (ii) If $k = 7$

The spokes (x_i, v), $i = n, n - 1, n - 2$ are assigned the next consecutive colors taken in order of the form (4, 6), (5, 7), (6, 1) respectively.

Case (iii) If $k \geq 8$

The spokes (x_i, v), $i = n, n - 1, n - 2$ are assigned the next consecutive colors taken in order of the form $(i, i + 2)$, i = 4, 5, 6 respectively.

Output: Skew edge coloring of $DF(n)$.

Proof of correctness**:** As 'k' edges are incident on the vertex u, at least 'k' distinct colors are needed for skew edge coloring of $DF(n)$. Thus colors of the form (i, i) are assigned to the spokes (x_i, u), $i = 1, 2, 3,\ldots, k$. The edges (x_i, x_{i+1}), $i = 1, 2, 3,\ldots, n - 4$ of the cycle C_n are adjacent to the spokes (x_i, u) and (x_{i+1}, u), $i = 1, 2, 3, \ldots, n - 4$. Hence (x_i, x_{i+1}), $i =1, 2, 3,\ldots, n - 4$ are assigned the colors $(i, i + 1)$, $i = 3, 4,\ldots, k - 1$. The next edge (x_{n-4}, x_{n-5}) is assigned the color $(k, 1)$ which is adjacent to the spokes (x_{n-5}, u) and (x_{n-4}, u). The remaining $n - k$ edges of the cycle C_n are assigned the colors $(i, i + 2)$, i = 1, 2, 3 respectively. As 'k' colors are used, depending on the value of 'k', the spokes (x_i, v), $i = n, n - 1, n - 2$ are assigned the colors of the form $(i, i + 2)$, i = 4, 5, 6 taken in order for $k \geq 8$ thus resulting in a skew edge coloring of $DF(n)$.

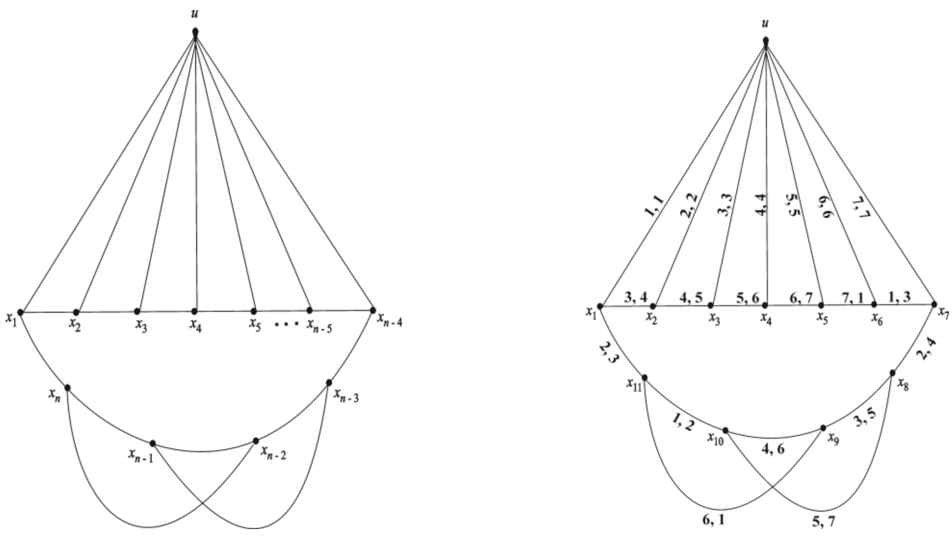
Theorem 4.3: Let G be a $DF(n)$, $n \geq 9$. Then $s(DF(n)) = n - 3$.

Proof: Proof follows directly from the algorithm.

(d) Circular Fan with Two Chords

In this section, we obtain the skew chromatic index of circular fan with two chords $F(n, 2)$.

Definition 4.5: [8] Let $C: x_1x_2x_3\ldots x_nx_1$ be a cycle on *n* vertices. Let *u* be a new vertex. The graph obtained by adding edges $(x_i, u), i = 1, 2, \ldots, n-4$ to *C* along with two chords (x_{n-1}, x_{n-3}) and (x_n, x_{n-2}) is called a circular fan with two chords and is denoted by *F*(*n*, 2). The new edges are called spokes of *F*(*n*, 2). See Figure 7.



(a) (b)

Figure 7: (a) *F*(*n*, 2) (b) skew edge colored *F*(11, 2)

Now, we give an algorithm to skew edge color *F*(*n*,2) for $n \geq 12$.

***Algorithm 4.4**:*

Input: A circular fan with two chords *F*(*n*, 2), $n \geq 12$.

Step 1: Find deg(*u*). Let it be '*k*' (say).

Step 2: Consider the list of colors {1, 2, 3, …, *k*} so that $\binom{k+1}{2}$ pairs are available for skew edge coloring of *F*(*n*, 2).

Step 3: Assign the pairs of colors of the form (*i*, *i*) to the spokes (x_i, *u*), *i* = 1, 2, 3,…, *n* – 4.

Step 4: Color the edges of the cycle $C_n : x_1x_2x_3\ldots x_nx_1$.

The first *k* – 1 edges of the cycle $C_n : x_{n-1}x_nx_1x_2x_3\ldots x_{n-3}x_{n-2}x_{n-1}$ are assigned the colors of the form (*i*, *i* +1), *i* = 1, 2, 3,…*k* –1. The next edge is assigned the color (*k*, 1). The remaining *n* – *k* edges of the cycle C_n are assigned the colors of the form (*i*, *i* + 2), *i* = 1, 2, 3, 4.

Step 5: The two chords (x_{n-1}, x_{n-3}) and (x_n, x_{n-2}) are assigned the pairs of colors (i, i + 2), i = 5, 6 respectively.

Output: Skew edge coloring of $F(n, 2)$.

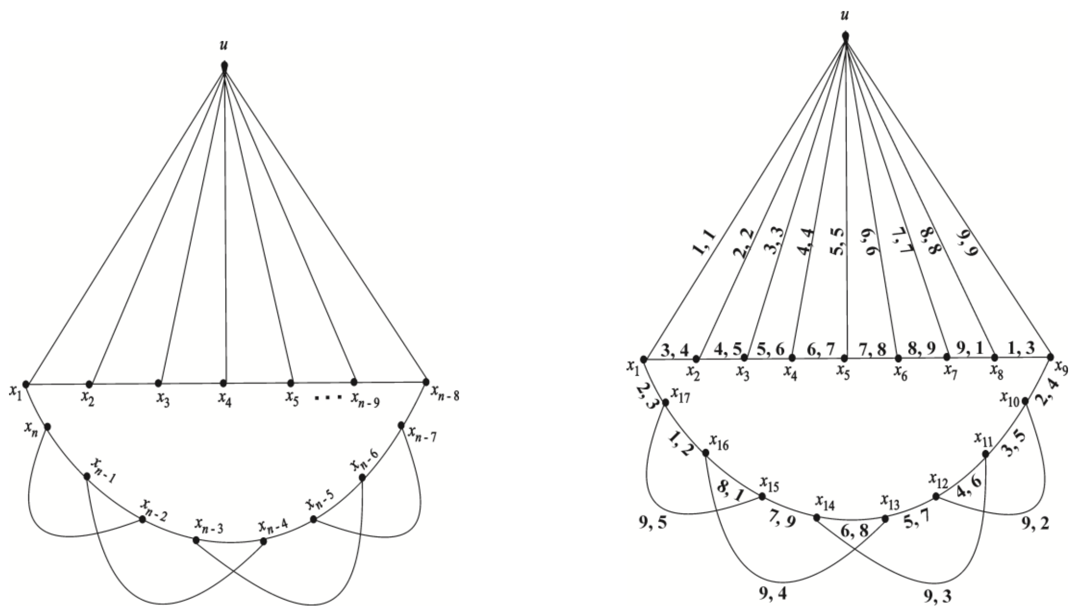
***Proof of correctness*:** Proof follows directly from the algorithm.

Theorem 4.4: Let G be a $F(n, 2)$, $n \geq 12$. Then $s(G) = n - 4$.

(e) Circular Fan with Four Chords

In this section, we obtain the skew chromatic index of circular fan with four chords $F(n, 4)$.

Definition 4.6: [8] Let $C: x_1x_2x_3\ldots x_nx_1$ be a cycle on n vertices. Let u be a new vertex. The graph obtained by adding edges $(x_i, u), i = 1, 2, \ldots, n-8$ to C along with four chords $(x_n, x_{n-2}), (x_{n-1}, x_{n-4}), (x_{n-3}, x_{n-6})$ and (x_{n-5}, x_{n-7}) is called a circular fan with four chords and is denoted by $F(n, 4)$. The new edges are called spokes of $F(n, 4)$. See Figure 8.



(a) (b)

Figure 8: (a) $F(n, 4)$ (b) skew edge colored $F(17, 4)$

Now, we give an algorithm to skew edge color $F(n, 4)$ for $n \geq 17$.

Algorithm 4.5:

Input: A circular fan with four chords $F(n, 4)$, $n \geq 17$.

Step 1: Find deg(v). Let it be 'k' (say).

Step 2: Consider {1, 2, 3,…, k} as the list of colors so that $\binom{k+1}{2}$ pairs are available for skew edge coloring of $F(n, 4)$.

Step 3: Assign the pairs of the form (i, i) to the spokes (x_i, u), i = 1, 2, 3,…, n – 8.

Step 4: Color the edges of the cycle $C_n : x_1x_2x_3\ldots x_nx_1$.

The first k –1 edges of the cycle $C_n : x_{n-1}x_nx_1x_2x_3\ldots x_{n-3}x_{n-2}x_{n-1}$ are assigned the colors of the form (i, i +1), i = 1, 2, 3,…, k –1. The next edge (x_{n-6}, x_{n-7}) is assigned the color (k, 1). The remaining n – k edges of the cycle C_n are assigned the colors of the form (i, i + 2), i = 1, 2, 3, ...,8.

Step 5: The four chords (x_{n-7}, x_{n-5}), (x_{n-6}, x_{n-3}), (x_{n-4}, x_{n-1}) and (x_{n-2}, x_n) are assigned the pairs of colors (k, i), i = 2, 3, 4, 5 respectively.

Output: Skew edge coloring of $F(n, 4)$.

Theorem 4.5: Let G be a $F(n, 4)$, $n \geq 17$. Then $s(G) = n - 4$.

***Time Complexity analysis of algorithm*:**

The time complexity analysis of the above mentioned algorithms depends on the size of the graph. The algorithm may be implemented to run in $O(m)$, where m is the number of edges of G.

CONCLUSION

In this paper we have developed algorithms for certain cycle related graphs and obtained an optimal solution for $s(G)$. We have already obtained the same for comb, ladder, Mobius ladder and Circular ladder graphs. Finding efficient algorithms to determine skew edge coloring for other interconnection networks is quite challenging.

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