Skew Chromatic Index of Cycle Related Graphs

Joice Punitha M. and S. Rajakumari

1Department of Mathematics, Bharathi Women’s College(Autonomous), Chennai - 600 108, Tamil Nadu, India.
2Department of Mathematics, R. M. D. Engineering College, Kavaraipettai - 601 206, Tamil Nadu, India.

Abstract

A skew edge coloring of a graph \( G \) is defined as a set of two edge colorings such that no two edges are assigned the same unordered pair of colors. The skew chromatic index \( s(G) \) is the minimum number of colors required for a skew edge coloring of \( G \). In this paper, we develop algorithms for skew edge coloring of certain cycle related graphs. Also the skew chromatic index of those cycle related graphs is solved in polynomial time.

Keywords: skew edge coloring; skew chromatic index; cycle related graphs; NP - complete.

1. INTRODUCTION

Interconnection networks are becoming increasingly pervasive in many different applications. The advent of very large scale integrated circuits has enabled the construction of complex interconnection networks. Graph theory is a fundamental and powerful mathematical tool for designing and analyzing topological structure of interconnection networks. An interconnection network can be modeled as a simple graph whose vertices represent components of a network and whose edges represent physical communication links. Topologically, graphs and interconnection networks are the same [12].

Graph coloring problems are fundamental and important problems of graph theory. It has a wide range of applications in scheduling, register allocation in compilers,
bandwidth allocation, timetable designing and pattern matching, and so on. Edge coloring problems have a variety of applications and are well studied in both computer science and mathematics [5]. Edge coloring problems are used to model various scheduling problems. In cellular communication, frequency reusing is done by modeling it as an edge coloring problem to avoid co-channel interference [9].

Let $G = (V, E)$ be a finite, simple, connected graph with vertex set $V$ and edge set $E$. A proper edge coloring of a graph $G$ is an assignment of colors to the edges of $G$ so that no two adjacent edges are assigned the same color [1]. The minimum number of colors required for such an edge coloring of $G$ is the edge chromatic number or the chromatic index and is denoted by $\chi'(G)$. The degree of a vertex of a graph $G$ is the number of edges incident to the vertex [2]. The maximum degree of a graph $G$ is denoted by $\Delta(G)$. Vizing [11] has shown that for any graph $G$, $\chi'(G)$ is either $\Delta(G)$ or $\Delta(G) + 1$. The problem of determining the chromatic index of an arbitrary graph is a difficult task. Holyer [7] has proved that edge coloring problem is $NP$-complete.

2. AN OVERVIEW OF THE PAPER

In this paper, we consider skew edge coloring problems that are inspired from the study of skew Room squares by R. A. Braudil [6]. The concept of skew chromatic index was introduced by Marsha. F. Foregger and better upper bounds for $s(G)$ in terms of its order, was discussed when $G$ is cyclic, cubic or bipartite [6].

A skew edge coloring of a graph $G$ is defined to be a set of two edge colorings such that no two edges are assigned the same unordered pair of colors. Equivalently, a skew edge coloring of $G$ is an assignment of an ordered pair of colors $(a_i, b_i)$ to each edge $e_i$ of $G$ such that

(i) the $a_i$’s form an edge coloring of $G$,

(ii) the $b_i$’s form an edge coloring of $G$, and

(iii) the pairs $\{a_i, b_i\}$ are all distinct.

The two edge colorings are referred to as component colorings of the skew edge coloring. The skew chromatic index $s(G)$ is the minimum number of colors required for a skew edge coloring of $G$. We see that $s(K_3) = 3$ where the two component colorings of $K_3$ are 1, 2, 3 and 2, 3, 1 respectively and not less than three colors can be used for skew edge coloring of $K_3$. See Figure 1.
The terminologies used in this paper are as in [3]. In this paper, algorithms are determined for skew edge coloring of a cycle \( C_n \) of length of \( n \) with parallel chords, a cycle \( C_n \) with parallel \( P_l \) chords, where \( P_l \) denotes a path of order \( l \), \( 3 \leq l \leq 10 \), a double headed circular fan, a circular fan with two chords, a circular fan with four chords. We have also solved the skew chromatic index of those cycle related graphs in polynomial time.

### 3. LOWER BOUND ON \( s(G) \)

Skew chromatic index, \( s(G) \) is defined as the minimum number of colors used in two edge colorings of \( G \) such that no two edges are assigned the same unordered pair of colors. Since each component coloring of a skew edge coloring is itself an edge coloring, it is obvious that \( s(G) \geq \chi'(G) \). By Vizing's theorem, \( \Delta(G) \leq \chi'(G) \leq \Delta(G) + 1 \), where \( \Delta(G) \) is the maximum degree of vertices in \( G \). Hence we have \( s(G) \geq \Delta(G) \). If \( 'k' \) colors are used for skew edge coloring, then there are \( \binom{k+1}{2} \) unordered pairs of colors [4] and this number must be at least as large as the number of edges in \( G \). Let \( k(m) \) denote the smallest integer \( 'k' \) satisfying \( \binom{k+1}{2} \geq m \) where \( 'm' \) denotes the number of edges in \( G \). Thus we have \( s(G) \geq k(\lfloor |E(G)| \rfloor) \). Hence the best lower bound for \( s(G) \) is \( s(G) \geq \max \{\Delta(G), k(\lfloor |E(G)| \rfloor)\} \) as stated in [6]. In this paper, we prove that the bound on skew chromatic index given here is sharp for the certain cycle related graphs like cycle \( C_n \) of length of \( n \) with parallel chords, a cycle \( C_n \) with parallel \( P_l \) chords, where \( P_l \) denotes a path of order \( l \), \( 3 \leq l \leq 10 \), a double headed circular fan, a circular fan with two chords and a circular fan with four chords.
4. CYCLE RELATED GRAPHS

In this section, we determine the skew chromatic index of a cycle $C_n$ with parallel chords and with parallel $P_l$ chords for $3 \leq l \leq 10$, a double headed circular fan, a circular fan with two chords and a circular fan with four chords. Here $C_n$ denote a cycle of length $n$ and $P_l$ denotes a path of order $l$.

(a) Cycle with parallel chords

In this section, we determine the skew chromatic index of cycle $C_n$ with parallel chords for $n \geq 8$.

Definition 4.1: [10] A chord of a cycle $C$ is an edge joining two non-adjacent vertices of a cycle.

Definition 4.2: [10] A graph $G$ is called a cycle with parallel chords if $G$ is obtained from the cycle $C_n: v_0v_1v_2\ldots v_{n-1}v_0, (n \geq 6)$ by adding the chords between each pair of non-adjacent vertices $v_0, v_1, v_2, \ldots, v_{n-1}, v_0$, $\alpha = \left\lfloor \frac{n}{2} \right\rfloor - 1$, $\beta = \left\lceil \frac{n}{2} \right\rceil + 1$, when $n$ is even and $\alpha = \left\lceil \frac{n}{2} \right\rceil - 1$, $\beta = \left\lceil \frac{n}{2} \right\rceil + 2$, when $n$ is odd. See Figure 2.

![Figure 2](image)

(a) Cycle $C_n$ with parallel chords when $n$ is even

(b) Cycle $C_n$ with parallel chords when $n$ is odd.
Remark 4.1: It is a graph with \( n \) vertices and \( \frac{3n - 2}{2} \) edges if \( n \) is even and \( \frac{3n - 3}{2} \) edges if \( n \) is odd. It is observed that \( v_\beta = v_{n-\alpha} \), and \( v_{\beta - 1} = v_{\alpha + 1} \) or \( v_{\alpha + 2} \) depending on whether \( n \) is even or odd.

We now give an algorithm for skew edge coloring of a cycle \( C_n \) with parallel chords, \( n \geq 8 \).

**Algorithm 4.1**

**Input**: A cycle \( C_n \) with parallel chords, \( n \geq 8 \).

**Step 1**: Find the smallest positive integer ‘\( k \)’ satisfying \( \left( \frac{1}{2} \right) k + m \geq \frac{k+1}{2} \), where \( m \) denotes the number of edges of the graph.

**Step 2**: Consider the set of colors \( \{1, 2, 3, \ldots, k\} \), then there are \( \left( \frac{k+1}{2} \right) \) unordered pairs of colors available for skew edge coloring of the graph.

**Step 3**: Consider the vertex \( v_0 \). The edge \( (v_0, v_1) \) is colored first, followed by the coloring of the edge \( (v_0, v_{n-1}) \). Next the chord \( (v_i, v_{j+1}) \) is colored. In this manner, the three edges viz., \( (v_i, v_{i+1}) \), \( (v_j, v_{j+1}) \) and the chord \( (v_{i+1}, v_{j+1}) \), \( i = 1, 2, 3, \ldots, \alpha - 1, j = n - i \) are colored. If \( n \) is even, the edges \( (v_{\alpha}, v_{\alpha+1}) \) and \( (v_{\beta}, v_{\beta-1}) \) are colored taken in order. If \( n \) is odd, the edges \( (v_{\alpha}, v_{\alpha+1}) \), \( (v_{\beta}, v_{\beta-1}) \) and \( (v_{\alpha+1}, v_{\beta-1}) \) are colored taken in order.

Assignment of colors in this step is done using the following Algorithms A or B depending upon the nature of ‘\( k \)’ which may be odd or even.

**Algorithm A**: \( k \) is odd.

**Case (i)**: \( m = 0 \mod k \)

In the aforementioned method, the first ‘\( k \)’ edges are assigned colors from the set \( \{(c, c), c = 1, 2, \ldots, k\} \) taken in order. The remaining edges are assigned colors from the set

\[
\{(c, c + d), c = 1, 2, \ldots, (k-d)\} \cup \{(k-d) + i, i = 1, 2, \ldots, d\} \quad \text{for each } d=1, 2, 3, \ldots, \left\lfloor \frac{k}{2} \right\rfloor
\]

taken in order. Here all the \( \left( \frac{k+1}{2} \right) \) pairs of colors are used.

**Case (ii)**: \( m = r \mod k \), \( r = 1, 2, 3, \ldots, k - 1 \).

In the aforementioned method, the first ‘\( k \)’ edges are assigned colors from the set \( \{(c, c), c = 1, 2, \ldots, k\} \) taken in order. The remaining edges are assigned colors from the set
Algorithm B: $k$ is even.

Case (i): $m = r \pmod{(k + 1)}$, $r = 0, 1$.

In the aforementioned method, the first ‘$k$’ edges are assigned colors from the set \{(c, c), c = 1, 2, ..., k\} taken in order. The remaining edges are assigned colors from the set \{(c, c + d), c = 1, 2, ..., (k - d)\} \cup \{(k - d) + i, i, i = 1, 2, ..., d\} for each $d = 1, 2, 3, ..., \left\lfloor \frac{k}{2} \right\rfloor$ taken in order. Here all the \(\binom{k+1}{2}\) pairs of colors are used.

Case (ii): $m = r \pmod{(k + 1)}$, $r = 2, 3, ..., k$.

In the aforementioned method, the first ‘$k$’ edges are assigned colors from the set \{(c, c), c = 1, 2, ..., k\} taken in order. The remaining edges are assigned colors from the set \{(c, c + d), c = 1, 2, ..., (k - d)\} \cup \{(k - d) + i, i, i = 1, 2, ..., d\} for each $d = 1, 2, 3, ..., \left\lfloor \frac{k}{2} \right\rfloor$ taken in order. Here the last \((k + 1) - r\) pairs of colors are left unused.

Output: Skew edge coloring of cycle $C_n$ with parallel chords.

Proof of correctness: As there are $m$ edges, fix ‘$k$’ in such a way that \(\binom{k+1}{2}\geq m\), so that \(\binom{k+1}{2}\) pairs are available for skew edge coloring of the graph. First ‘$k$’ edges are assigned the colors in such a way that all ordered pairs of the form (c, c), c = 1, 2, 3, ..., k are used. In the second set of ‘$k$’ edges, the first $k - 1$ edges are assigned the colors of the form (c, c + 1), c = 1, 2, 3, ..., k - 1. The $k^{th}$ edge is assigned the color (k, 1) as the ordered pair (k, k + 1) is not permissible. In the third set of ‘$k$’ edges, the first $k - 2$ edges are assigned the colors of the form (c, c + 2), c = 1, 2, 3, ..., k - 2. The $(k - 1)^{th}$ edge and the $k^{th}$ edge are assigned the colors ($(k - 2) + i, i, i = 1, 2$ respectively as the ordered pairs $(k - 1, k + 1)$ and $(k, k + 2)$ are not permissible. This process is continued till all the edges are colored and the above method of edge coloring results in skew edge coloring of the given graph. See Figure 3.
Theorem 4.1: Let $G$ be a cycle $C_n$, $n \geq 8$ with parallel chords and $m = \frac{3n-2}{2}$ edges, $n$ is even. Then $s(G) = k = \left\lfloor \frac{-1 + \sqrt{12n-7}}{2} \right\rfloor$.

Proof: As there are $m$ edges, at least $m$ pairs of colors are required for skew edge coloring of $G$. If ‘$k$’ colors are used for coloring, then there are $\binom{k+1}{2}$ unordered pairs. This $\binom{k+1}{2}$ must be at least as large as the number of edges in $G$. Therefore fix ‘$k$’ in such a way that $\binom{k+1}{2} \geq m$. i.e. $\frac{(k+1)k}{2} \geq \frac{3n-2}{2}$. It follows that $k^2+k \geq 3n-2$. i.e. $k^2+k-(3n-2) \geq 0$. Thus solving for ‘$k$’ and taking the positive root, we obtain $k = \frac{-1+\sqrt{12n-7}}{2} \geq 0$, and its greatest integer $k = \left\lfloor \frac{-1+\sqrt{12n-7}}{2} \right\rfloor$ will be minimum number of colors used in skew edge coloring. Hence $s(G) = \left\lfloor \frac{-1+\sqrt{12n-7}}{2} \right\rfloor$.

Theorem 4.2: Let $G$ be a cycle $C_n$, $n \geq 8$ with parallel chords and $m = \frac{3n-3}{2}$ edges, $n$ is odd. Then $s(G) = k = \left\lfloor \frac{-1+\sqrt{12n-11}}{2} \right\rfloor$. 

Figure 3: (a) Skew edge coloring of $C_{10}$ with parallel chords and $s(G) = 5$. (b) Skew edge coloring of $C_{13}$ with parallel chords and $s(G) = 6$. 


Proof: Proof is similar to theorem 4.1.

Thus based on the lower bound for $s_t(G)$, we obtain an optimal solution for skew chromatic index of cycle $C_n$ with parallel chords for $n \geq 8$.

(b) Cycle with parallel $P_l$ chords

In this section, we determine the skew chromatic index of cycle $C_n$ with parallel $P_l$ chords for $3 \leq l \leq 10$, where $C_n$ denotes a cycle of length $n$ and $P_l$ denotes a path of order $l$.

Definition 4.3: [10] A cycle with parallel $P_l$ chords is defined as a graph obtained from a cycle $C_n$, $n \geq 6$ with consecutive vertices $v_0, v_1, v_2, \ldots, v_n$ by adding disjoint paths(chords) of order $l$ namely $P_l$, $l \geq 3$ between each pair of non-adjacent vertices $(v_{\alpha}, v_{\beta})$, $v_{\alpha} = \left\lfloor \frac{n}{2} \right\rfloor - 1$, $v_{\beta} = \left\lceil \frac{n}{2} \right\rceil + 1$, when $n$ is even and

$\alpha = \left\lfloor \frac{n}{2} \right\rfloor - 1$, $\beta = \left\lceil \frac{n}{2} \right\rceil + 2$, when $n$ is odd. See Figure 4.

Figure 4: (a) A cycle $C_n$ with parallel $P_l$ chords when $n$ is even
(b) A cycle $C_n$ with parallel $P_l$ chords when $n$ is odd.
Remark 4.2: It is a graph with \( \frac{(n - 2)l + 4}{2} \) vertices and \( \frac{n + l(n - 2) + 2}{2} \) edges if \( n \) is even and \( \frac{(n - 3)l + 6}{2} \) vertices and \( \frac{n + l(n - 3) + 3}{2} \) edges if \( n \) is odd.

Parallel \( P_l \) chord coloring procedure

**Procedure 4.1:** The parallel \( P_l \) chords joining vertices \( v_i \) and \( v_{n-i} \) of the cycle \( C_n \) are colored in the following manner.

The chord joining vertices \( v_i \) and \( v_{n-i} \), \( i = 1, 2, 3, \ldots, \alpha \) is a path on \( l \) vertices. Let the internal vertices of path \( P_l \) be labeled as \( v_{i, 1}, v_{i, 2}, v_{i, 3}, \ldots, v_{i, u-2} \). In the parallel \( P_l \) chords, the right most edge \( (v_{i, 1}, v_i) \) is colored first, followed by the coloring of the leftmost edge \( (v_{n-i, u-1}, v_{n-i}) \). Next the second rightmost edge \( (v_{i, 2}, v_{i, 1}) \) is colored, followed by the coloring of the second leftmost edge \( (v_{(u-2), 1}, v_{(u-2)}) \). This manner of coloring is continued till all the edges of the chord are colored.

We now give an algorithm for skew edge coloring of a cycle \( C_n \) with parallel \( P_l \) chords, \( 3 \leq l \leq 10 \).

**Algorithm 4.2**

*Input:* A cycle \( C_n \) with parallel \( P_l \) chords.

**Step 1:** Find the smallest positive integer ‘\( k \)’ satisfying \( \binom{k+1}{2} \geq m \), where \( m \) denotes the number of edges of the given graph.

**Step 2:** Consider the set of colors \{1, 2, 3, \ldots, \( k \)\} so that \( \binom{k+1}{2} \) unordered pairs of colors are available for skew edge coloring of the given graph.

**Step 3:** Consider the vertex \( v_0 \). The edge \( (v_0, v_i) \) is colored first, followed by the coloring of the edge \( (v_0, v_{n-i}) \). Next the \( P_l \) chord \( (v_i, v_{n-i}) \) is colored as per the procedure 4.1. In this manner, for each \( i = 1, 2, 3, \ldots, \alpha \), \( j = n-i \), the edges \( (v_i, v_{i+j}), (v_j, v_{j-i}) \) and the \( P_l \) chord \( (v_{i+j}, v_{j-i}) \) are colored taken in order. Assignment of colors in this step is done using the **Algorithms A or B** of step 3 of **Algorithm 4.1** depending upon the nature of ‘\( k \)’ which may be odd or even.
*Output:* Skew edge coloring of the cycle $C_n$ with parallel $P_l$ chords.

**Remark 4.3:** The above algorithm holds good for all cycles $C_n$ with parallel $P_l$ chords where $l = 2i + 1$ and $l = 2i + 2$, for $n \geq 2i + 8$, $i = 1, 2, 3, 4$. See Figure 5.

![Figure 5](image)

(a) A cycle $C_{10}$ with parallel $P_4$ chords and $s(G) = 7$.

(b) A cycle $C_{11}$ with parallel $P_3$ chords and $s(G) = 6$.

**Theorem 4.3:** Let $G$ be a cycle $C_n$ with parallel $P_l$ chords, $3 \leq l \leq 10$ and $n$ is even. Then
\[
s(G) = \left\lceil \frac{-1 + \sqrt{4n - 4l(2 - n) + 9}}{2} \right\rceil.
\]

**Theorem 4.4:** Let $G$ be a cycle $C_n$ with parallel $P_l$ chords, $3 \leq l \leq 10$ and $n$ is odd. Then
\[
s(G) = \left\lceil \frac{-1 + \sqrt{4n - 4l(3 - n) + 13}}{2} \right\rceil.
\]

(c) **Double Headed Circular Fan**

In this section, we obtain the skew chromatic index of double headed circular fan $DF(n)$. 
Definition 4.4: [8] Let \( C : x_1, x_2, x_3, \ldots, x_n \) be a cycle on \( n \) vertices. For \( x_i \in V(C) \), the graph obtained by adding edges \((x_i, u), i = 1, 2, \ldots, n-3\) and \((x_i, v), i = n-2, n-1, n\) to \( C \) is called a double headed circular fan. It is denoted by \( DF(n) \). The new edges are called spokes of \( DF(n) \). See Figure 6.

Now, we give an algorithm for skew edge coloring of \( DF(n) \) for \( n \geq 9 \).

Algorithm 4.3:

Input: A double headed circular fan \( DF(n) \), \( n \geq 9 \).

Step 1: Find \( \text{deg}(u) \). Let it be ‘\( k \)’ (say).

Step 2: Consider the list of colors \( \{1, 2, 3, \ldots, k\} \) so that \( \binom{k+1}{2} \) pairs are available for skew edge coloring of \( DF(n) \).

Step 3: Assign the pairs of colors of the form \((i, i)\) to the spokes \((x_i, u), i = 1, 2, 3, \ldots, n-3\).

Step 4: Color the edges of the cycle \( C_z : x_1, x_2, x_3, \ldots, x_n \).

The first \( k - 1 \) edges of the cycle \( C_z : x_1, x_2, x_3, \ldots, x_n \) are assigned the colors of the form \((i, i+1), i = 1, 2, 3, \ldots, k-1\). The next edge is assigned the color \((k, 1)\). The
remaining \(n - k\) edges of the cycle \(C_n\) are assigned the colors of the form \((i, i + 2)\), \(i = 1, 2, 3\).

**Step 5:**

*Case (i) If \(k = 6\)*

The spokes \((x_i, v), i = n, n - 1, n - 2\) are assigned the next consecutive colors taken in order of the form \((4, 6), (5, 1), (6, 2)\) respectively.

*Case (ii) If \(k = 7\)*

The spokes \((x_i, v), i = n, n - 1, n - 2\) are assigned the next consecutive colors taken in order of the form \((4, 6), (5, 7), (6, 1)\) respectively.

*Case (iii) If \(k \geq 8\)*

The spokes \((x_i, v), i = n, n - 1, n - 2\) are assigned the next consecutive colors taken in order of the form \((i, i + 2), i = 4, 5, 6\) respectively.

Output: Skew edge coloring of \(DF(n)\).

**Proof of correctness:** As ‘\(k\)’ edges are incident on the vertex \(u\), at least ‘\(k\)’ distinct colors are needed for skew edge coloring of \(DF(n)\). Thus colors of the form \((i, i)\) are assigned to the spokes \((x_i, u), i = 1, 2, 3, \ldots, k\). The edges \((x_i, x_{i+1}), i = 1, 2, 3, \ldots, n - 4\) of the cycle \(C_n\) are adjacent to the spokes \((x_i, u)\) and \((x_{i+1}, u), i = 1, 2, 3, \ldots, n - 4\). Hence \((x_i, x_{i+1}), i = 1, 2, 3, \ldots, n - 4\) are assigned the colors \((i, i + 1), i = 3, 4, \ldots, k - 1\). The next edge \((x_{n-4}, x_{n-5})\) is assigned the color \((k, 1)\) which is adjacent to the spokes \((x_{n-5}, u)\) and \((x_{n-4}, u)\). The remaining \(n - k\) edges of the cycle \(C_n\) are assigned the colors \((i, i + 2), i = 1, 2, 3\) respectively. As ‘\(k\)’ colors are used, depending on the value of ‘\(k\)’, the spokes \((x_i, v), i = n, n - 1, n - 2\) are assigned the colors of the form \((i, i + 2), i = 4, 5, 6\) taken in order for \(k \geq 8\) thus resulting in a skew edge coloring of \(DF(n)\).

**Theorem 4.3:** Let \(G\) be a \(DF(n), n \geq 9\). Then \(s(DF(n)) = n - 3\).

**Proof:** Proof follows directly from the algorithm.

**(d) Circular Fan with Two Chords**

In this section, we obtain the skew chromatic index of circular fan with two chords \(F(n, 2)\).
Definition 4.5: [8] Let $C : x_1x_2x_3 \ldots x_nx_1$ be a cycle on $n$ vertices. Let $u$ be a new vertex. The graph obtained by adding edges $(x_i, u), i = 1, 2, \ldots, n-4$ to $C$ along with two chords $(x_{n-1}, x_{n-3})$ and $(x_n, x_{n-2})$ is called a circular fan with two chords and is denoted by $F(n, 2)$. The new edges are called spokes of $F(n, 2)$. See Figure 7.

![Figure 7: (a) $F(n, 2)$ (b) skew edge colored $F(11, 2)$](image-url)

Now, we give an algorithm to skew edge color $F(n, 2)$ for $n \geq 12$.

Algorithm 4.4:

*Input:* A circular fan with two chords $F(n, 2), n \geq 12$.

*Step 1:* Find $\text{deg}(u)$. Let it be ‘$k$’ (say).

*Step 2:* Consider the list of colors $\{1, 2, 3, \ldots, k\}$ so that $\binom{k+1}{2}$ pairs are available for skew edge coloring of $F(n, 2)$.

*Step 3:* Assign the pairs of colors of the form $(i, i)$ to the spokes $(x_i, u), i = 1, 2, 3, \ldots, n-4$.

*Step 4:* Color the edges of the cycle $C_n : x_1x_2x_3 \ldots x_nx_1$.

The first $k-1$ edges of the cycle $C_n : x_{n-1}x_1x_2x_3 \ldots x_{n-3}x_{n-2}x_{n-1}$ are assigned the colors of the form $(i, i+1), i = 1, 2, 3, \ldots k-1$. The next edge is assigned the color $(k, 1)$. The remaining $n-k$ edges of the cycle $C_n$ are assigned the colors of the form $(i, i+2), i = 1, 2, 3, 4$. 
Step 5: The two chords \((x_{n-1}, x_{n-3})\) and \((x_n, x_{n-2})\) are assigned the pairs of colors \((i, i + 2)\), \(i = 5, 6\) respectively.

Output: Skew edge coloring of \(F(n, 2)\).

Proof of correctness: Proof follows directly from the algorithm.

Theorem 4.4: Let \(G\) be a \(F(n, 2)\), \(n \geq 12\). Then \(s(G) = n - 4\).

(e) Circular Fan with Four Chords

In this section, we obtain the skew chromatic index of circular fan with four chords \(F(n, 4)\).

Definition 4.6: [8] Let \(C : x_1x_2x_3 \cdots x_n\) be a cycle on \(n\) vertices. Let \(u\) be a new vertex. The graph obtained by adding edges \((x_i, u), i = 1, 2, \ldots, n-8\) to \(C\) along with four chords \((x_i, x_{i+2}), (x_i, x_{i+4}), (x_i, x_{i+3}), (x_i, x_{i+5})\) and \((x_{n-5}, x_{n-7})\) is called a circular fan with four chords and is denoted by \(F(n, 4)\). The new edges are called spokes of \(F(n, 4)\). See Figure 8.

![Figure 8: (a) F(n, 4) (b) skew edge colored F(17, 4)](attachment://figure8.png)
Now, we give an algorithm to skew edge color $F(n, 4)$ for $n \geq 17$.

**Algorithm 4.5:**

*Input: A circular fan with four chords $F(n, 4)$, $n \geq 17$.*

*Step 1:* Find $\text{deg}(v)$. Let it be ‘$k$’ (say).

*Step 2:* Consider $\{1, 2, 3, \ldots, k\}$ as the list of colors so that $\binom{k+1}{2}$ pairs are available for skew edge coloring of $F(n, 4)$.

*Step 3:* Assign the pairs of the form $(i, i)$ to the spokes $(x_i, u)$, $i = 1, 2, 3, \ldots, n-8$.

*Step 4:* Color the edges of the cycle $C_n : x_1x_2x_3 \ldots x_n$.

The first $k-1$ edges of the cycle $C_n : x_{i-1}x_i x_{i+1}x_{i+2} \ldots x_{n-2}x_{n-1}$ are assigned the colors of the form $(i, i+1)$, $i = 1, 2, 3, \ldots, k-1$. The next edge $(x_{n-6}, x_{n-7})$ is assigned the color $(k, 1)$. The remaining $n-k$ edges of the cycle $C_n$ are assigned the colors of the form $(i, i + 2)$, $i = 1, 2, 3, \ldots, 8$.

*Step 5:* The four chords $(x_{n-7}, x_{n-6})$, $(x_{n-6}, x_{n-5})$, $(x_{n-4}, x_{n-3})$, and $(x_{n-2}, x_n)$ are assigned the pairs of colors $(k, i)$, $i = 2, 3, 4, 5$ respectively.

*Output:* Skew edge coloring of $F(n, 4)$.

**Theorem 4.5:** Let $G$ be a $F(n, 4)$, $n \geq 17$. Then $s(G) = n - 4$.

**Time Complexity analysis of algorithm:**

The time complexity analysis of the above mentioned algorithms depends on the size of the graph. The algorithm may be implemented to run in $O(m)$, where $m$ is the number of edges of $G$.

**CONCLUSION**

In this paper we have developed algorithms for certain cycle related graphs and obtained an optimal solution for $s(G)$. We have already obtained the same for comb, ladder, Mobius ladder and Circular ladder graphs. Finding efficient algorithms to determine skew edge coloring for other interconnection networks is quite challenging.
REFERENCES


