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Comparing Different Estimators of three Parameters for Transmuted Weibull Distribution

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Abstract

This paper deals with expanding the family of Weibull distribution, through introducing another parameter (β), to the original Weibull family which have one scale parameter (θ), and one shape parameter (λ). This expansion necessary to construct different model which will be add flexibility in representing the data, the family is called transmuted Weibull, here we explain the cumulative distribution function and the probability density function, and deriver ' th moments about origin, and then estimating parameters by different methods like maximum likelihood method, and moment estimators.

The comparison between estimators is done through simulation procedure using different sample size n=100, 150, 250 and different sets of inhial values for (λ,θ,β), through applying the generation formula which is derived in this paper. The comparison between estimators is done using statistical measure mean square error (MSE), and all result is explained in tables.

Keywords: Transmuted Weibull distribution, Maximum likelihood method and Moment estimators, Three Parameters.

1. INTRODUCTION

The modification on distributions are introduced, like using weighted distribution and transmuted distribution [12], through introducing a new parameter, to obtain a new family distribution (or modified formula) as compared with original probability distribution.

The modification on models helps the researcher to have another model that deals with estimation, and computing confidence interval for estimated parameters, also for modeling lifetime data.

Also the generalized Weibull distributions, was studied by Murthy.D.N.P,Xie.M, & Jiang.R. [10] and indicate to various properties and various extensions which can be applied on this distribution we know that Weibull distribution have a potential contribution to equipment maintenance and life policies . We continue the research about the distribution (Weibull) due to its benefits, in many statistical application and work on finding a formula for the p.d .f and C.D.F. of transmuted Weibull distribution and then work on estimating its three parameters (θ,β.λ) using different methods of mounts and maximum likelihood.

2. THE THEORETICAL SIDE

Let T be A random variable distributed Weibull distribution with two parameters (λ,θ) defined by equation bellow :

$$f(t;\lambda,\theta) = \frac{\lambda}{\theta} t^{\lambda-1}\, e^{-\frac{t^{\lambda}}{\theta}} \ldots (1)$$

0= 0.w t<0

λ: is shape parameter, θ: is scale parameter with C.D.F. given in equation (2)

$$F(t,\lambda,\theta) = 1 - e^{-\frac{t^{\lambda}}{\theta}} \ldots (2)$$

Apply equation (3) to the C.D.F given in (2), to obtain G(t): $|\beta| \leq 1$

$G(t) = (1+\beta)\, F(t) - \beta (F(t))^2 \;\ldots (3)$

Using equation (3), so that the new C.D.F of transmuted Weibull(1) ,Which is

$G(t) = (1+\beta)\,(F(t)) - \beta (F(t))^2$

$$G(t) = (1+\beta)\left(\left(1 - e^{-\frac{t^{\lambda}}{\theta}}\right) - \beta\left[(1 - e^{-\frac{t^{\lambda}}{\theta}})^2\right]\right)$$

$$G(t) = (1+\beta)\left(1 - e\frac{-t^{\lambda}}{\theta}\right) - \beta\left[(1 - 2e^{\frac{-t^{\lambda}}{\theta}} + e^{-2\frac{t^{\lambda}}{\theta}})\right]$$

This lead to the new C.D.F of transmuted

$$G(t)=\left(1-e^{\frac{-t^{\lambda}}{\theta}}\right)+\beta\left[\left(e^{\frac{-t^{\lambda}}{\theta}}-e^{-2\frac{t^{\lambda}}{\theta}}\right)\right]\ldots 4$$

Deriving equation (4), gives us the p.d.f

g(t) od the new model, which is given in equation(5)

$$g(t)=(1-\beta)\frac{\lambda}{\theta}t^{\lambda-1}e^{-\frac{t^{\lambda}}{\theta}}+2\frac{\beta\lambda}{\theta}t^{\lambda-1}e^{-2\frac{t^{\lambda}}{\theta}}\ldots(5)$$

Then we can simplify the formula in equation (5), for the p.d.f of transmuted Wiebull

$$g(t)=\frac{\lambda\beta}{\theta}t^{\lambda-1}e^{-\frac{t^{\lambda}}{\theta}}\left[\left(\frac{1}{\beta}-1\right)+2e^{-\frac{t^{\lambda}}{\theta}}\right]\ldots(6)$$

$\theta > 0$, $t > 0$, $\beta \leq 1$

And the C.D.F of this new generated transmuted wiebull is:

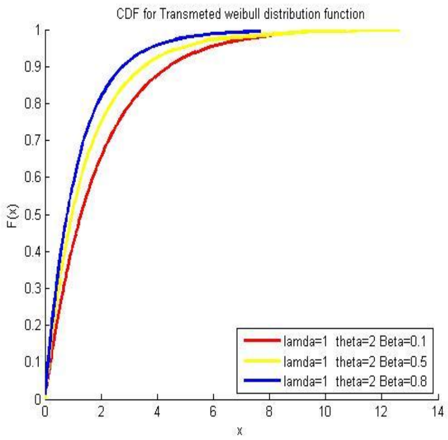
$$G(t)=\left(1-e^{-\frac{t^{\lambda}}{\theta}}\right)+\beta\left[e^{-\frac{t^{\lambda}}{\theta}}-e^{-2\frac{t^{\lambda}}{\theta}}\right]$$

$$let\ Z=e^{-\frac{t^{\lambda}}{\theta}}$$

$$\therefore G(t)=1+(\beta-1)z-\beta z^{2}$$

$$G(t)=\left(1-e^{-\frac{t^{\lambda}}{\theta}}\right)+\left[1+\beta e^{-\frac{t^{\lambda}}{\theta}}\right]\ldots(7)$$

While the p.d.f of this new generated transmuted Weibull distribution which is given in equation (5)


CDF for Transmeted weibull distribution function
F(x)
x
lamda=1 theta=2 Beta=0.1
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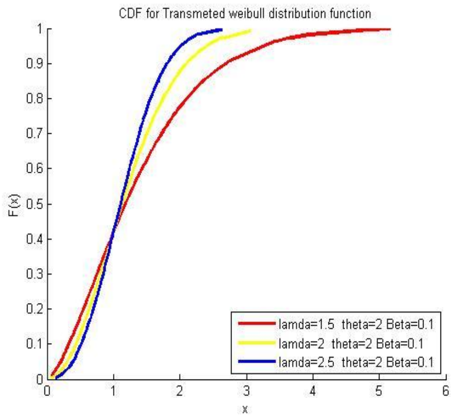
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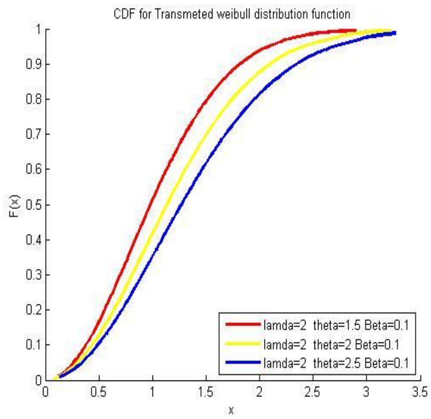
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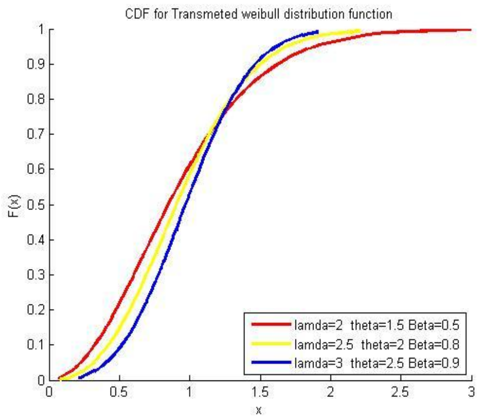
CDF for Transmeted weibull distribution function
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x
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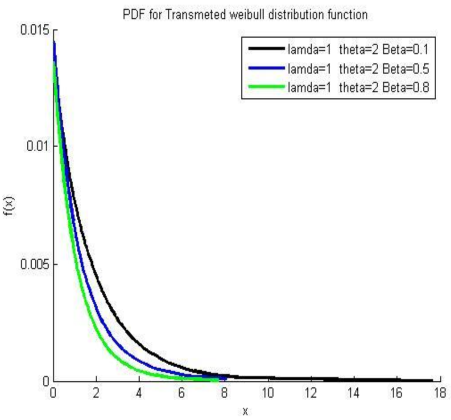
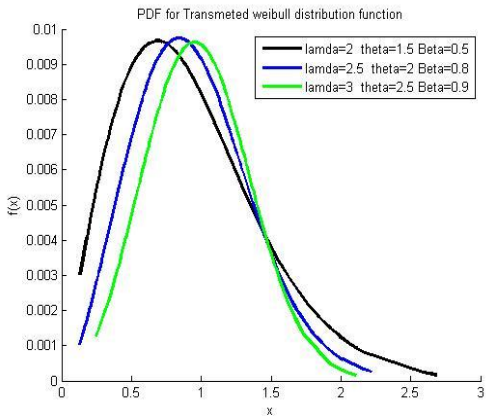
lamda=3 theta=2.5 Beta=0.9


PDF for Transmeted weibull distribution function
f(x)
x
lamda=1 theta=2 Beta=0.1
lamda=1 theta=2 Beta=0.5
lamda=1 theta=2 Beta=0.8


PDF for Transmeted weibull distribution function
f(x)
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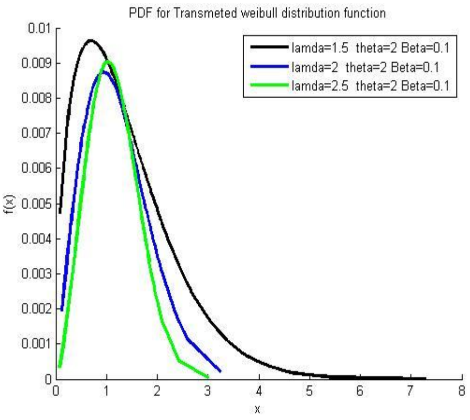

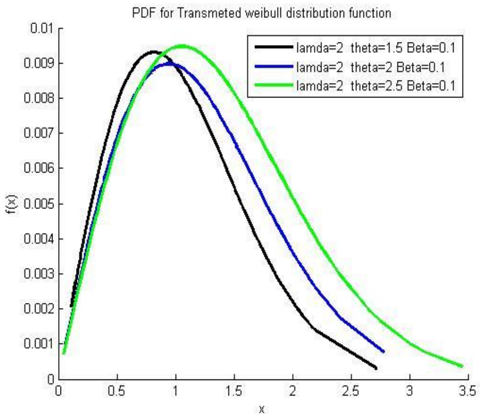

3. GENERATION OF RANDOM NUMBERS

The generation of random numbers for estimation:

Normally we solve the equation of this type:

Let $G(t) = u$

$$Z^2 + \left(\frac{1-\beta}{\beta}\right)Z - \frac{u}{\beta} = 0$$

$$Z = \frac{\left(\frac{\beta-1}{\beta}\right) \mp \sqrt{\left(\frac{1-\beta}{\beta}\right)^2 + \frac{4u}{\beta}}}{2}$$

$$\therefore Z = \frac{(\beta-1) \mp \sqrt{(1-\beta)^2 + 4\beta u}}{2\beta}$$

$$\therefore t_i = \left\{\theta\left(-ln\left[\frac{(\beta-1) + \sqrt{(1-\beta)^2 + 4\beta u_i}}{2\beta}\right]\right)\right\}^{1/\lambda} \quad ... \ (8)$$

$$0 \leq ui \leq 1$$

Equation (8) can be used for generating values of observations ti, from this new transmuted Weibull.

4. METHODS OF SOLVING FORMULA

1.The r ’ th moments:

To derive a formula about origin, i.e

$E(t^r), we\ apply$

$$M'r = E(t^r) = \int_0^\infty t^r g(t)dt$$

After some steps of simplification, we have:

$$E(t^r) = (1-\beta)\theta^{\frac{r}{\lambda}}\Gamma\left(\frac{r}{\lambda}+1\right) + \frac{\beta\theta^{\frac{r}{\lambda}}}{2^{\frac{r}{\lambda}}}\Gamma\left(\frac{r}{\lambda}+1\right)$$

$$\square(t^r) = \Gamma\left(\frac{r}{\lambda}+1\right)\theta^{\frac{r}{\lambda}}\left[(1-\beta)+\frac{\beta}{2^{\frac{r}{\lambda}}}\right]...(9)$$

Therefore the formula for E(tr) can be used for obtaining moment estimators from solving

$$M'r = E(t^r)$$

$$\sum_{i=1}^{n}\frac{t_i^r}{n}\Gamma\left(\frac{r}{\lambda}+1\right)\theta^{\frac{r}{\lambda}}\left[(1-\beta)+\frac{\beta}{2^{\frac{r}{\lambda}}}\right]...(10)$$

$$for\ r = 1,2,3$$

So that the r , th moment about origin is

$$M'r = (1-\beta)\theta^{\frac{r}{\lambda}}\Gamma\left(\frac{r}{\lambda}+1\right) + \beta\left(\frac{\theta}{2}\right)^{\frac{r}{\lambda}}\Gamma\left(\frac{r}{\lambda}+1\right)$$

$$M'r = (1-\beta)\theta^{\frac{r}{\lambda}}\Gamma\left(\frac{r}{\lambda}+1\right) + \beta\left(\frac{\theta}{2}\right)^{\frac{r}{\lambda}}\Gamma\left(\frac{r}{\lambda}+1\right)$$

Hence $M'_1 = m_1 = \frac{\sum xi}{n}$

$$\therefore \Gamma\left(\frac{1}{\lambda}+1\right)\left[(1-\beta)\theta^{\frac{1}{\lambda}} + \beta\left(\frac{\theta}{2}\right)^{\frac{1}{\lambda}}\right] = \frac{\sum xi}{n}...(11)$$

And from $M'_2 = m_2 = \frac{\sum xi^2}{n}$

$$\therefore \Gamma\left(\frac{2}{\lambda}+1\right)\left[(1-\beta)\theta^{\frac{2}{\lambda}} + \beta\left(\frac{\theta}{2}\right)^{\frac{2}{\lambda}}\right] = \frac{\sum xi^2}{n}$$

$$\Gamma\left(\frac{2}{\lambda}+1\right)\theta^{\frac{2}{\lambda}}\left[(1-\beta)+\beta\, 2^{-\frac{2}{\lambda}}\right] = \frac{\sum xi^2}{n}. . .(12)$$

From the equation (11)

Also we have

$$\Gamma\left(\frac{1}{\lambda}+1\right)\theta^{\frac{1}{\lambda}}\left[(1-\beta)+\beta\, 2^{-\frac{1}{\lambda}}\right] = \bar{X}$$

The third equation which is needed to obtain:

$$\Gamma\left(\frac{3}{\lambda}+1\right)\left[(1-\beta)\theta^{\frac{3}{\lambda}} + \beta\left(\frac{\theta}{2}\right)^{\frac{3}{\lambda}}\right] = \frac{\sum xi^3}{n}$$

$$\therefore \Gamma\left(\frac{3}{\lambda}+1\right)\theta^{\frac{3}{\lambda}}[(1-\beta)]+\beta\, 2^{-\frac{3}{\lambda}}] = \frac{\sum xi^3}{n} \ldots..(13)$$

These three equation which is obtained from ($M'r = m_r$)are solved in order to find $\hat{\lambda}_{Mom}$, $\hat{\theta}_{Mom}$, $\hat{\beta}_{Mom}$.

Maximum likelihood method.

The estimators $\hat{\lambda}_{MLE}$, $\hat{\theta}_{MLE}$, $\hat{\beta}_{MLE}$.

From the new transmuted Weibull (equ.6) are obtained from

Maximizing log L=$\log \prod_{i=1}^{n} g(t_i)$

$$L = \lambda^n \beta^n \theta^{-n} \prod_{i=1}^{n} (t_i)^{\lambda-1} e^{\frac{\sum (t_i)^\lambda}{\theta}} . \prod_{i=1}^{n}\left[\left(\frac{1}{\beta}-1\right)+e^{\frac{(t_i)^\lambda}{\theta}}\right] \ldots (14)$$

$$\text{Let } k_i = \left[\left(\frac{1}{\beta}-1\right)+e^{\frac{(t_i)^\lambda}{\theta}}\right]$$

$$\log L = n\log\lambda + n\log\beta - n\log\theta + (\lambda-1)\sum_{i=1}^{n}\log t_i - \sum_{i=1}^{n}\frac{(t_i)^\lambda}{\theta} + \sum_{i=1}^{n}\log k_i \ldots (15)$$

If we derivative this equation three times by λ ,β and θ in order to obtained three equations as bellow:

$$\frac{n}{\lambda}+\sum_{i=1}^{n}\log t_i-\frac{1}{\theta}\sum_{i=1}^{n}(t_i)^{\lambda}\log t_i=\frac{2}{\theta}\sum_{i=1}^{n}\frac{\log t_i e^{-\frac{t_i^{\lambda}}{\theta}}}{\left(\frac{1}{\beta}-1\right)+2e^{-\frac{t_i}{\theta}}} \quad \ldots (16)$$

$$\frac{\partial \log L}{\partial \lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\log t_i-\sum_{i=1}^{n}\frac{(t_i)^{\lambda}}{\theta}(1)\log t_i+\sum_{i=1}^{n}\frac{1}{k_i}\left(2e^{-\frac{t_i}{\theta}}.\left(-\frac{1}{\theta}\right)(1)\log t_i\right)=0$$

Which is an implicit function of λ.

Then:

$$\frac{\partial \log L}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\frac{\left(\frac{-1}{\beta^2}\right)}{k_i}=0$$

$$=\frac{n}{\beta}-\frac{1}{\beta^2}\sum_{i=1}^{n}\left[\left(\frac{1}{\beta}-1\right)+2e^{-\frac{t_i^{\lambda}}{\theta}}\right]=0$$

$$\hat{\beta}_{MLE}=\frac{\sum_{i=1}^{n}[(\frac{1}{\beta}-1)+2\,e^{-\frac{t_i^{\lambda}}{\theta}}]^{-1}}{n} \quad . . . (17)$$

Solved numerically to find $\hat{\beta}_{MLE}$ and while

$$\frac{\partial logL}{\partial\theta}=\frac{-n}{\theta}+\frac{\sum t_i^{\lambda}}{\theta^2}+\sum\frac{1}{ki}\left[2\,e^{-\frac{t_i^{\lambda}}{\theta}}\left(\frac{t_i^{\lambda}}{\theta^2}\right)=0\right]$$

We obtain:

$$-n\theta+\sum t_i^{\lambda}+2\sum\frac{t_i^{\lambda}e^{-\frac{t_i^{\lambda}}{\theta}}}{\left[\left(\frac{1}{\beta}-1\right)+2e^{-\frac{t_i^{\lambda}}{\theta}}\right]}=0 \quad . . . (18)$$

Solved numerically to find $\hat{\theta}_{MLE}$.

5. **CONCLUSION**

From the results of simulation, we find the best estimator for different sets of initial values (β,η,Q)and also different set of sample size (n=100, 150, 250), are moment estimators, first and maximum like likelihood second (i. e)

$Table\ (1)\ where\ \lambda = 1\ \ \theta = 2\ \ \beta = 0.1$							
n	**Method**	λ	$MSE(\lambda)$	θ	$MSE(\theta)$	β	$MSE(\beta)$
100	**MLE**	1.07971	0.020978	2.113155	1.303171	0.178522	0.697366
	Moment	0.995733	0.000666	2.021724	1.045673	0.104153	0.802544
BEST		Mom		Mom		MLE	
150	**MLE**	1.066099	0.017574	2.068542	1.209383	0.15402	0.728907
	Moment	0.993644	0.000559	2.025385	1.052958	0.104507	0.80191
BEST		Mom		Mom		MLE	
250	**MLE**	1.032916	0.010909	1.991473	1.032243	0.132541	0.761422
	Moment	0.989989	0.001495	2.036162	1.08667	0.104889	0.801234
BEST		Mom		MLE		MLE	

$Table\ (1)\ where\ \lambda = 1\ \ \theta = 2\ \ \beta = 0.5$							
n	**Method**	λ	$MSE(\lambda)$	θ	$MSE(\theta)$	β	$MSE(\beta)$
100	**MLE**	1.067264	0.02357	1.842773	0.791121	0.132642	0.76349
	Moment	0.996917	0.000504	2.019676	1.041471	0.521892	0.228651
BEST		Mom		MLE		Mom	
150	**MLE**	1.059098	0.016581	1.779604	0.640207	0.149966	0.737991
	Moment	0.981452	0.002706	2.02775	1.059133	0.526771	0.224125
BEST		Mom		MLE		Mom	
250	**MLE**	0.995913	0.012291	1.720249	0.544712	0.192939	0.67278
	Moment	0.987433	0.000909	2.022297	1.046595	0.525651	0.22507
BEST		Mom		MLE		Mom	

Table (1) *where* $\lambda = 1$ $\theta = 2$ $\beta = 0.8$							
n	**Method**	λ	$MSE(\lambda)$	θ	$MSE(\theta)$	β	$MSE(\beta)$
100	**MLE**	1.063256	0.022799	1.488076	0.278039	0.18045	0.692978
	Moment	0.98601	0.001407	2.023028	1.049012	0.839828	0.025824
BEST		Mom		MLE		Mom	
150	**MLE**	1.024093	0.017123	1.433803	0.215451	0.258761	0.586024
	Moment	0.989122	0.001276	2.020627	1.043486	0.840077	0.025718
BEST		Mom		MLE		Mom	
250	**MLE**	1.00586	0.011891	1.462011	0.23206	0.31999	0.510796
	Moment	0.992124	0.000607	2.017143	1.035808	0.839065	0.025971
BEST		Mom		MLE		Mom	

Table (1) *where* $\lambda = 1.5$ $\theta = 2$ $\beta = 0.1$							
n	**Method**	λ	$MSE(\lambda)$	θ	$MSE(\theta)$	β	$MSE(\beta)$
100	**MLE**	1.480902	0.030336	1.415202	0.1097	0.305915	1.52937
	Moment	1.48847	0.002216	2.024619	0.277068	0.835294	0.441974
BEST		Mom		MLE		Mom	
150	**MLE**	1.450706	0.021652	1.465996	0.098385	0.369279	1.389329
	Moment	1.486772	0.002559	2.023193	0.280288	0.839375	0.436563
BEST		Mom		MLE		Mom	
250	**MLE**	1.428652	0.022149	1.479297	0.116298	0.425148	1.263559
	Moment	1.484908	0.001251	2.018496	0.26979	0.838917	0.437078
BEST		Mom		MLE		Mom	

$Table\ (1)\ where\ \lambda = 2\ \ \theta = 2\ \ \beta = 0.5$							
n	**Method**	λ	$MSE(\lambda)$	θ	$MSE(\theta)$	β	$MSE(\beta)$
100	**MLE**	1.929781	0.040379	1.705662	0.26881	0.368094	2.742997
	Moment	1.973206	0.008261	2.023413	0.002347	0.524194	2.17811
BEST		Mom		Mom		Mom	
150	**MLE**	1.898322	0.025738	1.72495	0.267703	0.351805	2.799576
	Moment	1.9842	0.001922	2.017832	0.001444	0.522892	2.181901
BEST		Mom		Mom		Mom	
250	**MLE**	1.902926	0.034388	1.685888	0.277782	0.359036	2.760898
	Moment	1.980908	0.001869	2.024545	0.001719	0.52248	2.183101
BEST		Mom		Mom		Mom	

$Table\ (1)\ where\ \lambda = 2.5\ \ \theta = 2\ \ \beta = 0.8$							
n	**Method**	λ	$MSE(\lambda)$	θ	$MSE(\theta)$	β	$MSE(\beta)$
100	**MLE**	2.378908	0.068307	1.900365	0.600425	0.774849	3.059782
	Moment	2.470776	0.010545	2.019433	0.23278	0.838187	2.761806
BEST		Mom		Mom		Mom	
150	**MLE**	2.385457	0.066937	1.85641	0.605125	0.741752	3.179136
	Moment	2.476763	0.002922	2.02674	0.225673	0.836975	2.765736
BEST		Mom		Mom		Mom	
250	**MLE**	2.41956	0.033727	1.986879	0.368897	0.801434	2.93893
	Moment	2.471297	0.003367	2.01832	0.232856	0.838828	2.759542
BEST		Mom		Mom		Mom	

Table (1) where $\lambda = 1.5$ $\theta = 1.5$ $\beta = 0.1$							
n	**Method**	λ	$MSE(\lambda)$	θ	$MSE(\theta)$	β	$MSE(\beta)$
100	**MLE**	1.55055	0.024024	1.517881	0.087765	0.143731	1.881452
	Moment	1.48631	0.003064	1.525887	0.001945	0.104253	1.948117
BEST		Mom		Mom		MLE	
150	**MLE**	1.543378	0.016455	1.531201	0.063799	0.15428	1.851061
	Moment	1.487496	0.002448	1.521755	0.002903	0.104482	1.947472
BEST		Mom		Mom		MLE	
250	**MLE**	1.491042	0.017287	1.550058	0.146371	0.252493	1.637832
	Moment	1.476441	0.00402	1.522963	0.00165	0.10477	1.946669
BEST		Mom		Mom		MLE	

Table (1) where $\lambda = 2$ $\theta = 2$ $\beta = 0.5$							
n	**Method**	λ	$MSE(\lambda)$	θ	$MSE(\theta)$	β	$MSE(\beta)$
100	**MLE**	1.911905	0.029101	1.614688	0.300492	0.276182	3.034896
	Moment	1.965801	0.013273	2.030139	0.00497	0.524533	2.177212
BEST		Mom		Mom		Mom	
150	**MLE**	1.910616	0.032289	1.706694	0.262349	0.349414	2.801987
	Moment	1.983545	0.003284	2.026716	0.002204	0.52287	2.181972
BEST		Mom		Mom		Mom	
250	**MLE**	1.922759	0.022918	1.744811	0.221103	0.350738	2.789928
	Moment	1.960961	0.009615	2.039576	0.016143	0.524009	2.178594
BEST		Mom		Mom		Mom	

Table (1) where $\lambda = 2.5$ $\theta = 2.5$ $\beta = 0.8$							
n	**Method**	λ	$MSE(\lambda)$	θ	$MSE(\theta)$	β	$MSE(\beta)$
100	**MLE**	2.33505	0.145036	1.700179	1.024004	0.525276	4.028033
	Moment	2.48006	0.006174	2.520366	0.003448	0.837385	2.764377
BEST		Mom		Mom		Mom	
150	**MLE**	2.264643	0.134589	1.572841	1.14498	0.499034	4.102253
	Moment	2.437227	0.019755	2.540256	0.005097	0.841583	2.750579
BEST		Mom		Mom		Mom	
250	**MLE**	2.243644	0.178377	1.536017	1.292867	0.583901	3.764464
	Moment	2.469335	0.003766	2.528441	0.002424	0.837829	2.762872
BEST		Mom		Mom		Mom	

n	λ_{MLE}	θ_{MLE}	β_{MLE}
100	0	33.3%	22.2%
150	0	33.3%	22.2%
250	0	44.4%	22.2%

n	λ_{Mom}	θ_{Mom}	β_{Mom}
100	100%	66.7%	77.8%
150	100%	66.7%	77.8%
250	100%	55.6%	77.8%

We find the percentage of preface of (moment estimator) is higher than percentage of (maximum likelihood estimator) this is due to closed form of moment estimators, while MLE estimators need iterations

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