

Submersion of Semi-invariant Submanifolds of Contact Manifolds

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Abstract

In this paper, we discuss submersion of semi-invariant submanifolds of contact manifolds and derive some results on its geometry. We also derive some curvature relations.

Keywords: Semi-invariant submanifold, almost contact manifold, Submersion.

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1. INTRODUCTION

The field of submersion has been enriched by works of many authors. Papaghuice [1] studied the submersion of semi-invariant submanifolds of a Sasakian manifold whereas Kobayashi [2] studied submersion of CR -submanifolds and obtained interesting results. Motsumoto et al. [3] studied the submersion of semi-invariant submanifolds of trans-Sasakian manifold. Submersion of CR -submanifolds of nearly trans-Sasakian manifold were studied by Mohammed Jamali and Mohammad Hasan Shahid [4]. If the characteristic vector field ξ , belongs to the (k, μ) -nullity distribution, then a contact metric manifold is called (k, μ) -contact metric manifold. In 1995, Blair, Koufogiorgas and Papantonion [5] introduced the notion of (k, μ) -contact metric manifold and a full classification of such a manifold is given by E. Boeckx (2000) [6]. A (k, μ) -contact metric manifold with $\mu=0$ is called $N(k)$ -

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contact metric manifold [12]. Our aim in this paper is to study the submersion of semi-invariant submanifolds of contact manifolds.

2. PRELIMINARIES

Let \bar{M} be an almost contact metric manifold [7] with an almost contact metric structure (ϕ, ξ, η, g) , where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$(2.1) \quad \phi^2 = -I + \eta \otimes \xi,$$

$$(2.2) \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0,$$

$$(2.3) \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

$$(2.4) \quad g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X),$$

for all $X, Y \in T\bar{M}$. It is known that on a contact metric manifold, N^3 vanishes if and only if ξ is a killing vector field. A contact metric structure for which ξ is a killing vector field is called a K -contact metric manifold. As the tensor N^3 satisfies many important properties in a contact metric manifold, we define a tensor field h by $h = \frac{1}{2} \mathcal{L}_\xi \phi = \frac{1}{2} N^3$, where \mathcal{L}_ξ denotes the operator of Lie-differentiation. The tensor field h is symmetric and satisfies the properties

$$(2.5) \quad h\xi = 0, \quad h\phi + \phi h = 0, \quad \nabla \xi = -\phi - \phi h, \quad \text{Trace} h = \text{Trace} \phi h = 0,$$

where ∇ is Levi-Civita connection.

The (k, μ) -nullity distribution on a contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$ is defined as [5]

$$\begin{aligned} N(k, \mu): p \rightarrow N_p(k, \mu) &= \{Z \in T_p M : R(X, Y)Z \\ &= k[g(Y, Z)X - g(X, Z)Y] + \mu[g(Y, Z)hX - g(X, Z)hY], \end{aligned}$$

for any $X, Y \in T_p \bar{M}$. Hence if the characteristic vector field belongs to the (k, μ) -nullity distribution, then we have

$$(2.6) \quad R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],$$

where k, μ are constants. A contact metric manifold satisfying the equation (2.6) is called a (k, μ) -contact metric manifold and the examples (for both Sasakian and non-Sasakian cases) of such manifolds are given in [5]. A contact metric manifold with ξ , belonging to the (k, μ) -nullity distribution has been studied in [6],[8],[9],[10],[11]. In a (k, μ) -contact metric manifold the following relations hold:

$$(2.7) \quad h^2 = (k-1)\phi^2,$$

$$(2.8) \quad (\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX),$$

$$(2.9) \quad (\nabla_X \eta)Y = g(X + hX, \phi Y).$$

In particular, if $\mu = 0$ then, we obtain the condition of k -nullity distribution introduced by Tanno [12].

$$N(k): p \rightarrow N_p(k) = \{Z \in T_p M : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$

for any $X, Y \in T_p \bar{M}$. Also if the characteristic vector field ξ belongs to the k -nullity distribution, then we have

$$(2.10) \quad R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y].$$

A contact metric manifold with $\xi \in N(k)$, is called a $N(k)$ -contact metric manifold. Thus a $N(k)$ -contact metric manifold is a contact metric manifold satisfying the relation (2.10). In a $N(k)$ -contact metric manifold equations (2.7), (2.8) and (2.9) also holds.

Let M be a n -dimensional isometrically immersed submanifold of \bar{M} and tangent to ξ . Let g be the metric tensor field on M as well as the induced metric on M .

Definition 2.1: A m -dimensional Riemannian submanifold M of a (k, μ) -contact metric manifold \bar{M} is called a semi-invariant submanifold if ξ is tangent to M and it is endowed with a pair of orthogonal differentiable distributions (D, D^\perp) , which satisfies

- (1) $TM = D \oplus D^\perp \oplus \xi$,, where \oplus denotes the orthogonal direct sum,
- (2) the distribution $D_x : x \rightarrow D \subset T_x M$ is invariant under ϕ i.e $\phi D_x \subset D_x$ for each $x \in M$
- (3) the orthogonal complementary distribution $D^\perp : x \rightarrow D^\perp \subset T_x M$ of the distribution D on M is totally

real i.e. $\phi D^\perp \subset T_x^\perp M$, where $T_x M$ and $T_x^\perp M$ are the tangent space and the normal space of M at x respectively.

Let the dimension of D (resp. D^\perp) be $2p$ (resp. q) where $2p+q=m-1$. If $p=0$ (resp. $q=0$) the submanifold M becomes anti-invariant (resp. invariant) submanifold. A generic submanifold M satisfies $D^\perp = \dim T_x^\perp M$. A submanifold is called proper if it is neither invariant nor anti-invariant. It is easy to see that any hypersurface to which the vector field ξ is tangent is a typical example of semi-invariant submanifold.

Where D and D^\perp are the horizontal and vertical distribution respectively. Let $\bar{\nabla}$ (resp. ∇) be the covariant differentiation with respect to Levi-Civita connection on \bar{M} (resp. M). The Gauss and Weingarten formulas for M are respectively given by

$$(2.11) \quad \bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$$

$$(2.12) \quad \text{and } \bar{\nabla}_X N = -A_N X + \nabla_X^\perp N,$$

for $X, Y \in TM$, $N \in T^\perp M$, where σ (resp. A) is the second fundamental form (resp. tensor) of M in \bar{M} and $\bar{\nabla}$ denote the operator of the normal connection.

$$(2.13) \quad g(\sigma(X, Y), N) = g(A_N X, Y).$$

The projection of TM to D and D^\perp are denoted by h and v respectively i.e., for any $X \in TM$, we have

$$(2.14) \quad X = \sigma X + vX + \eta(X)\xi.$$

The normal bundle to M has the decomposition

$$(2.15) \quad T^\perp M = \phi D^\perp \oplus n_1,$$

where $g(\phi D^\perp, n_1) = 0$. For any $U \in T^\perp M$, we put

$$(2.17) \quad U = nU + mU,$$

where $nU \in \phi D^\perp$, $mU \in n_1$. From the above equation we have

$$\phi U = \phi nU + \phi mU, U \in T^\perp M, \phi nU \in D^\perp, \phi mU \in n_1.$$

Definition 2.2: Let M be a semi-invariant submanifold of a (k, μ) -contact metric manifold \bar{M} and M' be an almost contact metric manifold with the almost contact metric structure (ϕ', ξ', η', g') . Assume that there is a submersion $\pi: M \rightarrow M'$ such that

- (i) $D^\perp = \ker \pi_*: TM \rightarrow TM'$ is the tangent mapping to π
- (ii) $\pi_*: D_p \oplus \{\xi\} \rightarrow T_{\pi(p)}M'$ is an isometry for each $p \in M$ which satisfies $\pi_* \circ \phi = \phi' \circ \pi_*$; $\eta = \eta' \circ \pi_*$; $\pi_*(\xi_p) = \xi'_{\pi(p)}$, where $T_{\pi(p)}M'$ denotes the tangent space of M' at $\pi(p)$.

A vector X on M is said to be basic if, $X \in D_p \oplus \xi$ and X is π -related to a vector field on M' i.e there exists a vector field $X_* \in TM'$ such that $\pi_*(X_p) = X_{*\pi(p)}$ for each $p \in M$. Note that, by condition (ii) of the above definition 2.2, we have that the structural vector field ξ is a basic vector field.

Lemma 2.3: Let X, Y be basic vector fields on M . Then

- (i) $g(X, Y) = g'(X_*, Y_*) \circ \pi$,
- (ii) the component $\sigma([X, Y]) + \eta([X, Y]\xi) = [X_*, Y_*]$,
- (iii) $[U, X] \in D^\perp$ for any $U \in D^\perp$,
- (iv) $\sigma(\nabla_X Y) + \eta(\nabla_X Y)\xi$ is a basic vector field corresponding to $\nabla_{X_*}^* Y_*$, where ∇^* denotes the Levi-Civita connection on M' .

For basic vector fields on M , we define the operator $\tilde{\nabla}^*$ corresponding to ∇^* by setting $\tilde{\nabla}_X^* Y = \sigma([X, Y]) + \eta([X, Y]\xi)$ for $X, Y \in (D_p \oplus \{\xi\})$. By (iv) of Lemma 2.3, $\nabla_X^* Y$ is a basic vector field and we have

$$(2.17) \quad \pi_*(\tilde{\nabla}_X^* Y) = \nabla_{X_*}^* Y_*$$

Define the tensor field C by

$$(2.18) \quad \nabla_X Y = \tilde{\nabla}_X^* Y + C(X, Y),$$

$X, Y \in (D_p \oplus \{\xi\})$, where $C(X, Y)$ is the verticle part of $\nabla_X Y$. It is known that C is skew-symmetric and satisfies

$$(2.19) \quad C(X, Y) = \frac{1}{2}v[X, Y],$$

where $X, Y \in (D_p \oplus \{\xi\})$.

The curvature tensor R, R^* of the connection ∇, ∇^* on M and M' respectively are related by

$$(2.20) \quad \begin{aligned} R(X, Y, Z, W) &= R^*(X_*, Y_*, Z_*, W_*) - g(C(Y, Z), C(X, W)) \\ &+ g(C(X, Z), C(Y, W)) + 2g(C(X, Y), C(Z, W)), \end{aligned}$$

for $X, Y, Z, W \in (D_p \oplus \{\xi\})$, where $\pi_*X = X_*, \pi_*Y = Y_*, \pi_*Z = Z_*$ and $\pi_*W = W_* \in \chi M'$. For (k, μ) -contact metric manifold we prove:

Proposition 2.4: Let $\pi: M \rightarrow M'$ be a submersion of semi-invariant submanifold of a (k, μ) -contact metric manifold \bar{M} on to almost contact metric manifold M' . Then we have

$$(2.21) \quad (\tilde{\nabla}_X^* \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX),$$

$$(2.22) \quad C(X, \phi Y) = \phi n\sigma(X, Y),$$

$$(2.23) \quad \phi C(X, Y) = n\sigma(X, \phi Y),$$

$$(2.24) \quad \phi mh(X, \phi Y) = m\sigma(X, \phi Y),$$

Proof: For any $X, Y \in (D_p \oplus \{\xi\})$ and by using Gauss formula (2.11), decomposition equation (2.16) and (2.18), we obtain

$$(2.25) \quad \begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + \sigma(X, Y) = \nabla_X Y + n\sigma(X, Y) + m\sigma(X, Y) \\ &= \tilde{\nabla}_X^* Y + C(X, Y) + n\sigma(X, Y) + m\sigma(X, Y). \end{aligned}$$

And

$$(2.26) \quad \phi \bar{\nabla}_X Y = \phi \tilde{\nabla}_X^* Y + \phi C(X, Y) + \phi n\sigma(X, Y) + \phi m\sigma(X, Y).$$

Putting $Y = \phi Y$ in equation (2.25), we get

$$(2.27) \quad \tilde{\nabla}_X \phi Y = \tilde{\nabla}_X^* \phi Y + C(X, \phi Y) + n\sigma(X, \phi Y) + m\sigma(X, \phi Y).$$

Using the definition of (k, μ) -contact metric manifold, we find

$$(2.28) \quad (\tilde{\nabla}_X^* \phi)Y = \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X^* Y = g(X + hX, Y)\xi - \eta(Y)(X + hX).$$

Substituting (2.26) and (2.27) in (2.28), we get

$$(2.29) \quad \begin{aligned} & \tilde{\nabla}_X^* \phi Y + C(X, \phi Y) + n\sigma(X, \phi Y) + m\sigma(X, \phi Y) - \phi \tilde{\nabla}_X^* Y - \phi C(X, Y) \\ & - \phi n\sigma(X, Y) - \phi m\sigma(X, Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX). \end{aligned}$$

Comparing the components of $(D_p \oplus \{\xi\})$, D^\perp , ϕD^\perp and n_1 respectively on both sides in the above equation, we get the required results.

Corollary (2.5) : Let $\pi : M \rightarrow M'$ be a submersion of semi-invariant submanifold of a $N(k)$ -contact metric manifold \bar{M} on to almost contact metric manifold M' . Then we have

- (i) $(\tilde{\nabla}_X^* \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX),$
- (ii) $C(X, \phi Y) = \phi n\sigma(X, Y),$
- (iii) $\phi C(X, Y) = n\sigma(X, \phi Y),$
- (iv) $\phi mh(X, \phi Y) = m\sigma(X, \phi Y),$

Proposition 2.6: Let $\pi : M \rightarrow M'$ be a submersion of semi-invariant submanifold of a (k, μ) -contact metric manifold \bar{M} on to almost contact metric manifold M' . Then M' is also a (k, μ) -contact metric manifold.

Proof: From equation (2.21) of proposition (2.4), we have

$$(2.30) \quad (\tilde{\nabla}_X^* \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX).$$

Applying π_* to the above equation and using Lemma 2.3, equation (2.17) and definition of submersion, we derive

$$(2.31) \quad (\tilde{\nabla}_X^* \phi')Y_* = g(X_* + h'X_*, Y_*)\xi - \eta'(Y_*)(X_* + h'X_*).$$

Hence M' is (k, μ) -contact metric manifold.

Corollary 2.7: Let $\pi : M \rightarrow M'$ be a submersion of semi-invariant submanifold of a $N(k)$ -contact metric manifold \bar{M} on to almost contact metric manifold M' . Then M' is also a $N(k)$ -contact metric manifold.

Proposition 2.8: Let $\pi: M \rightarrow M'$ be a submersion of semi-invariant submanifold of a (k, μ) -contact metric manifold on to almost contact metric manifold M' . Then

- (i) $n\sigma(\phi X, \phi Y) + n\sigma(\phi X, Y) = 0$,
- (ii) $n\sigma(\phi X, \phi Y) = n\sigma(\phi X, Y)$,
- (iii) $m\sigma(\phi X, \phi Y) - m\sigma(X, Y)$,
- (iv) $C(\phi X, \phi Y) = C(X, Y)$, for any $X, Y \in (D_p \oplus \{\xi\})$.

Proof: (i) Interchanging X and Y in equation (2.23) gives

$$(2.32) \quad \phi C(Y, X) = n\sigma(Y, \phi X) = n\sigma(X, \phi Y),$$

Then

$$\begin{aligned} n\sigma(X, \phi Y) + n\sigma(\phi X, Y) &= \phi C(Y, X) + \phi C(X, Y) \\ &= \phi C(Y, X) - \phi C(Y, X) = 0. \end{aligned}$$

(ii) Putting $X = \phi X$ in (2.23), we get

$$(2.33) \quad n\sigma(\phi X, \phi Y) = \phi C(\phi X, Y) = -\phi C\sigma(Y, \phi X)..$$

Using (2.22) in (2.33), we deduce

$$\begin{aligned} n\sigma(\phi X, \phi Y) &= -\phi C(Y, \phi X) = -\phi(\phi n\sigma(Y, X)) = \phi^2 n\sigma(Y, X). \\ &= n\sigma(Y, X) - \eta(\sigma(X, Y))\xi = n\sigma(Y, X). \end{aligned}$$

(iii) Putting $X = \phi X$ in (2.23) and using again the same equation, we find

$$\begin{aligned} m\sigma(\phi X, \phi Y) &= \phi m\sigma(\phi X, Y) = \phi m\sigma(Y, \phi X) \\ &= \phi^2 m\sigma(Y, X) = -m\sigma(X, Y). \end{aligned}$$

(iv) Putting $X = \phi X$ in (2.22) and then using (2.23) yields

$$\begin{aligned} C(\phi X, \phi Y) &= \phi n\sigma(\phi X, Y) = \phi n\sigma(Y, \phi X) \\ &= \phi^2 C(Y, X) = -C(Y, X) + \eta(C(Y, X))\xi = C(X, Y). \end{aligned}$$

Corollary 2.7: Let $\pi: M \rightarrow M'$ be a submersion of semi-invariant submanifold of a $N(k)$ -contact metric manifold \bar{M} on to almost contact metric manifold M' . Then we have

- (i) $n\sigma(\phi X, \phi Y) + n\sigma(\phi X, Y) = 0$,

- (ii) $n\sigma(\phi X, \phi Y) = n\sigma(\phi X, Y)$,
- (iii) $m\sigma(\phi X, \phi Y) = m\sigma(X, Y)$,
- (iv) $C(\phi X, \phi Y) = C(X, Y)$, for any $X, Y \in (D_p \oplus \{\xi\})$.

3. CURVATURE RELATION

Proposition 3.1. Let $\pi: M \rightarrow M'$ be a submersion of semi-invariant submanifold of a (k, μ) -contact metric manifold \bar{M} on to almost contact metric manifold M' . Then the ϕ -sectional curvature of \bar{M} of M' are related by

$$\begin{aligned} \bar{B}(X, Y) &= B'(X_*, Y_*) - 2\|n\sigma(X, Y)\|^2 - 2\|n\sigma(X, \phi Y)\|^2 \\ &\quad - 2(n\sigma(X, X), n\sigma(Y, Y)) + 2\|m\sigma(X, Y)\|^2, \end{aligned}$$

where $X, Y \in (D_p \oplus \{\xi\})$.

Proof: We know

$$\bar{B}(X, Y) = \bar{R}(X, \phi X, \phi Y, Y).$$

Putting $Y = \phi X, Z = \phi Y, W = Y$ in Gauss equation

$$\begin{aligned} \bar{R}(X, Y, Z, W) &= R(X, Y, Z, W) - g(\sigma(X, W), \sigma(Y, Z)) \\ &\quad + g(\sigma(X, Z), \sigma(Y, W)), \end{aligned}$$

we get

$$\begin{aligned} \bar{R}(X, \phi X, \phi Y, Y) &= R(X, \phi X, \phi Y, Y) - g(\sigma(X, Y), \sigma(\phi X, \phi Y)) \\ &\quad + g(\sigma(X, \phi Y), \sigma(\phi X, Y)). \end{aligned}$$

Substituting $\sigma = n\sigma + m\sigma$, in above equation, we get

$$\begin{aligned} \bar{R}(X, \phi X, \phi Y, Y) &= R(X, \phi X, \phi Y, Y) - g(n\sigma(X, Y) + m\sigma(X, Y), n\sigma(\phi X, \phi Y) + m\sigma(\phi X, \phi Y)) \\ &\quad + g(n\sigma(X, \phi Y) + m\sigma(X, \phi Y), n\sigma(\phi X, Y) + m\sigma(\phi X, Y)), \\ &= R(X, \phi X, \phi Y, Y) - g(n\sigma(X, Y), n\sigma(\phi X, \phi Y)) - g(n\sigma(X, Y), m\sigma(\phi X, \phi Y)) \\ &\quad - g(m\sigma(X, Y), n\sigma(\phi X, \phi Y)) - g(m\sigma(X, Y), m\sigma(\phi X, \phi Y)) + g(n\sigma(X, \phi Y), n\sigma(\phi X, Y)) \\ &\quad + g(n\sigma(X, \phi Y), m\sigma(\phi X, Y)) + g(m\sigma(X, \phi Y), n\sigma(\phi X, \phi Y)) + g(m\sigma(X, \phi Y), m\sigma(\phi X, Y)), \end{aligned}$$

$$\begin{aligned} \bar{R}(X, \phi X, \phi Y, Y) &= R(X, \phi X, \phi Y, Y) - g(n\sigma(X, Y), n\sigma(\phi X, \phi Y)) \\ &- g(m\sigma(X, Y), m\sigma(\phi X, \phi Y)) + g(n\sigma(X, \phi Y), n\sigma(X, \phi Y)) \\ &+ g(\phi m\sigma(X, Y), \phi m\sigma(X, Y)), \end{aligned}$$

$$(3.1) \quad \bar{R}(X, \phi X, \phi Y, Y) = R(X, \phi X, \phi Y, Y) - \|n\sigma(X, Y)\|^2 + 2\|m\sigma(X, Y)\|^2 - \|n\sigma(X, \phi Y)\|^2,$$

Putting $Y = \phi X, Z = \phi Y, W = Y$ in equation (2.20) it follows

$$(3.2) \quad \begin{aligned} R(X, \phi X, \phi Y, Y) &= R^*(X_*, \phi'X_*, \phi'Y_*, Y_*) - g(C(\phi X, \phi Y), C(X, Y)) \\ &+ g(C(X, \phi Y), C(\phi X, Y)) + 2g(C(X, \phi Y), C(\phi Y, Y)). \end{aligned}$$

Applying ϕ on both side of equation (2.23), we obtain

$$(3.3) \quad C(X, Y) = -\phi n\sigma(X, \phi Y).$$

Using equation (3.3) in equation (3.2) give

$$\begin{aligned} R(X, \phi X, \phi Y, Y) &= R^*(X_*, \phi'X_*, \phi'Y_*, Y_*) - \|n\sigma(X, Y)\|^2 \\ &- \|n\sigma(X, \phi Y)\|^2 - 2g(n\sigma(X, X), n\sigma(Y, Y)). \end{aligned}$$

Putting the value of $R(X, \phi X, \phi Y, Y)$ in (3.1) we obtain

$$\begin{aligned} \bar{R}(X, \phi X, \phi Y, Y) &= R^*(X_*, \phi'X_*, \phi'Y_*, Y_*) - \|n\sigma(X, Y)\|^2 \\ &- \|n\sigma(X, \phi Y)\|^2 - 2g(n\sigma(X, X), n\sigma(Y, Y)) - \|n\sigma(X, Y)\|^2 \\ &+ 2\|m\sigma(X, Y)\|^2 - \|n\sigma(X, \phi Y)\|^2, \end{aligned}$$

or

$$\begin{aligned} \bar{B}(X, Y) &= B'(X_*, Y_*) - 2\|n\sigma(X, Y)\|^2 - 2\|n\sigma(X, \phi Y)\|^2 \\ &- 2g(n\sigma(X, X), n\sigma(Y, Y)) + 2\|m\sigma(X, Y)\|^2. \end{aligned}$$

Corollary 3.2. Let $\pi: M \rightarrow M'$ be a submersion of semi-invariant submanifold of a

$N(k)$ -contact metric manifold \bar{M} on to almost contact metric manifold M' . Then the ϕ -sectional curvature of \bar{M} of M' are related by

$$\begin{aligned}\bar{B}(X, Y) &= B'(X_*, Y_*) - 2\|n\sigma(X, Y)\|^2 - 2\|n\sigma(X, \phi Y)\|^2 \\ &\quad - 2g(n\sigma(X, X), n\sigma(Y, Y)) + 2\|m\sigma(X, Y)\|^2.\end{aligned}$$

where $X, Y \in (D_p \oplus \{\xi\})$.

Proposition 3.3. Let $\pi: M \rightarrow M'$ be a submersion of semi-invariant submanifold of a (k, μ) -contact metric manifold \bar{M} on to almost contact metric manifold M' . Then the ϕ -sectional curvature of \bar{M} of M' are related by

$$\bar{H}(X) = H'(X_*) - 4\|n\sigma(X, X)\|^2 + 2\|m\sigma(X, X)\|^2,$$

where $X, Y \in (D_p \oplus \{\xi\})$.

Proof: Putting $X = Y$ in equation (3.4) we obtain

$$\begin{aligned}\bar{B}(X, X) &= \bar{H}(X) = H'(X_*) - 2\|n\sigma(X, X)\|^2 - 2\|n\sigma(X, \phi X)\|^2 \\ &\quad - 2g(n\sigma(X, X), n\sigma(X, X)) + 2\|m\sigma(X, X)\|^2 \\ &= H'(X_*) - 4\|n\sigma(X, X)\|^2 - 2\|n\sigma(X, \phi X)\|^2 \\ &\quad + 2\|m\sigma(X, X)\|^2\end{aligned}$$

Putting $Y = X$ in (2.23), we get

$$n\sigma(X, \phi X) = \phi C(X, X) = 0.$$

Thus we get

$$\bar{H}(X) = H'(X_*) - 4\|n\sigma(X, X)\|^2 + 2\|m\sigma(X, X)\|^2.$$

Corollary 3.4. Let $\pi: M \rightarrow M'$ be a submersion of semi-invariant submanifold of a $N(k)$ -contact metric manifold \bar{M} on to almost contact metric manifold M' . Then the ϕ -sectional curvature of \bar{M} of M' are related by

$$\bar{H}(X) = H'(X_*) - 4\|n\sigma(X, X)\|^2 + 2\|m\sigma(X, X)\|^2,$$

where $X, Y \in (D_p \oplus \{\xi\})$.

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