k-Regular and k-Duo $\Gamma$-Semirings

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Abstract

The concept of a k-duo $\Gamma$-semiring is introduced. Several characterizations of a k-duo $\Gamma$-semiring in a k-regular $\Gamma$-semiring are furnished. Further characterizations of a k-regular and k-duo $\Gamma$-semiring are studied by using different kinds of k-ideals in a $\Gamma$-semiring.

Keywords: k-ideal, k-bi-ideal, k-quasi-ideal, k-regular $\Gamma$-semiring, k-duo $\Gamma$-semiring.

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1. INTRODUCTION

The notion of a $\Gamma$-semiring was introduced by Rao [9] as a generalization of a semiring and studied it. Dutta and Sardar [1] discussed semiprime ideals in a $\Gamma$-semiring. Author studied quasi-ideals and minimal quasi-ideals of a $\Gamma$-semiring in [3] and bi-ideals of a $\Gamma$-semiring in [6]. In general ring ideal does not coincides with semiring ideal. Hence Henriksen [2] defined more restricted class of ideals in a semiring known as k-ideals. Sen and Adhikari [10, 11] studied k-ideals of semirings. Properties of k-ideals in a $\Gamma$-semiring were discussed by Rao [9] and Dutta and Sardar [1]. Also Author studied k-ideals and full k-ideals in $\Gamma$-semirings in [5].

Neumann [9] gave the definition of a regular ring. Analogously the concept of a regular semiring was introduced by Zelznikov [13]. This concept of regularity was extended to a $\Gamma$-semiring by Rao [9]. Author furnished some characterizations of regular $\Gamma$-semirings in [4]. In [7] Author gave definitions of k-quasi-ideal, k-bi-ideal and k-regular $\Gamma$-semiring and then some characterizations of k-regular $\Gamma$-semirings.
are furnished. The concept of a duo semiring was considered by Shabir, Ali and Batool [12] and proved some characterizations of it. In [4] author introduced the concept of a duo $\Gamma$-semiring and gave some characterizations of it.

In this paper the notions of a left $k$-duo $\Gamma$-semiring, right $k$-duo $\Gamma$-semiring and $k$-duo $\Gamma$-semiring are defined. Various characterizations of a $k$-duo $\Gamma$-semiring in a $k$-regular $\Gamma$-semiring are proved. Further some characterizations of a $k$-regular and $k$-duo $\Gamma$-semiring are discussed by using $k$-ideals, $k$-bi-ideals, $k$-quasi-ideals in a $\Gamma$-semiring.

2. PRELIMINARIES:

For the basic concepts of $\Gamma$-semirings we follow Dutta and Sardar [1].

**Definition 2.1:** Let $S$ and $\Gamma$ be two additive commutative semigroups. $S$ is called a $\Gamma$-semiring if there exists a mapping $S \times \Gamma \times S \to S$ whose image is denoted by $a\alpha b$; for all $a, b \in S$ and for all $\alpha \in \Gamma$ satisfying the following conditions:

(i) $a(a+b) = (a\alpha b) + (a\alpha c)$
(ii) $(b+c)a = (b\alpha a) + (c\alpha a)$
(iii) $a(a+b) = (a\alpha c) + (a\beta c)$
(iv) $aa(b+c) = (a\alpha b)\beta c$; for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

**Definition 2.2:** An element $0 \in S$ is said to be an absorbing zero if $0\alpha a = 0 = a\alpha 0, a + 0 = 0 + a = a$; for all $a \in S$ and for all $\alpha \in \Gamma$.

**Definition 2.3:** A non-empty subset $T$ of a $\Gamma$-semiring $S$ is said to be a sub-$\Gamma$-semiring of $S$ if $(T, +)$ is a subsemigroup of $(S, +)$ and $a\alpha b \in T$; for all $a, b \in T$ and for all $\alpha \in \Gamma$.

**Definition 2.4:** A non-empty subset $T$ of a $\Gamma$-semiring $S$ is called a left (respectively right) ideal of $S$ if $T$ is a subsemigroup of $(S, +)$ and $x\alpha a \in T$ (respectively $a\alpha x \in T$) for all $a \in T, x \in S$ and for all $\alpha \in \Gamma$.

**Definition 2.5:** If a non-empty subset $T$ of a $\Gamma$-semiring $S$ is both left and right ideal of $S$, then $T$ is known as an ideal of $S$.  


**Definition 2.6:** A right ideal $I$ of a $\Gamma$-semiring $S$ is said to be a right $k$-ideal if $a \in I$ and $x \in S$ such that $a + x \in I$, then $x \in I$.

Similarly we define a left $k$-ideal of a $\Gamma$-semiring $S$.

If an ideal $I$ is both right $k$-ideal and left $k$-ideal of a $\Gamma$-semiring $S$, then $I$ is known as a $k$-ideal of $S$.

**Examples:**
(1) Let $N_0$ denotes the set of all positive integers with zero. $S = N_0$ is a semiring and with $\Gamma = S$, $S$ forms a $\Gamma$-semiring. A subset $I = 3N_0 \setminus \{3\}$ of $S$ is an ideal of $S$ but not a $k$-ideal. Since $6, 9 = 3 + 6 \in I$ but $3 \notin I$.

(2) If $S = N$ is the set of all positive integers, then $(S, \max., \min.)$ is a semiring and with $\Gamma = S$, $S$ forms a $\Gamma$-semiring. $I_n = \{1, 2, 3, \ldots, n\}$ is a $k$-ideal for any $n \in I$.

**Definition 2.7:** For a subset $I$ of a $\Gamma$-semiring $S$ define
\[ \overline{I} = \{a \in S \mid a + x \in I, \text{for some } x \in I\} \]
\[ \overline{I} \] is called a $k$-closure of $I$.

**Definition 2.8 [7]:** A non-empty subset $B$ of a $\Gamma$-semiring $S$ is said to be a $k$-bi-ideal of $S$ if $B$ is a sub-$\Gamma$-semiring of $S$, $B \Gamma S \Gamma B \subseteq B$ and if $a \in B$ and $x \in S$ such that $a + x \in B$, then $x \in B$.

**Definition 2.9 [7]:** A subsemigroup $Q$ of $(S, +)$ is a $k$-quasi-ideal of $S$ if $\overline{(STQ)} \cap (QTS) \subseteq Q$ and if $a \in Q$ and $x \in S$ such that $a + x \in Q$, then $x \in Q$.

**Definition 2.10 [7]:** An element $a$ of a $\Gamma$-semiring $S$ is said to be $k$-regular if $a \in a\Gamma S \Gamma a$.

If all elements of a $\Gamma$-semiring $S$ are $k$-regular, then $S$ is known as a $k$-regular $\Gamma$-semiring.

Now onwards $S$ denotes a $\Gamma$-semiring with absorbing zero unless otherwise stated.

Some basic properties of $k$-closure are given in the following lemma.

**Lemma 2.11:** For non-empty subsets $A$ and $B$ of $S$ we have,
1) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$. 
2) $\tilde{A}$ is the smallest (left $k$-ideal, right $k$-ideal, $k$-quasi-ideal, $k$-bi-ideal) $k$-ideal containing (left $k$-ideal, right $k$-ideal, $k$-quasi-ideal, $k$-bi-ideal) $k$-ideal $A$ of $S$.

3) $\tilde{A} = A$ if and only if $A$ is (left $k$-ideal, right $k$-ideal, $k$-quasi-ideal, $k$-bi-ideal) $k$-ideal of $S$.

4) $\tilde{\tilde{A}} = \tilde{A}$, where $A$ is (left $k$-ideal, right $k$-ideal, $k$-quasi-ideal, $k$-bi-ideal) $k$-ideal of $S$.

5) $\tilde{A} \Gamma \tilde{B} = \tilde{AB}$, where $A$ and $B$ are (left $k$-ideals, right $k$-ideals, $k$-quasi-ideals, $k$-bi-ideals) $k$-ideals of $S$.

**Theorem 2.12 [7]:** In $S$ following statements are equivalent.

1) $S$ is $k$-regular.

2) For every left $k$-ideal $L$ and right $k$-ideal $R$ of $S$, $R \Gamma L = R \cap L$.

3) For every left $k$-ideal $L$ and right $k$-ideal $R$ of $S$,
   
   (i) $R^2 = R \Gamma R = R$

   (ii) $L^2 = L \Gamma L = L$

   (iii) $R \Gamma L = R \cap L$ is a $k$-quasi-ideal of $S$.

4) Every $k$-quasi-ideal $Q$ of $S$ is of the form $Q \Gamma S \Gamma Q = Q$.

3. **k-DUO $\Gamma$-SEMIRING** :

Now we define a k-duo $\Gamma$-semiring as follows.

**Definition 3.1:** A $\Gamma$-semiring $S$ is said to be left (right) k-duo $\Gamma$-semiring if every left (right) $k$-ideal of $S$ is a right (left) $k$-ideal.

A $\Gamma$-semiring $S$ is said to be a k-duo $\Gamma$-semiring if every one sided $k$-ideal of $S$ is a two sided $k$-ideal. That is a $\Gamma$-semiring $S$ is said to be a k-duo $\Gamma$-semiring if it is both left k-duo and right k-duo.

**Theorem 3.2:** If $S$ is $k$-regular, then $S$ is left k-duo if and only if for any two left $k$-ideals $A$ and $B$ of $S$, $A \cap B = \tilde{A} \Gamma \tilde{B}$.

**Proof:** Let $S$ be a k-regular $\Gamma$-semiring. Suppose that $S$ is a left k-duo $\Gamma$-semiring. Let $A$ and $B$ be any two left $k$-ideals of $S$. Hence $A$ is a right $k$-ideal of $S$. Therefore by Theorem 2.12, $A \cap B = \tilde{A} \Gamma \tilde{B}$. Conversely, by assumption $L \Gamma S = L \cap$
$S \subseteq L$. This shows that $L$ is a right $k$-ideal of $S$. Therefore $S$ is a left $k$-duo $\Gamma$-semiring.

Similar to Theorem 3.2 we have following theorem.

**Theorem 3.3:** If $S$ is $k$-regular, then $S$ is right $k$-duo if and only if for any two right $k$-ideals $A$ and $B$ of $S$, $A \cap B = \overline{A \Gamma B}$.

**Theorem 3.4:** If $S$ is $k$-regular, then $S$ is left $k$-duo if and only if every $k$-quasi-ideal of $S$ is a right $k$-ideal of $S$.

**Proof:** Let $S$ be a $k$-regular $\Gamma$-semiring and $Q$ be any $k$-quasi-ideal of $S$. Suppose that $S$ is left $k$-duo. Then there exists a right $k$-ideal $R$ and a left $k$-ideal $L$ of $S$ such that $Q = R \cap L$. Therefore $Q = R \cap L$ is a right $k$-ideal of $S$. Conversely, suppose that every $k$-quasi-ideal of $S$ is a right $k$-ideal of $S$. Let $L$ be a left $k$-ideal of $S$. Hence $L$ is a $k$-quasi-ideal of $S$. Therefore by assumption $L$ is a right $k$-ideal of $S$. Hence $S$ is a left $k$-duo $\Gamma$-semiring.

Proofs of the following theorems are similar to above theorem hence omitted.

**Theorem 3.5:** If $S$ is $k$-regular, then $S$ is right $k$-duo if and only if every $k$-quasi-ideal of $S$ is a left $k$-ideal of $S$.

**Theorem 3.6:** A $k$-regular $\Gamma$-semiring $S$ is $k$-duo if and only if every $k$-quasi-ideal of $S$ is a $k$-ideal of $S$.

**Theorem 3.7:** If $S$ is $k$-regular, then $S$ is left $k$-duo if and only if every $k$-bi-ideal of $S$ is a right $k$-ideal of $S$.

**Theorem 3.8:** If $S$ is $k$-regular, then $S$ is right $k$-duo if and only if every $k$-bi-ideal of $S$ is a left $k$-ideal of $S$.

**Theorem 3.9:** If $S$ is $k$-regular, then $S$ is $k$-duo if and only if every $k$-bi-ideal of $S$ is a $k$-ideal of $S$. 

4. k-REGULAR AND k-DUO Γ-SEMIRING

In this section characterizations of a k-regular and k-duo Γ-semiring are furnished.

**Theorem 4.1:** Following statements are equivalent in $S$.

1. $S$ is k-regular and left k-duo.
2. For any k-bi-ideal $B$ and a left k-ideal $L$ of $S$, $B \cap L = B \Gamma L$.
3. For any k-quasi-ideal $Q$ and a left k-ideal $L$ of $S$, $Q \cap L = Q \Gamma L$.

**Proof:**

(1) $\Rightarrow$ (2)

Let $B$ be a k-bi-ideal and $L$ be a left k-ideal of $S$. Then by Theorem 3.7, $B$ is a right k-ideal of $S$. Therefore $B \Gamma L \subseteq B$ and $B \Gamma L \subseteq L$. Hence $B \Gamma L \subseteq B \cap L$.

Let $a \in B \cap L$. Hence $a \in a \Gamma S \Gamma a$. Therefore $a \Gamma S \Gamma a \subseteq B \Gamma S \Gamma L \subseteq B \Gamma L$. Thus $B \cap L \subseteq B \Gamma L$. Hence we get $B \cap L = B \Gamma L$.

(2) $\Rightarrow$ (3)

Implication holds as every k-quasi-ideal of $S$ is a k-bi-ideal of $S$.

(3) $\Rightarrow$ (1)

Let $R$ be a right k-ideal and $L$ be a left k-ideal of $S$. Then by (3), $R \cap L = R \Gamma L$. This shows that $S$ is k-regular by Theorem 2.12. For $L = S$, we have $L \cap S = L \Gamma S$. Therefore $L = L \Gamma S$. Hence $L$ is a right k-ideal. Thus $S$ is left k-duo.

**Theorem 4.2:** Following statements are equivalent in $S$.

1. $S$ is k-regular and right k-duo.
2. For any k-bi-ideal $B$ and a right k-ideal $R$ of $S$, $B \cap R = R \Gamma B$.
3. For any k-quasi-ideal $Q$ and a right k-ideal $R$ of $S$, $Q \cap R = R \Gamma Q$.

**Proof:**

(1) $\Rightarrow$ (2)

Let $B$ be a k-bi-ideal and $R$ be a right k-ideal of $S$. Then by Theorem 3.8, $B$ is a left k-ideal of $S$. Therefore $R \Gamma B \subseteq B$ and $R \Gamma B \subseteq R$. Hence $R \Gamma B \subseteq B \cap R$. Let $a \in B \cap R$. Hence $a \in a \Gamma S \Gamma a$. Therefore $a \Gamma S \Gamma a \subseteq R \Gamma S \Gamma B \subseteq R \Gamma B$. Thus $B \cap R \subseteq R \Gamma B$. Hence we get $B \cap R = R \Gamma B$.

(2) $\Rightarrow$ (3)

Implication follows as every k-quasi-ideal of $S$ is a k-bi-ideal of $S$.

(3) $\Rightarrow$ (1)

Let $R$ be a right k-ideal and $L$ be a left k-ideal of $S$. Then by (3), $R \cap L = R \Gamma L$. This shows that $S$ is k-regular. For $R = S$, we have $R \cap S = S \Gamma R$. Therefore $R = S \Gamma R$. Hence $R$ is a left k-ideal. Therefore $S$ is right k-duo.
Theorem 4.3: In S following conditions are equivalent.

(1) S is k-regular and k-duo.
(2) For any two k-quasi-ideals Q₁ and Q₂ of S, \( Q₁ \cap Q₂ = \overline{Q₁ \Gamma Q₂} \).
(3) For a left k-ideal L and a right k-ideal R of S, \( L \cap R = \overline{L \Gamma R} \).

Proof: (1) \( \Rightarrow \) (2)

Let \( Q₁ \) and \( Q₂ \) be any two k-quasi-ideals of S. Therefore \( Q₁ = R₁ \cap L₁ \) and \( Q₂ = R₂ \cap L₂ \), where \( R₁ \) and \( R₂ \) are right k-ideals and \( L₁ \), \( L₂ \) are left k-ideals of S. Therefore \( Q₁ = R₁ \cap L₁ \) and \( Q₂ = R₂ \cap L₂ \) are k-ideals of S. Hence by Theorem 2.12, \( Q₁ \cap Q₂ = \overline{Q₁ \Gamma Q₂} \).

(2) \( \Rightarrow \) (3)

Let \( R \) be a right k-ideal and \( L \) be a left k-ideal of S. Therefore \( R \) and \( L \) are k-regular and k-duo of S. Hence by (2), we have \( L \cap R = \overline{L \Gamma R} \).

(3) \( \Rightarrow \) (1)

For \( R = S \), \( \overline{L \Gamma S} = L \cap S = L \). This shows that a left k-ideal \( L \) is a right k-ideal of S. Similarly we can show that a right k-ideal \( R \) is a left k-ideal of S. Thus every one sided k-ideal of S is a k-ideal. Hence S is a k-duo \( \Gamma \)-semiring. Then clearly \( R \cap L = \overline{R \Gamma L} \). Hence S is k-regular (see Theorem 2.12).

Theorem 4.4: In S following conditions are equivalent.

(1) S is k-regular and k-duo.
(2) \( I \cap B = \overline{I \Gamma B \Gamma I} \), for every k-ideal I and every k-bi-ideal B of S.
(3) \( I \cap Q = \overline{I \Gamma Q \Gamma I} \), for every k-ideal I and every k-quasi-ideal Q of S.

Proof: (1) \( \Rightarrow \) (2)

Let I be a k-ideal and B be a k-bi-ideal of S. Hence by Theorem 3.9, B is a k-ideal of S. Therefore \( \overline{I \Gamma B \Gamma I} \subseteq I \) and \( \overline{I \Gamma B \Gamma I} \subseteq B \). Hence \( \overline{I \Gamma B \Gamma I} \subseteq I \cap B \). Take any \( a \in I \cap B \). Hence \( a \in \overline{a \Gamma S \Gamma a} \). Therefore \( \overline{a \Gamma S \Gamma a} \subseteq \overline{a \Gamma S \Gamma (a \Gamma S \Gamma a)} \subseteq \overline{I \Gamma S \Gamma (B \Gamma S \Gamma I)} \subseteq \overline{I \Gamma B \Gamma I} \). Thus \( I \cap B \subseteq \overline{I \Gamma B \Gamma I} \). Therefore \( I \cap B = \overline{I \Gamma B \Gamma I} \).

(2) \( \Rightarrow \) (3)

As every k-quasi-ideal of S is a k-bi-ideal of S, implication holds.

(3) \( \Rightarrow \) (1)

For a left k-ideal \( L \) and a right k-ideal \( R \) of S, by (3) we have \( L = S \cap L = \overline{S \Gamma L \Gamma S} \) and \( R = S \cap R = \overline{S \Gamma R \Gamma S} \). Now \( \overline{L \Gamma S} = \overline{S \Gamma L \Gamma S} \subseteq \overline{S \Gamma L \Gamma S} = L \) and \( \overline{S \Gamma R} = \overline{S \Gamma R \Gamma S} \subseteq \overline{S \Gamma R \Gamma S} = R \).

\[ \square \]
\( S = R \cap L \subseteq S \Gamma R \subseteq R \cap L \). Therefore \( R \cap L = R \Gamma L \). Hence by Theorem 2.12, \( S \) is k-regular.

**Theorem 4.5:** \( S \) is k-regular and k-duo if and only if \( L \cap R = L \Gamma R \subseteq S \Gamma S \), for a left k-ideal \( L \) and a right k-ideal \( R \) of \( S \).

**Proof:** Assume that \( S \) is a k-regular and k-duo \( \Gamma \)-semiring. Let \( R \) be a right k-ideal and \( L \) be a left k-ideal of \( S \). Hence \( R \) is a left \( k \)-ideal and \( L \) is a right \( k \)-ideal of \( S \). Therefore \( L \Gamma R \subseteq L \) and \( L \Gamma R \subseteq L \Gamma L \subseteq R \). Thus we get \( L \Gamma R \subseteq L \cap R \). Let \( a \in L \cap R \). Hence \( a \in S \). Therefore \( S 

**Theorem 4.6:** \( S \) is k-regular and k-duo if and only if \( L \cap R = L \Gamma R \subseteq S \Gamma R \), for a left k-ideal \( L \) and a right k-ideal \( R \) of \( S \).

**Proof:** Assume that \( S \) is a k-regular and k-duo \( \Gamma \)-semiring. Let \( R \) be a right k-ideal and \( L \) be a left k-ideal of \( S \). Therefore \( R \) is a left \( k \)-ideal and \( L \) is a right \( k \)-ideal of \( S \). Hence \( L \subseteq L \Gamma L \subseteq L \). Therefore \( L \Gamma L \subseteq L \cap I \). Similarly we can show that \( L \subseteq L \Gamma L \subseteq R \cap I \). Take any \( a \in L \cap I \). Hence \( a \in S \Gamma S \). Therefore \( S \Gamma L \subseteq L \Gamma L \subseteq L \Gamma L \). Hence \( L \cap I \subseteq L \Gamma L \). In the same way we can show that \( R \cap I = R \Gamma R \). Conversely, let \( R \) be a right \( k \)-ideal and \( L \) be a left \( k \)-ideal of \( S \). Hence by assumption, \( L \cap S = L \Gamma S \) and \( S \cap R = S \Gamma R \). Therefore \( L = L \Gamma R \). This shows that \( L \) is a right \( k \)-ideal and \( R \) is a left \( k \)-ideal of \( S \). Therefore \( S \) is a k-regular \( \Gamma \)-semiring.
Then clearly $R \cap L = \overline{RL}$ holds by assumption. Therefore $S$ is a k-regular $\Gamma$-semiring (see Theorem 2.12).

**Theorem 4.8**: Following statements are equivalent in $S$.

1. $S$ is k-regular and k-duo.
2. For any k-bi-ideals $A$ and $B$ of $S$, $A \cap B = \overline{AB\Gamma S}$.
3. For any k-bi-ideals $A$ and $B$ of $S$, $A \cap B = \overline{ST\Gamma B}$.
4. For any k-bi-ideal $B$ and a k-quasi-ideal $Q$ of $S$, $B \cap Q = \overline{B\Gamma Q\Gamma S}$.
5. For any k-bi-ideal $B$ and a k-quasi-ideal $Q$ of $S$, $B \cap Q = \overline{S\Gamma B\Gamma Q}$.
6. For any k-bi-ideal $B$ and a k-quasi-ideal $Q$ of $S$, $B \cap Q = \overline{Q\Gamma B\Gamma S}$.
7. For any k-bi-ideal $B$ and a k-quasi-ideal $Q$ of $S$, $B \cap Q = \overline{S\Gamma Q\Gamma B}$.
8. For any k-quasi-ideals $Q_1$ and $Q_2$ of $S$, $Q_1 \cap Q_2 = \overline{Q_1\Gamma Q_2\Gamma S}$.
9. For any k-quasi-ideals $Q_1$ and $Q_2$ of $S$, $Q_1 \cap Q_2 = \overline{S\Gamma Q_1\Gamma Q_2}$.

**Proof**: (1) $\Rightarrow$ (2)

Let $A$ and $B$ be any two k-bi-ideals of $S$. Hence by Theorem 3.9, both $A$ and $B$ are k-ideals of $S$. Therefore $\overline{AB\Gamma S} \subseteq A$ and $\overline{AB\Gamma S} \subseteq \overline{AB} \subseteq B$. Hence $\overline{AB\Gamma S} \subseteq A \cap B$. Let $a \in A \cap B$. Therefore $a \in \overline{a\Gamma S\Gamma a}$. Hence $a \Gamma S\Gamma a \subseteq (a\Gamma S\Gamma a) \overline{S\Gamma a} \subseteq (\overline{A\Gamma S\Gamma B})\overline{S} \subseteq \overline{AB\Gamma S}$. Thus, we get $A \cap B = \overline{AB\Gamma S}$.

(2) $\Rightarrow$ (4), (4) $\Rightarrow$ (8), (2) $\Rightarrow$ (6), (6) $\Rightarrow$ (8)

Implications follow as every k-quasi-ideal of $S$ is a k-bi-ideal of $S$.

(8) $\Rightarrow$ (1)

Let $R$ be a right k-ideal and $L$ be a left k-ideal of $S$. Then both $R$ and $L$ are k-quasi-ideals of $S$. Hence by (8), $L \cap R = \overline{L\Gamma R\Gamma S}$. Therefore by Theorem 4.5, $S$ is a k-regular and k-duo $\Gamma$-semiring.

(1) $\Rightarrow$ (3)

Let $A$ and $B$ be any two k-bi-ideals of $S$. Therefore by Theorem 3.9, both $A$ and $B$ are k-ideals of $S$. Then $\overline{ST\Gamma A} \subseteq B$ and $\overline{ST\Gamma A} \subseteq \overline{ST\Gamma B} \subseteq \overline{AB}$. Thus we get $\overline{ST\Gamma A} \subseteq A \cap B$. Take any $a \in A \cap B$. Therefore $a \in \overline{a\Gamma S\Gamma a}$. Hence $a \Gamma S\Gamma a \subseteq (a\Gamma S\Gamma a) \overline{S\Gamma a} \subseteq (\overline{A\Gamma S\Gamma B})\overline{S} \subseteq \overline{ST\Gamma A\Gamma B}$. Thus, we get $A \cap B \subseteq \overline{ST\Gamma A\Gamma B}$. Therefore $A \cap B = \overline{ST\Gamma A\Gamma B}$.
Clearly implications hold as every $k$-quasi-ideal of $S$ is a $k$-bi-ideal of $S$.

Let $R$ be a right $k$-ideal and $L$ be a left $k$-ideal of $S$. Then both $R$ and $L$ are $k$-quasi-ideals of $S$. Therefore by (9), $L \cap R = S \Gamma L \Gamma R$. This shows that $S$ is a $k$-regular and $k$-duo $\Gamma$-semiring by Theorem 4.6.

**Theorem 4.9:** In $S$ following statements are equivalent.

1. $S$ is $k$-regular and $k$-duo.
2. For any $k$-bi-ideals $A$, $B$ and a $k$-ideal $I$ of $S$, $A \cap B \cap I = A \Gamma B \Gamma I$.
3. For any $k$-bi-ideals $A$, $B$ and a $k$-ideal $I$ of $S$, $A \cap B \cap I = I \Gamma A \Gamma B$.
4. For any $k$-bi-ideal $B$, a $k$-quasi-ideal $Q$ and a $k$-ideal $I$ of $S$, $B \cap Q \cap I = B \Gamma Q \Gamma I$.
5. For any $k$-bi-ideal $B$, a $k$-quasi-ideal $Q$ and a $k$-ideal $I$ of $S$, $B \cap Q \cap I = I \Gamma Q \Gamma B$.
6. For any $k$-bi-ideal $B$, a $k$-quasi-ideal $Q$ and a $k$-ideal $I$ of $S$, $B \cap Q \cap I = Q \Gamma B \Gamma I$.
7. For any $k$-bi-ideal $B$, a $k$-quasi-ideal $Q$ and a $k$-ideal $I$ of $S$, $B \cap Q \cap I = Q \Gamma I \Gamma B$.
8. For any $k$-quasi-ideals $Q_1$, $Q_2$ and a $k$-ideal $I$ of $S$, $Q_1 \cap Q_2 \cap I = Q_1 \Gamma Q_2 \Gamma I$.
9. For any $k$-quasi-ideals $Q_1$, $Q_2$ and a $k$-ideal $I$ of $S$, $Q_1 \cap Q_2 \cap I = I \Gamma Q_1 \Gamma Q_2$.

**Proof:** (1) $\Rightarrow$ (2)

Let $A, B$ be any two $k$-bi-ideals and $I$ be a $k$-ideal of $S$. Therefore by Theorem 3.9, both $A$ and $B$ are $k$-ideals of $S$. Hence $A \Gamma B \Gamma I \subseteq A$ and $A \Gamma B \Gamma I \subseteq A \Gamma B \subseteq B$. Also $A \Gamma B \Gamma I \subseteq I$. Hence we get $A \Gamma B \Gamma I \subseteq A \cap B \cap I$. Let $\alpha \in A \cap B \cap I$. Hence $\alpha \in A \Gamma B \Gamma I$. Therefore $a \Gamma S \Gamma a \subseteq (a \Gamma S \Gamma a) \Gamma S \Gamma a \subseteq (A \Gamma S \Gamma B) \Gamma S \Gamma I \subseteq A \Gamma B \Gamma I$. Thus we get $A \cap B \cap I \subseteq A \Gamma B \Gamma I$. Hence $A \cap B \cap I = A \Gamma B \Gamma I$.

(2) $\Rightarrow$ (4), (4) $\Rightarrow$ (8), (2) $\Rightarrow$ (6), (6) $\Rightarrow$ (8)
Implications follow as every k-quasi-ideal of \( S \) is a k-bi-ideal of \( S \).

\((8) \Rightarrow (1)\)

Let \( R \) be a right k-ideal and \( L \) be a left k-ideal of \( S \). Then both \( R \) and \( L \) are k-quasi-ideals of \( S \). Therefore by (8), \( L \cap R = L \Gamma R \Gamma S \). Hence \( S \) is a k-regular and k-duo \( \Gamma \)-semiring by Theorem 4.5.

\((1) \Rightarrow (3)\)

Let \( A, B \) be any two k-bi-ideals and \( I \) be a k-ideal of \( S \). Hence by Theorem 3.9, both \( A \) and \( B \) are k-ideals of \( S \). Therefore \( I \Gamma A \Gamma B \subseteq B \) and \( I \Gamma A \Gamma B \subseteq A \). Also \( I \Gamma A \Gamma B \subseteq I \). Take any \( a \in A \cap B \cap I \). Hence \( a \in a \Gamma ST \Gamma a \). Therefore \( a \Gamma ST \Gamma a \subseteq a \Gamma ST (a \Gamma ST \Gamma a) \subseteq I \Gamma ST (A \Gamma ST \Gamma B) \subseteq I \Gamma A \Gamma B \). Hence \( A \cap B \cap I \subseteq I \Gamma A \Gamma B \). Thus we get \( A \cap B \cap I = I \Gamma A \Gamma B \).

\((3) \Rightarrow (5), (5) \Rightarrow (9), (3) \Rightarrow (7), (7) \Rightarrow (9)\)

As every k-quasi-ideal of \( S \) is a k-bi-ideal of \( S \), implications hold.

\((9) \Rightarrow (1)\)

Let \( R \) be a right k-ideal and \( L \) be a left k-ideal of \( S \). Then both \( R \) and \( L \) are k-quasi-ideals of \( S \). Therefore by (9), \( L \cap R = L \Gamma R \Gamma S \). Hence \( S \) is a k-regular and k-duo \( \Gamma \)-semiring (see Theorem 4.6).

Proof of following theorem is straightforward so omitted.

**Theorem 4.10:-** In \( S \) following statements are equivalent.

1. \( S \) is k-regular and k-duo.
2. For every k-bi-ideals \( A \) and \( B \) of \( S \), \( A \cap B = A \Gamma B \).
3. For every k-bi-ideal \( B \) and a k-quasi-ideal \( Q \) of \( S \), \( B \cap Q = B \Gamma Q \).
4. For every k-bi-ideal \( B \) and a right k-ideal \( R \) of \( S \), \( B \cap R = B \Gamma R \).
5. For every k-quasi-ideal \( Q \) and a k-bi-ideal \( B \) of \( S \), \( Q \cap B = Q \Gamma B \).
6. For every k-quasi-ideals \( Q_1 \) and \( Q_2 \) of \( S \), \( Q_1 \cap Q_2 = Q_1 \Gamma Q_2 \).
7. For every k-quasi-ideal \( Q \) and a right k-ideal \( R \) of \( S \), \( Q \cap R = Q \Gamma R \).
8. For every left k-ideal \( L \) and a k-bi-ideal \( B \) of \( S \), \( L \cap B = L \Gamma B \).
9. For every left k-ideal \( L \) and a right k-ideal \( R \) of \( S \), \( L \cap R = L \Gamma R \).
REFERENCES


