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Fixed Point Theorem in Hilbert Space using Weak and Strong Convergence

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Abstract

In this paper we first give a weak convergence theorem for pseudo nonspreading mappings and then we establish strong convergence for these mappings which is the generalization of the work recently done by Kurokawa and Takahashi.(2010). The results are the improvement of the work done by previous authors.

Keywords: Nonspreading mapping, fixed point, demiclosed principle, strong convergence, weak convergence.

Mathematics Subject Classifications: 47H1054H25

INTRODUCTION AND PRELIMINARIES:

In this paper, Let H be a real Hilbert space and C be a complex convex subset of H .Throughout the paper ,we denote "$x_n \to x$ *and* $x_n \rightharpoonup x$" the strong and weak convergence of $\{x_n\}$, respectively. Denote by $F(t)$ the set of fixed points of a mapping T.

Definition 1.

Let $T: C \to C$ be a mapping

(1) T is said to be non expansive, if $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$.

(2) T is said to be quasinonexpansive , if $F(T)$ is nonempty and

$\|Tx - p\| \leq \|x - p\|, \quad \forall\, x \in C, p \in F(T).$ ……………………………..(1)

(3) T is said to be non spreading , if

$2\,\|\mathrm{Tx} - \mathrm{Ty}\|^2 \leq \|\mathrm{Tx} - \mathrm{y}\|^2 + \|\mathrm{Ty} - \mathrm{p}\|^2 \;\; \forall x, y \; \in C.$ ……………… ….(2)

It is easy to prove that $T: C \to C$ is nonspreading if and only if

$\|\mathrm{Tx} - \mathrm{Ty}\|^2 \leq \|\mathrm{x} - \mathrm{y}\|^2 + 2\,\langle x - Tx, y - Ty\rangle\, \forall x, y \; \in C$ ....…………………..(3)

(4) $T: C \to H$ is said to be $k - strictly\, pseudononspreading$ in the terminology of Browder-Petryshyn , if there exists $k \in [0,1)$ such that

(5) $\|\mathrm{Tx} - \mathrm{Ty}\|^2 \leq \|\mathrm{x} - \mathrm{y}\|^2 + \mathrm{k}\|\mathrm{x} - \mathrm{Tx} - (\mathrm{y} - \mathrm{Ty})\|^2 + 2\langle x - Tx, y - Ty\rangle \; \forall x, y \; \in C$

Definition 2.

(1) If $T: C \to C$ is a nonspreading mapping with $F(t) \neq \emptyset$, then T is quasinonexpansive and

$F(T)$ is closed and convex.

(2)Clearly every nonspreading mapping is $k - strictly\, pseudononspreading$ with *k=0,*but the inverse is not true. This can be seen from the following example.

Example: Let $\mathcal{R}$ denote the set of all real numbers. Let $T: \mathcal{R} \to \mathcal{R}$ be a mapping defined by

$$Tx = \begin{cases} x, & x \in (-\infty, 0) \\ -2x, & x \in [0, \infty). \end{cases} \quad \text{...(5)}$$

It is easy to see that k- strictly pseudononspreading mapping with$k \in [0,1)$, but it is not nonspreading.

Definition 3.

(1)C be a mapping I-T is said to be demiclosed at), if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x\; and \|(I - T)x_n\| \to 0, we\ have\ x^* = Tx^*.$

(2) A Banach space E is said to have Opial’s property , if for any sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x^*$, we have $\lim_{n\to\infty} inf\|x_n - x^*\| < \lim_{n\to\infty} inf\|x_n - y\|, \forall y \neq x^*$……(6)

It is well known that each Hilbert space possesses Opial property.

(3) A mapping Let $S: C \to C$ is said to be semicompact , if for any bounded sequence $\{x_n\} \subset C$ with

$\lim_{n\to\infty}\|x_n - Sx_n\|$=0, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $x^* \in C$.

Lemma 1. Let E be a uniformly convex Banach space and let $B_r(0) = \{x \in E: \|x\| \leq r\}$ be a closed ball with center 0 and radius $r > 0$.For any given sequence $\{x_1, x_2, \ldots\ldots, x_n \ldots\ldots\} \subset B_r(0)$ and any given number sequence $\{\lambda_1, \lambda_2, \ldots\ldots\ldots \lambda_n, \ldots\ldots\ldots\}$ with $\lambda_i \geq 0, \sum_{i=1}^{\infty}\lambda_i = 1$, there exists a continuous strictly increasing and convex function $g: [0,2r) \to [0,\infty) with\ g(0) = 0$ such that for any $i, j \in N, i < j$ the following holds:

$\|\sum_{n=1}^{\infty}\lambda_n x_n\|^2 \leq \sum_{n=1}^{\infty}\lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|)$ ……………………..…(7)

Lemma 2. Let H be a real Hilbert space , C be a nonempty and closed convex subset of H , and $T: C \to C$ be a strictly pseudononspreading mapping,

(i) If $F(T) \neq \emptyset$, then it is closed and convex.

(ii) $(I - T)$ is demiclosed at origin.

Lemma 3. Let $T: C \to C$ be a strictly pseudononspreading mapping with $k \in [0,1)$. Denote $T_\beta = \beta I + (1-\beta)T$, where $\beta \in [k, 1)$, then

(i) $F(T) = F(T_\beta)$;

(ii) The following inequality holds

$\|T_\beta x - T_\beta y\|^2 \leq \|x - y\|^2 + \frac{2}{1-\beta}\langle x - T_\beta x, y - T_\beta y\rangle, \forall\, x, y \in C$, ………(8)

(iii) T_β is a quasinonexpansive mapping , that is

(iv) $\|T_\beta x - p\|^2 \leq \|x - p\|^2, \forall\, x \in C, p \in F(T)$………………………………..(9)

Lemma 4. Let C be a nonempty set and closed convex subset of a Hilbrt space H and let $\emptyset: C \times C \to \mathcal{R}$

Be a bifunction satisfying conditions (A1), (A2), (A3), and (A4). Then for any r> 0 and $x \in C$, there exists $z \in C$ such that

$\emptyset(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C$………………………………………..(10)

Furthermore if for given *r> 0*

$T_r: C \to C$ by $T_r(x) = \{z \in C: \emptyset(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\}$………………………...(11)

Then the following hold;

(1) T_r is single valued;

(2) T_r is firmly non expansive , that is $\|T_r\mathrm{x} - T_r\mathrm{y}\|^2 \le \langle T_r x - T_r y, x - y\rangle$;

(3)$F(T_r) = \Omega$, where Ω is the set of solutions of the equilibrium problem;

(4) Ω is a closed and convex subset of C.

Concerning the weak and strong convergence problem for some kinds of iterative algorithms for nonspreading mappings, k-strictly pseudononspreading mappings and other kind of non-linear mappings have been considered in Osilike and Isiogugu[4],Igarashie at al, Iemoto and Takahashi, Kurokawa and Takahashi . The purpose of this paper is to propose anew iterative algorithm as a generalization for pseudononspreading mappings using weak and strong convergence. Under suitable conditions , the weak and strong convergence are proved. The results are extension and generalization of previous results.

MAIN RESULTS

We assume the following conditions satisfied, throughout this section.

(1)H is a Hilbert space , C be nonempty and closed convex subset of H.

(2) For each $S_i: C \to C, i = 1,2, \dots \dots$ & $R_j: C \to C, j = 1,2, \dots \dots$ are k_i& k_j strictly pseudo nonspreading mapping with $= sup_{i,\ge 1}\, k_i \in (0,1)$ $and\ similarly\ k = sup_{j\ge 1}\, k_j \in (0,1)$. For given $\beta, \gamma \in [k, 1), denoted\ by\ S_i\beta = \beta I + (1 - \beta) S_i\ and\ \ R_j\gamma = \gamma I + (1 - \gamma)\, R_j$

For each $i = 1,2 \dots \dots$ & $j = 1,2 \dots \dots .., it\ follows\ from\ (8) that$

$$\left\|S_{i,\beta}x - S_{i,\beta}y\right\|^2 + \left\|R_{j,\gamma}a - R_{j,\gamma}b\right\|^2 \le \|x - y\|^2 + \|a - b\|^2 + \frac{2}{1-\beta}\langle x - S_{i,\beta}x, y - S_{i,\beta}\, y\rangle + \frac{2}{1-\gamma}\langle a - R_{j,\gamma}a, b - R_{j,\gamma}b\rangle\ \ \forall x, y, a, b \in C.$$

(3) $\emptyset: C \times C \to \mathcal{R}$ & $\varphi: C \times C \to \mathcal{R}$ are bifunction satisfying the conditions (A1) - (A4).The it follows from lemma 4 that the mapping defined in (11) is single valued, $z = T_r x\ , v = Q_r a\ , F(T_r) = \Omega, \mathrm{G}(Q_r) = \xi$ where Ω and ξ are the solution set of the equilibrium problem and Ω and ξ are closed and convex subset of C.

Theorem: Let $H, C, \{\,S_i\}\left\{\,R_j\right\}, k, \beta, \gamma, \{S_i\beta\}, \left\{\,R_j\gamma\right\}, \emptyset, \varphi, T_r, Q_r, \Omega, \xi$ be the same as above .Let $\{x_n\}, \{a_n\}, \{u_n\} and\ \{c_n\}$ be the sequences defined by

$$\begin{cases} x_1, a_1 \in C, & \textit{chosen arbitrarily} \\ \emptyset(x,y) + \varphi(a,b) + \frac{1}{r}\langle y-u_n, u_n-x_n\rangle + \frac{1}{r}\langle b-c_n, c_n-a_n\rangle \geq 0, \forall y,b \in C \\ x_{n+1}+a_{n+1} = \alpha_{0,n}u_n + \delta_{0,n}c_n + \sum_{i=1}^{\infty}\alpha_{i,n} S_{i,\beta}u_n + \sum_{j=1}^{\infty}\delta_{j,n}\ R_{j\gamma}c_n \end{cases}$$

…………(A)

Where $\{\alpha_{i,n}\} \subset (0,1)$& $\{\delta_{j,n}\} \subset (0,1)$ and $\{r_n\}$ & $\{d_n\}$ satisfies the following conditions

(a) $\sum_{i=0}^{\infty}\alpha_{i,n} = 1$ & $\sum_{j=0}^{\infty}\delta_{j,n}$ =1 for each n≥ 1

(b) $for\ each\ i,j \geq 1, \lim inf_{n\to\infty}\ \ \alpha_{0,n}\ \ \alpha_{i,n} > 0$ & $\lim inf_{n\to\infty}\delta_{0,n}\ \ \delta_{j,n} > 0$

(c)$\{r_n\} \subset (0,\infty), and\ \{d_n\} \subset (0,\infty)\ and \lim inf_{n\to\infty} r_n > 0, \lim inf_{n\to\infty} d_n > 0$

(I) If $\mathcal{F} = (\cap_{i=1}^{\infty}F(S_i)) \cap\ \Omega \neq 0$ and $\mathcal{F} = (\cap_{j=1}^{\infty}F(R_j)) \cap \delta\ \neq 0$ then $\{x_n\},\{a_n\},\{u_n\}\ and\ \{c_n\}$

Converge weakly to some point $x^*\ and\ a^*$ respectively where $x^*, a^* \in\ \mathcal{F}$.

(II) In addition , if there exists some positive integer m such that S_m & R_m such that S_m & R_m are semicompact, then the sequences $\{x_n\},\{u_n\}$ converge strongly to $x^* \in\ \mathcal{F}$ and the sequences

$\{a_n\}and\ \{c_n\}$ converge strongly to $a^* \in\ \mathcal{F}$.

Proof: First we prove the conclusion (I). The proof here is divided into three steps.

Step1. We prove that the sequence $\{x_n\},\{a_n\},\{u_n\}and\ \{c_n\}$, $\{S_{i,\beta}u_n\}$,$\{\ R_{j\gamma}c_n\}, i,j \geq 1$ are all bounded,and for each $p,q \in \mathcal{F}$ the limits $\lim_{n\to\infty}\|x_n-p\|$, $\lim_{n\to\infty}\|u_n-p\|$, $\lim_{n\to\infty}\|a_n-q\|$, $\lim_{n\to\infty}\|c_n-q\|$ exists and

$$\lim_{n\to\infty}\|x_n-p\| + \lim_{n\to\infty}\|a_n-q\| = \lim_{n\to\infty}\|u_n-p\| + \lim_{n\to\infty}\|c_n-q\| \quad \text{………..(12)}$$

In fact it follows from lemma 4 that$u_n = T_{r_n}x_n$, $p = T_{r_n}p$, $c_n = Q_{r_n}a_n$, $q = Q_{r_n}q$ and

$$\begin{aligned}\|u_n-p\| + \|c_n-q\| &= \|T_{r_n}x_n - T_{r_n}p\| + \|Q_{r_n}a_n - Q_{r_n}q\| \\ &\leq \|x_n-p\| + \|c_n-q\|\ \forall n \geq 1\end{aligned}$$

…………(13)

Since $p,q \in\ \mathcal{F}$ by lemma 3 $p \in \cap_{i=1}^{\infty}F(S_{i,\beta})$ & $q \in \cap_{i=1}^{\infty}F(R_{j,\gamma})$

Hence it follows that

$\|x_{n+1}-p\|+\|a_{n+1}-q\|$

$$=\left\|\alpha_{0,n}u_n+\sum_{i=1}^{\infty}\alpha_{i,n}\,S_{i,\beta}u_n-p\right\|+\left\|\delta_{0,n}c_n+\sum_{j=1}^{\infty}\delta_{j,n}\;R_{j\gamma}c_n-q\right\|$$

$\leq\ \alpha_{0,n}\|u_n-p\|+\sum_{i=1}^{\infty}\alpha_{i,n}\left\|S_{i,\beta}u_n-p\right\|+\delta_{0,n}\|c_n-q\|+\sum_{j=1}^{\infty}\delta_{j,n}\left\|\,R_{j\gamma}c_n-q\right\|$ ……………………………………………………………(14)

$$\leq\ \alpha_{0,n}\|u_n-p\|+\sum_{i=1}^{\infty}\alpha_{i,n}\,\|u_n-p\|+\delta_{0,n}\|c_n-q\|+\sum_{j=1}^{\infty}\delta_{j,n}\,\|c_n-q\|$$

$$=\|u_n-p\|+\|c_n-q\|\leq\|x_n-p\|+\|c_n-q\|\forall n\geq 1$$

This implies that for each $p,q\in\mathcal{F}$, the limits $\lim\limits_{n\to\infty}\|x_n-p\|$, $\lim\limits_{n\to\infty}\|a_n-q\|$, $\lim\limits_{n\to\infty}\|c_n-q\|$ exists and so $\{x_n\},\{a_n\},\{u_n\}\ and\ \{c_n\}$, are all bounded and (12) holds.

Furthermore by (9) it is easy to see that for each $i,j\geq 1$, $\{S_{i,\beta}u_n\}$,$\{S_{i,\beta}x_n\}$,$\{R_{j\gamma}c_n\}$, $\{R_{j\gamma}a_n\}$ are also bounded.

Step2. Next we prove that for each $i,j\geq 1$the following holds

$\lim\limits_{n\to\infty}\|x_n-S_ix_n\|+\lim\limits_{n\to\infty}\left\|a_n-R_ja_n\right\|=\lim\limits_{n\to\infty}\|u_n-S_iu_n\|+\lim\limits_{n\to\infty}\left\|c_n-R_jc_n\right\|=0$ ……(15)

Infact by lemma5. for any positive integer $i,j\geq 1\ and p,q\in\ \mathcal{F}$, we have

$$\|x_{n+1}-p\|^2+\|a_{n+1}-q\|^2$$

$$=\left\|\alpha_{0,n}(u_n-p)+\sum_{i=1}^{\infty}\alpha_{i,n}\,(S_{i,\beta}u_n-p)\right\|^2+\left\|\delta_{0,n}(c_n-q)+\sum_{j=1}^{\infty}\delta_{j,n}\;(R_{j\gamma}c_n-q)\right\|^2$$

$\leq\ \alpha_{0,n}\|u_n-p\|^2+\sum_{i=1}^{\infty}\alpha_{i,n}\left\|(S_{i,\beta}u_n-p)\right\|^2+\delta_{0,n}\|c_n-q\|^2+$
$\sum_{j=1}^{\infty}\delta_{j,n}\left\|\,(R_{j\gamma}c_n-\ q)\right\|^2-\ \alpha_{0,n}\,\alpha_{i,n}g\big(\left\|u_n-S_{i,\beta}u_n\right\|\big)-\delta_{0,n}\delta_{j,n}\,h(\left\|c_n-R_{j\gamma}c_n\right\|)$ ………………………(15)

$$\leq \alpha_{0,n}\|u_n - p\|^2 + \sum_{i=1}^{\infty} \alpha_{i,n} \|(u_n - p)\|^2 + \delta_{0,n}\|c_n - q\|^2 + \sum_{j=1}^{\infty} \delta_{j,n} \|(c_n - q)\|^2 - \alpha_{0,n}\, \alpha_{i,n}\, g(\|u_n - S_{i,\beta}u_n\|) - \delta_{0,n}\delta_{j,n}\, h(\|c_n - R_{j,\gamma}c_n\|)$$

$$\leq \|u_n - p\|^2 + \|c_n - q\|^2 - \alpha_{0,n}\, \alpha_{i,n} g(\|u_n - S_{i,\beta}u_n\|) - \delta_{0,n}\delta_{j,n}\, h(\|c_n - R_{j\gamma}c_n\|)$$

$$\leq \|x_n - p\|^2 + \|a_n - q\|^2 - \alpha_{0,n}\, \alpha_{i,n} g(\|u_n - S_{i,\beta}u_n\|) - \delta_{0,n}\delta_{j,n}\, h(\|c_n - R_{j\gamma}c_n\|)$$

This shows that $\alpha_{0,n}\, \alpha_{i,n}\, g(\|u_n - S_{i,\beta}u_n\|) + \delta_{0,n}\delta_{j,n}\, h(\|c_n - R_{j,\gamma}c_n\|) \leq \|x_n - p\|^2 + \|a_n - q\|^2 - \|x_{n+1} - p\|^2 - \|a_{n+1} - q\|^2 \to 0\ as(n \to 0)$

……………………………………………………….(16)

Since g, h are continuous and strictly increasing function with $g(0) = 0$. By condition (b), it yields that $\lim_{n\to\infty}\|u_n - S_{i,\beta}u_n\| + \lim_{n\to\infty}\|c_n - R_{j,\gamma}c_n\| = 0$

……………………………………………….(17)

Therefore we have

$$\lim_{n\to\infty}\|u_n - S_i u_n\| + \lim_{n\to\infty}\|c_n - R_j c_n\| = \lim_{n\to\infty} \frac{1}{1-\beta} \|u_n - S_{i,\beta}u_n\| + \lim_{n\to\infty} \frac{1}{1-\gamma} \|c_n - R_{j,\gamma}c_n\|=0$$

…………………(18)

On the other hand, it follows from lemma4 that $u_n = T_{r_n}x_n, c_n = Q_{r_n}a_n$ and for each $p, q \in \mathcal{F}$

$$\|u_n - p\|^2 + \|c_n - q\|^2 = \|T_{r_n}x_n - T_{r_n}p\|^2 + \|Q_{r_n}a_n - Q_{r_n}q\|^2$$

$$\leq \langle T_{r_n}x_n - T_{r_n}p,\ x_n - p\rangle + \langle Q_{r_n}a_n - Q_{r_n}q,\ a_n - q\rangle$$

$$= \langle u_n - p,\ x_n - p\rangle + \langle c_n - q,\ a_n - q\rangle$$

$$= \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 + \|c_n - q\|^2 + \|a_n - q\|^2 - \|x_n - u_n\|^2 - \|a_n - c_n\|^2\}$$

This shows that

$$\|u_n - p\|^2 + \|c_n - q\|^2 \leq \|x_n - p\|^2 + \|a_n - q\|^2 - \|x_n - u_n\|^2 - \|a_n - c_n\|^2$$

………………………….(19)

In view of (15) & (19)

$\|x_{n+1}-p\|^2+\|a_{n+1}-q\|^2 \le \|u_n-p\|^2+\|c_n-q\|^2 \le \|x_n-p\|^2+\|a_n-q\|^2-\|x_n-u_n\|^2-\|a_n-c_n\|^2$

…………………………………………………………………(20)

That is,

$$\|x_n-u_n\|^2+\|a_n-c_n\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\|a_n-q\|^2-\|a_{n+1}-q\|^2 \to 0 \ as\ (n\to\infty)$$

…………………………………………………………………(21)

In view of (21), (17) and (A) and noting that $\{x_n-S_{i,\beta}u_n\}$ & $\{a_n-R_{j,\gamma}c_n\}$ are bounded ,we have

$$\begin{aligned}\|x_n-S_{i,\beta}u_n\}\|+\|a_n-R_{j,\gamma}c_n\| &\le \|x_n-u_n\|+\|a_n-c_n\|+\|u_n-S_{i,\beta}u_n\|+\|c_n-R_{j,\gamma}c_n\| \\ &\quad +\|S_{i,\beta}u_n-S_{i,\beta}x_n\|+\|R_{j,\gamma}c_n-R_{j,\gamma}a_n\| \\ &\le \|x_n-u_n\|+\|a_n-c_n\|+\|u_n-S_{i,\beta}u_n\|+\|c_n-R_{j,\gamma}c_n\|+\end{aligned}$$

$\{\|x_n-u_n\|^2+\frac{1}{1-\beta}|\langle u_n-S_{i,\beta}u_n,\ x_n-S_{i,\beta}u_n\rangle|\}^{1/2}+\{\|a_n-c_n\|^2+\frac{1}{1-\gamma}|\langle c_n-R_{j,\gamma}c_n, a_n-R_{j,\gamma}c_n\rangle|\}^{1/2} \to 0\ as\ (n\to\infty)$

…………………………………………………………………(22)

Therefore we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n-S_ix_n\| &+\lim_{n\to\infty}\|a_n-R_ja_n\| \\ &=\lim_{n\to\infty}\frac{1}{1-\beta}\|x_n-S_{i,\beta}x_n\|+\lim_{n\to\infty}\frac{1}{1-\gamma}\|a_n-R_{j,\gamma}a_n\|=0\end{aligned}$$

…………………………………………………………………(23)

The conclusion is proved.

Step3. Next we prove that the weak-accumulation point set $W_w(x_n)$ & $V_v(a_n)$ of the sequence$\{x_n\},\{a_n\}$ are singleton set and $W_w(x_n)\subset \mathcal{F},\ V_v(a_n)\subset \mathcal{F}$

Infact, for any $w \in W_w(x_n)$, $v \in V_v(a_n)$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}, \{a_{n_j}\} \subset \{a_n\}$ such that $x_{n_i} \rightharpoonup w, a_{n_j} \rightharpoonup v$. It follows from (19) that $u_{n_i} \rightharpoonup w, c_{n_j} \rightharpoonup v$ .Since $u_n = T_{r_n} x_n , c_n = Q_{r_n} a_n$ we have from condition A2 that

$$\langle y - u_{n_i}, \frac{1}{r_{n_i}}(u_{n_i} - x_{n_i})\rangle + \langle b - c_{n_j}, \frac{1}{d_{n_j}}(c_{n_j} - a_{n_j})\rangle \geq \emptyset(y, u_{n_i}) + \varphi\left(b - c_{n_j}\right) \forall y, b \in C$$

……(24)

Since $({}^{1}/_{r_{n_i}})\left(u_{n_i} - x_{n_i}\right) + ({}^{1}/_{d_{n_j}})\ (c_{n_j} - a_{n_j}) \to 0$ as(n $\to \infty$) and that $u_{n_i} \rightharpoonup w, c_{n_j} \rightharpoonup v$, it follows from condition A4, that $\emptyset(y, w) + \varphi(b, v) \leq 0, \forall\ y, b \in C$

…………………………………………(25)

For any $t \in (0,1), y, b \in C, letting\ y_t = t_y + (1-t)w, b_t = b_t + (1-t)v\ \ then\ y_t, b_t \in C$

$$0 = \emptyset(y_t, y_t) + \varphi(b_t, b_t) \leq\ t\{\emptyset(y_t, y) + \varphi(b_t, b)\} + (1-t)\{\emptyset(y_t, w) + \varphi(b_t, v)\} \leq t\{\emptyset(y_t, y) + \varphi(b_t, b)\}\ldots$$

…………………………………….(26)

This implies that $\emptyset(w, y) + \varphi(v, b) \geq 0\ \forall y, b \in C$

This shows that $w, v \in C$ are the solution to the equillibrium problem and that is $w \in \Omega$, v $\in \xi$.

On the other hand,by lemma2. for each $i \geq 1, I - S_i$ is demiclosed at $0.$ In view of (15),we know that

w, v $\in\ \mathcal{F}$.Due to the arbitrariness of $w \in W_w(x_n)$, $v \in V_v(a_n)$ with $x^* \neq y^*$ & $a^* \neq b^*$.Therefore there exists subsequences $\{x_{n_k}\}, \{x_{n_l}\}\ in\ \{x_n\}, \{a_{n_k}\}, \{a_{n_l}\} in\ \{a_n\}$ such that $x_{n_k} \rightharpoonup x^*\ and\ x_{n_j} \rightharpoonup y^*, and\ a_{n_k} \rightharpoonup a^*$ & $a_{n_l} \rightharpoonup b^*$.

Since x^*, y^*, a^*, $b^* \in C$. by (12) the limits $\lim_{n\to\infty}\|x_n - x^*\|$ & $\lim_{n\to\infty}\|x_n - y^*\|$ and $\lim_{n\to\infty}\|a_n - a^*\|$ & $\lim_{n\to\infty}\|a_n - b^*\|$ exists. By using Opial property of H , we have

$$\liminf_{n_k\to\infty}\left\|x_{n_k} - x^*\right\| + \liminf_{n_k\to\infty}\left\|a_{n_k} - a^*\right\| < \lim_{n_k\to\infty}\left\|x_{n_k} - y^*\right\| + \lim_{n_k\to\infty}\left\|a_{n_k} - b^*\right\| = \lim_{n\to\infty}\|x_n - y^*\| + \lim_{n\to\infty}\|a_n - b^*\|$$

$$< \lim_{n_j\to\infty} \left\|x_{n_j} - x^*\right\| + \lim_{n_j\to\infty} \left\|a_{n_j} - a^*\right\|$$

$$= \lim_{n\to\infty}\|x_n - x^*\| + \lim_{n\to\infty}\|a_n - a^*\|$$

$$= \liminf_{n_k\to\infty}\left\|x_{n_k} - x^*\right\| + \liminf_{n_k\to\infty}\left\|a_{n_k} - a^*\right\| \qquad \ldots\ldots(27)$$

This is a contradiction. Therefore , $W_w(x_n)$, $V_v(a_n)$ are singleton. Without loss of generality, we assume that $W_w(x_n) = \{x^*\}$, $V_v(a_n) = \{a^*\} and \quad x_n \rightharpoonup x^*$ & $a_n \rightharpoonup a^*$.By using (A) and(15)

,we have $u_n \rightharpoonup x^*$ & $a_n \rightharpoonup a^*$.

This completes the proof of the conclusion (I).

Next we prove the conclusion (II).

Without loss of generality, we can assume S_i & R_j are semicompact. From (15) we have that

$\|x_n - S_i x_n\| + \left\|a_n - R_J a_n\right\| \to 0\ as(n \to \infty)$……………………………………..(28)

Therefore, there exists a subsequence of $\{x_{n_i}\} \subset \{x_n\}$, & $\{a_{n_j}\} \subset \{a_n\}$ such that $x_{n_i} \to u^* \in C,$ & $a_{n_j} \to c^* \in C. Since \quad x_{n_i} \rightharpoonup x^*$ & $a_{n_j} \rightharpoonup a^*$ we have $x^* = u^*$ & $a^* = b^*$ and so $x_{n_i} \to x^* \in \mathcal{F},$ & $a_{n_j} \to a^* \in \mathcal{F}$.By virtue of (12), we have here

$$= \lim_{n\to\infty}\|u_n - x^*\| + \lim_{n\to\infty}\|c_n - a^*\| = 0\ \ and\ = \lim_{n\to\infty}\|x_n - x^*\| + \lim_{n\to\infty}\|a_n - a^*\| = 0$$

…………………………………………………………………………………(29)

This completes the proof of the theorem.

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