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Applications of Double - Framed Soft Ideal Structures over Gamma Near-Rings

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Abstract

In this paper, we discuss double-framed soft set theory with respect to Γ-near ring structure. Moreover, we investigate double-framed soft mapping with respect to soft image, soft pre-image and β-inclusion of soft sets. Finally, we give some applications of double-framed soft Γ-near ring to Γ-near ring theory.

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1. Introduction

In 1984, Satyanarayana introduced Γ-near-ring in his doctoral thesis and obtained some basic results [15]. To solve complicated problems in economics, engineering, environmental science and social science. Methods in classical mathematics are not always successful because of various types of uncertainties presented in these problems. While probability theory, fuzzy set theory [18], rough set theory [[12], [13]], and other mathematical tools are well known and often useful approaches to describing uncertainty, each of these theories has its inherent difficulties as pointed out in [[4], [5]]. In 1999, Molodtsov [4] introduced the concept of soft sets, which can be seen as a new mathematical tool for dealing with uncertainties. This so-called soft set theory is free from the difficulties affecting existing methods. Presently, works on soft set theory are progressing rapidly. Maji et al. [6] defined several operations on soft sets and made a theoretical study on the theory of soft sets. Atagün and Sezgin [2] defined the concepts of soft sub rings and ideals of a ring, soft subfields of a field and soft sub modules of a module and studied their related properties with respect to soft set operations. Also union soft sub structures of near-rings and near-ring modules are studied in [16]. K. Hayat et al. [7] defined applications of double-framed soft ideals in BE-algebra. Jun et al. [[9], [10]] introduced the notion of double-framed soft sets (briefly, DFS-sets), and applied it to BCK/BCI- algebras. They discussed double-framed soft algebras (briefly, DFS-algebras) and investigated related properties. A.R. Hadipour [4] defined Double-framed soft BF-algebras and Yongukcho et al. [15] studied on double-framed soft Near-rings. Cagman et al. defined two new types of group actions on a soft set, called group SI-action and group SU-action [11], which are based on the inclusion relation and the intersection of sets and union of sets, respectively. Ali et al. [3] introduced several operations of soft sets. Yongukcho et al. [17] studied on double-framed soft Near-rings derive some properties. In this paper, we discuss double-framed soft set theory with respect to Γ-near ring structure. Moreover, we investigate double-framed soft mapping with respect to soft image, soft pre-image and β-inclusion of soft sets. Finally, we give some applications of double-framed soft Γ-near ring to Γ-near ring theory.

2. Preliminaries

In this section, we recall basic definitions of soft set theory that are useful for subsequent sections. For more detail see the papers [[8], [9]].

Through out the paper, U refers to an initial universe, E is a set of parameters and $P(U)$ is the power set of U. $\subset$ and $\supset$ stand for proper subset and super set, respectively.

Definition 2.1. [9] For any subset A of E, a soft set λ_A over U is a set, defined by a function λ_A, representing the mapping $\lambda_A : E \to P(U)$. A soft set over U can also be represented by the set of ordered pairs $\lambda_A = \{(x, \lambda_A(x)); x \in E, \lambda_A(x) \in P(U)\}$. Note that the set of all soft sets over U will denoted by $S(U)$.

Definition 2.2. [9] Let $\lambda, \mu \in S(U)$. Then

(i) If $\lambda(e) = \emptyset$ for all $e \in E$, λ is said to be a null soft set, denoted by $\emptyset$.

(ii) If $\lambda(e) = U$ for all $e \in E$, λ is said to be an absolute soft set, denoted by U.

(iii) λ is a soft subset of μ, denoted $\lambda \subseteq \mu$, if $\lambda(e) \subseteq \mu(e)$ for all $e \in E$.

(iv) Soft union of λ and μ, denoted by $\lambda \cup \mu$, is a soft set over U and defined by $\lambda \cup \mu : E \to P(U)$ such that $(\lambda \cup \mu)(e) = \lambda(e) \cup \mu(e)$ for all $e \in E$.

(v) $\lambda = \mu$, if $\lambda \subseteq \mu$ and $\lambda \supseteq \mu$.

(vi) Soft intersection of λ and μ, denoted by $\lambda \cap \mu$, is a soft set over U and defined by $\lambda \cap \mu : E \to P(U)$ such that $(\lambda \cap \mu)(e) = \lambda(e) \cap \mu(e)$ for all $e \in E$.

(vii) Soft complement of λ is denoted by λ^C and defined by $\lambda^C : E \to P(U)$ such that $\lambda^C(e) = U/\lambda(e)$ for all $e \in E$.

Definition 2.3. Let E be a parameter set, $S \subset E$ and $\lambda : S \to E$ be an injection function. Then $S \cup \lambda(s)$ is called extended parameter set of S and denoted by ζ_s.

If $S = E$, then extended parameter set of S will be denoted by ζ.

Definition 2.4. [6] A double-framed pair $\langle(\overline{\alpha}, \overline{\lambda}) : G\rangle$ is called a double-framed soft set (briefly DFS-set) over U where $\overline{\alpha}$ and $\overline{\lambda}$ are mapping from A to $P(U)$.

For a DFS-set $\langle(\overline{\alpha}, \overline{\lambda}) : G\rangle$ over U and two subsets γ and δ of U, the $\gamma-$inclusive set and the δ-exclusive set of $\langle(\overline{\alpha}, \overline{\lambda}) : G\rangle$, denoted by $i_A(\overline{\alpha}; \gamma)$ and $e_A(\overline{\lambda}, \delta)$ respectively, are defined as follows.

$i_A\,(\overline{\alpha}; \gamma) = \{x \in A/\gamma \subseteq \overline{\alpha}(x)\}$ and $e_A(\overline{\lambda}, \delta) = \{x \in A/\delta \subseteq \overline{\lambda}(x)\}$ respectively.

The set $DF_A(\overline{\alpha}, \overline{\lambda})_{(\gamma,\delta)} = \{x \in A/\gamma \subseteq \overline{\alpha}(x), \delta \subseteq \overline{\lambda}(x)\}$ is called a double framed including set of $\langle(\overline{\alpha}, \overline{\lambda}) : G\rangle$. It is clear that $DF_A(\overline{\alpha}, \overline{\lambda})_{(\gamma,\delta)} = i_A\,(\overline{\alpha}; \gamma) \cap_A (\overline{\lambda}, \delta)$.

From now on, we will take G, as set of parameters, which is a group unless otherwise specified.

Note 2.5. Let $\lambda_S = (\overline{\alpha}_S, \overline{\beta}_S, E)$ be a double framed soft set over U. We will say that $\lambda_S(e) = (\overline{\alpha}_S(e), \overline{\beta}_S(e))$ is image of parameter $e \in E$.

Definition 2.6. Let λ_A and $\lambda_B \in DFS_E(U)$ then,

(i) If $\alpha_A(e) = \emptyset$ and $\beta_A(e) = U$ for all $e \in E$, λ_A is said to be a null double-framed soft set, denoted by $\emptyset_b = (\emptyset, U, E)$.

(ii) If $\alpha_A(e) = U$ and $\beta_A(e) = \emptyset$ for all $e \in E$, λ_A is said to be an absolute double-framed soft set, denoted by $\emptyset_b = (U, \emptyset, E)$.

(iii) λ_A is double-framed soft subset of λ_B, denoted by $\lambda_A \subseteq \lambda_B$, if $\alpha_A(e) \subseteq \alpha_B(e)$ and $\beta_A(e) \supseteq \beta_B(e)$ for all $e \in E$.

(iv) Double-framed soft union and intersection of λ_A and λ_B, denoted by $(\alpha_A \cup \alpha_B) : A \cup B \to P(U)$ such that $(\alpha_A \cup \alpha_B)(e) = \alpha_A(e) \cup \alpha_B(e)$ and $(\beta_A \cap \beta_B)(e) = \beta_A(e) \cap \beta_B(e)$ for all $e \in E$.
Also, $(\alpha_A \cap \alpha_B) : A \cap B \to P(U)$ such that $(\alpha_A \cap \alpha_B)(e) = \alpha_A(e) \cap \alpha_B(e)$ and $(\beta_A \cup \beta_B)(e) = \beta_A(e) \cup \beta_B(e)$ for all $e \in E$.

(v) Double-framed soft complement of λ_A is denoted by λ_A^C and defined by $\lambda_A^C : E \to P(U) \times P(U)$ such that $\lambda_A^C(e) = \{(e, \alpha_A(e), \beta_A(e)) : e \in E\}$.

Definition 2.7. Let R be a Γ-near ring and f_R be a soft set over X. Then, f_R is said to be double-framed soft Γ-near ring over U, if it satisfies the following conditions hold:

1. $f_R(x + y) \subseteq f_R(x) \cup f_R(y)$, $\quad f_R(x + y) \supseteq f_R(x) \cap f_R(y)$.
2. $f_R(-x) = -f_R(x)$.
3. $f_R(x\alpha y) \subseteq f_R(x) \cup f_R(y)$, $\quad f_R(x\alpha y) \supseteq f_R(x) \cap f_R(y)$ for all $x, y \in R$ and $\alpha \in \Gamma$.

Example 2.8. Let $R = \{0, 1, 2, 3\}$ and $\Gamma = \{\alpha, \beta\}$ be non-empty sets. The binary operations defined as

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

α	0	1	2	3
0	0	0	0	0
1	0	1	0	2
2	0	0	0	0
3	0	2	0	2

β	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	0
3	0	0	0	2

Clearly, $(R, +, \Gamma)$ is a Γ-near ring.

Assume that R is the set of parameters and $U = \left\{ \begin{pmatrix} x & x \\ x & 0 \end{pmatrix} \middle| x \in \mathbb{Z}_4 \right\}$, 2×2 matrices with $\mathbb{Z}_4$ terms, in the universal set. We construct a soft set f_R over by

$$f_R(0) = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$
$$f_R(1) = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 4 & 0 \end{pmatrix} \right\},$$
$$f_R(2) = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 3 & 0 \end{pmatrix} \right\},$$
$$f_R(3) = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 4 & 0 \end{pmatrix} \right\}.$$

Then, one can easily show that the soft set f_R is a soft union Γ-near over U.

Example 2.9. In example-2.8, assume that $R\{0, 1, 2, 3\}$ is again the set of parameters and $U = S_3$, symmetric group, is the universal set, we defined a soft set f_R by,

$$f_R(0) = \{(1\ \ 2), (2\ \ 3)\}$$
$$f_R(1) = \{(1\ \ 2), (1\ \ 3), (1\ \ 2\ \ 3)\}$$
$$f_R(2) = \{(1\ \ 2), (2\ \ 3), (1\ \ 2\ \ 3)\}$$
$$f_R(3) = \{(1\ \ 2), (1\ \ 3), (1\ \ 2\ \ 3)\}$$

f_R is not double-framed soft Γ-near ring, because

$$f_R(1+1) = f_R(0) = \{(1\ \ 2), (2\ \ 3)\} \not\subseteq \{(1\ \ 2), (1\ \ 3), (1\ \ 2\ \ 3)\}.$$

Note 2.10. If f_R is a double-framed soft Γ-near ring over U, then $f_R \subseteq f_R(y)$, $f_R \supseteq f_R(y)$, for all $y \in R$.

3. Results

Based on the above definition, we prove the following resuts which will be very useful for further study.

Theorem 3.1. Let R be a Γ-near ring and f_R a soft set over U. Then, f_R is a double-framed soft Γ-near ring if and only if $f_R(x-y) \subseteq f_R(x) \cup f_R(y)$ and $f_R(x\alpha y) \subseteq f_R(x) \cup f_R(y)$ and $f_R(x-y) \supseteq f_R(x) \cap f_R(y)$ and $f_R(x\alpha y) \supseteq f_R(x) \cap f_R(y)$, for all $x, y \in R$ and $\alpha \in \Gamma$.

Proof. Assume that f_R is a soft union Γ-near ring over U. Then, by definition of a soft union Γ-near ring, we have

$f_R(x-y) \subseteq f_R(x) \cup f_R(-y) = f_R(x) \cup f_R(y)$ and
$f_R(x\alpha y) \subseteq f_R(x) \cup f_R(y)$,

similarly,

$f_R(x-y) \supseteq f_R(x) \cap f_R(-y) = f_R(x) \cap f_R(y)$ and
$f_R(x\alpha y) \supseteq f_R(x) \cap f_R(y)$, for all $x, y \in R$ and $\alpha \in \Gamma$.

Conversely,
Assume that $f_R(x-y) \subseteq f_R(x) \cup f_R(y)$ and $f_R(x\alpha y) \subseteq f_R(x) \cup f_R(y)$, for all $x, y \in R$ and $\alpha \in \Gamma$. If we choose $x = 0$, then $f_R(0-y) = f_R(-y) \subseteq f_R(0) \cup f_R(y) = f_R(y)$. Now, $f_R(y) = f_R(-(-y)) \subseteq f_R(-y)$. Thus $f_R(y) = f_R(-y)$ for all $y \in R$. Also, by assumption, we have

$$f_R(x+y) = f_R(x-(-y)) \subseteq f_R(x) \cup f_R(-y) = f_R(x) \cup f_R(y).$$

Similarly, we can prove the second part. Thus f_R is a double-framed soft Γ-near ring over U. ∎

Note 3.2. Let f_R is a soft union $\Gamma-$ near ring over U.

(i) If $f_R(x - y) = 0$ for all $x, y \in R$, then $f_R(x) = f_R(y)$.

(ii) If $f_R(x - y) = f_R(0)$ for all $x, y \in R$, then $f_R(x) = f_R(y)$.

It is known that if $(R, +, \Gamma)$ is a Γ-near ring, then $(N, +)$ is a group but not necessarily abelian. That is, for any $x, y \in R$, $x + y$ needs not be equal to $y + x$. However we have the following.

Theorem 3.3. Let f_R be a double-framed soft Γ-near ring over U and $x \in R$. Then

$$f_R(x) = f_R(y) \Leftrightarrow f_R(x + y) = f_R(y + x).$$

Proof. Straight forward. ∎

Theorem 3.4. Let R be a Γ-near field and f_R a soft set over U. If $f_R(0) \subseteq f_R(1_R) = f_R(x)$ and $f_R(0) \supseteq f_R(1_R) = f_R(x)$ for all $0 \neq x \in R$, then f_R is a double-framed soft Γ-near ring over U.

Proof. Suppose that $f_R(0) \subseteq f_R(1_R) = f_R(x)$ for all $0 \neq x \in R$. In order to prove that f_R is a soft union Γ-near ring over U, It is enough to prove that $f_R(x-y) \subseteq f_R(x) \cup f_R(y)$ and $f_R(x\alpha y) \subseteq f_R(x) \cup f_R(y)$. Let $x, y \in R$ and $\alpha \in \Gamma$. Then, we have the following cases:
Case-1: Suppose that $x \neq 0$ and $y = 0$ or $x = 0$ and $y \neq 0$. Since R is a Γ-near field, so it follows that $x\alpha y = 0$ and $f_R(x\alpha y) = f_R(0)$. Since $f_R(0) \subseteq f_R(x)$ for all $x \in R$, so $f_R(x\alpha y) = f_R(0) \subseteq f_R(x)$ and $f_R(x\alpha y) = f_R(0) \subseteq f_R(y)$. This imply $f_R(x\alpha y) \subseteq f_R(x) \cup f_R(y)$.
Case-2: Suppose that $x \neq 0$ and $y \neq 0$. It follows that $x\alpha y \neq 0$.
Then $f_R(x\alpha y) = f_R(1_R) = f_R(x)$ and $f_R(x\alpha y) = f_R(1_R) = f_R(y)$.
This imply $f_R(x\alpha y) \subseteq f_R(x) \cup f_R(y)$.
Case-3: Suppose that $x = 0$ and $y = 0$, then clearly $f_R(x\alpha y) \subseteq f_R(x) \cup f_R(y)$.
Hence $f_R(x\alpha y) \subseteq f_R(x) \cup f_R(y)$ for all $x, y \in R$ and $\alpha \in \Gamma$.
Similarly we prove second part. ∎

Theorem 3.5. If f_R and f_S are double-framed soft Γ-near ring over U_1 and U_2, then $f_R \times f_S$ is also soft union Γ-near ring over $U_1 \times U_2$.

Proof. Let $(x_1, y_1), (x_2, y_2) \in R \times S$. Then

$$\begin{aligned} h_{R\times S}\{(x_1, y_1) - (x_2, y_2)\} &= h_{R\times S}(x_1 - x_2, y_1 - y_2) \\ &= h_R(x_1 - x_2) \times h_S(y_1 - y_2) \\ &\subseteq (h_R(x_1) \cup h_R(x_2)) \times (h_S(x_1) \cup h_S(x_2)) \\ &= (h_R(x_1) \times h_S(y_1)) \cup (h_R(x_2) \times h_S(y_2)) \\ &= h_{R\times S}(x_1, y_1) \cup h_{R\times S}(x_2, y_2). \end{aligned}$$

Let $(x_1, y_1), (x_2, y_2) \in R \times S$ and $(\alpha_1, \alpha_2) \in \Gamma_1 \times \Gamma_2$. Then

$$\begin{aligned} h_{R\times S}\{(x_1, y_1), (\alpha_1, \alpha_2), (x_2, y_2)\} &= h_{R\times S}(x_1\alpha_1x_2, y_1\alpha_2y_2) \\ &= h_R(x_1\alpha_1x_2) \times h_S(y_1\alpha_2y_2) \\ &\subseteq (h_R(x_1) \cup h_R(x_2)) \times (h_S(y_1) \cup h_S(y_2)) \\ &= (h_R(x_1) \times h_S(y_1)) \cup (h_R(x_2) \times h_S(y_2)) \\ &= h_{R\times S}(x_1, y_1) \cup h_{R\times S}(x_2, y_2). \end{aligned}$$

Hence $f_R \times f_S$ is soft union Γ-near ring over $U_1 \times U_2$. ■

Theorem 3.6. f_R and f_S are double-framed soft Γ-near ring over U, then $f_R \cup f_S$ is also soft union Γ-near ring over U.

Proof. Now, let $x, y \in R$ then

$$\begin{aligned} (f_R \cup f_S)(x - y) &= f_R(x - y) \cup f_S(x - y) \\ &\subseteq (f_R(x) \cup f_R(y)) \cup (f_S(x) \cup f_S(y)) \\ &= (f_R(x) \cup f_S(x)) \cup (f_R(y) \cup f_S(y)) \\ &= f_{R\cup S}(x) \cup f_{R\cup S}(y). \end{aligned}$$

Now, let $x, y \in R$ and $\alpha \in \Gamma$ then

$$\begin{aligned} (f_R \cup f_S)(x\alpha y) &= f_R(x\alpha y) \cup f_S(x\alpha y) \\ &\subseteq (f_R(x) \cup f_R(y)) \cup (f_S(x) \cup f_S(y)) \\ &= (f_R(x) \cup f_S(x)) \cup (f_R(y) \cup f_S(y)) \\ &= f_{R\cup S}(x) \cup f_{R\cup S}(y). \end{aligned}$$

Hence $f_R \cup f_S$ is soft union Γ-near ring over U. ■

Definition 3.7. Let R be a Γ-near ring and f_R be a double-framed soft Γ-near ring over U. Then f_R is said to be a double-framed soft Γ-ideal of R over U, if the following conditions hold:

$$\forall x, y, z \in R \text{ and } \alpha \in \Gamma$$

$$f_R(x + y - x) \subseteq f_R(x) \cup f_R(y). \qquad f_R(x + y - x) \supseteq f_R(x) \cap f_R(y).$$

$$f_R(x\alpha y) \subseteq f_R(x). \qquad f_R(x\alpha y) \supseteq f_R(x).$$

$$f_R(x\alpha(y + z) - x\alpha y) \subseteq f_R(z). \qquad f_R(x\alpha(y + z) - x\alpha y) \supseteq f_R(z).$$

If f_R is double-framed soft Γ-near ring over U and the conditions (i) and (iii) hold ,then f_R is a double-framed soft right ideal of R over U and if conditions (i) and (iii) hold, then f_R is called a double-framed soft Γ-ideal of R over U.

Example 3.8. Let $R = \{0, 1, 2, 3\}$ and $\Gamma = \{\alpha, \beta\}$ be non-empty sets. The binary operations defined as;

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

α	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	0	1	2	3
3	0	0	3	2

β	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	2	0	3
3	0	0	0	2

Clearly, $(R, +, \Gamma)$ is a Γ-near ring. Assume that R is the set of parameters and

$$U = D_3 = \{(x, y) : x^3 = y^3 = (xy)^2 = e, \quad xy = yx^2\} = \{x, x^2, y, yx, y^2x\}$$

dihedral group, the universal set. We define a soft set f_R over U by

$$f_R(0) = f_R(3) = D_3$$
$$f_R(2) = f_R(1) = \{e, x\}.$$

Then, clearly f_R is a double-framed soft left Γ-ideal and right Γ-ideal of R over U.

Theorem 3.9. Let R be a Γ-near field and f_R a double-framed soft Γ-ideal of R over U. then,
$f_R(0) \subseteq f_R(1_R) = f_R(x)$ and $f_R(0) \supseteq f_R(1_R) = f_R(x)$ for all $0 \neq x \in R$.

Proof. Suppose that f_R is a double-framed soft Γ-ideal of R over U, then f_R is a soft Γ-near ring of R over U. Since $f_R(0) \subseteq f_R(x)$, so in particular $f_R(0) \subseteq f_R(1_R)$.

Now, let $0 \neq x \in R$, then

$$f_R(x) = f_R(1_R \cdot x) \subseteq f_R(1_R) = f_R(x \cdot x^{-1}) \subseteq f_R(x)$$

$$\Rightarrow f_R(x) = f_R(1_R) \text{ for all } 0 \neq x \in R.$$

For a near-ring R, the symmetric part of R denoted by R_0 is defined by $R_0 = \{r \in / r0 = 0\}$. It is a zero-symmetric near-ring and $I_i \nabla R$, then $RI \supseteq R$. Hence we have an analog for this case. ■

Theorem 3.10. Let $R = R_0$ and f_R be a soft set of R over U. Then $f_R(x\alpha(y + z) - x\alpha y) \subseteq f_R(z)$, implies that $f_R(xz) \subseteq f_R(z)$ and $f_R(x\alpha(y + z) - x\alpha y) \supseteq f_R(Z)$, implies that $f_R(xz) \supseteq f_R(z)$ for all $x, y, z \in R$.

Theorem 3.11. If f_R and f_S are double-framed soft Γ-ideals over U, then $f_R \vee f_S$ is also double-framed soft Γ-ideal over U.

Proof. Let f_R and f_S are double-framed soft Γ-ideals over U, then, Let $(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3) \in R \times S$. and $(\delta_1, \delta_2) \in \Gamma_1 \times \Gamma_2$. then

$$\begin{aligned}(f_R \vee f_S)((x_1, y_1) + (x_2, y_2) - (x_1, y_1)) &= (f_R \vee f_S)(x_1 + x_2 - x_1 - y_1 + y_2 - y_1)\\ &= f_R(x_1 + x_2 - x_1) \cup f_S(y_1 + y_2 - y_1)\\ &\subseteq f_R(x_2) \cup f_S(y_2) = f_R \vee f_S(x_2, y_2)\end{aligned}$$

$$\begin{aligned}(f_R \vee f_S)((x_1, y_1)(x_2, y_2)) &= (f_R \vee f_S)(x_1, x_2, t_1, y_2)\\ &= f_R(x_1, x_2) \vee f_S(y_1, y_2)\\ &\subseteq f_R(x_1) \cup f_S(y_1) = (f_R \vee f_S)(x_1, y_1) \text{ and}\end{aligned}$$

$$(f_R \vee f_S)((x_1, y_1)(\sigma_1, \sigma_2)((x_2, y_2) + (x_3, y_3)) - (x_1, y_1)(\sigma_1, \sigma_2)(x_2, y_2))$$

$$\begin{aligned}&= (f_R \vee f_S)(x_1\sigma_1(x_2, x_3) - x_1\sigma_1 x_2, y_1\sigma_2(y_2 + y_3) - y_1\sigma_2 y_2)\\ &= f_R(x_1\sigma_1(x_1 + x_2) - x_1\sigma_1 x_2) \cup f_S(y_1\sigma_2(y_2 + y_3) - y_1\sigma_2 y_3)\\ &\subseteq f_R(x_3) \cup f_S(y_3) = (f_R \vee f_S)(x_3, y_3)\end{aligned}$$

Hence, $f_R \vee f_S$ is a double-framed soft Γ-ideal of $R \times S$ over U. ■

Theorem 3.12. If f_R and f_S are double-framed soft Γ-ideals over U_1 and U_2, then $f_R \times f_S$ is also double-framed soft Γ-ideal over $U_1 \times U_2$.

Proof. Similar to previous theorem. ■

Theorem 3.13. f_R is a double-framed soft Γ-ideal of Γ near-ring of R over U, then $R_f = \{x \in R : f_R(x) = f_R(0)\}$ is a Γ-ideal of R over U.

Proof. It is obvious that $0 \in R_f \subseteq R$. We need to prove that
$(i)\ x - y \in R_f\ \ (ii)\ n + x - n \in R_f\ \ (iii)\ x\alpha n \in R_f$ and $n\alpha(i + x) - n\alpha i \in R_f$.
For all $x, y \in R_f,\ \ n, i \in R_f$ and $\alpha \in \Gamma$.
If $x, y \in R_f$, then $f_R(x) = f_R(y) = f_R(0)$. So by remark 1, it follows that $\lambda_R(0) \subseteq f_R(n + x - x)$,
$\lambda_R(0) \subseteq f_R(xn\alpha)$ and $\lambda_R(0) \subseteq f_R(n\alpha(i + x) - n\alpha i)$
For all $x, y \in R_f,\ n, i \in R$ and $\alpha \in \Gamma$.

Since f_R is a soft union Γ-ideal of over U, so

(i) $f_R(x - y) \subseteq f_R(x) \cup f_R(y) = f_r(0)$

(ii) $f_R(n + x - n) \subseteq f_R(x) = f_R(0)$

(iii) $f_R(x\alpha n) \subset f_R(x) = f_R(0)$ and $f_R(n\alpha(i + x) - n\alpha i) \subseteq f_R(x) = f_R(0)$.

This implies that $(i)\ f_R(x-y) = f_R(0),\ (ii)\ f_R(n+x-x) = f_R(0),\ (iii)\ f_R(x\alpha n) = f_R(0)$ and $f_R(n\alpha(i + x) - n\alpha i) = f_R(0)$ for all $x, y \in R_f,\ \ n, i \in R$ and $\alpha \in \Gamma$
Thus, R_f is a Γ-ideal of R over U. ■

Theorem 3.14. Let f_R be a soft set over U and $\mathcal{B}$ be a subset of U such that $\emptyset \neq \mathcal{B} \supseteq f_R(0)$. If f_R is a double-framed soft Γ-ideal of R over U, then (i) $f_R{}^{\subseteq \mathcal{B}} = \{x \in R/f_R(x) \subseteq \mathcal{B}\}$, (ii) $f_R{}^{\supseteq \mathcal{B}} = \{x \in R/f_R(x) \supseteq \mathcal{B}\}$ is a Γ-ideal over X.

Proof. Since $f_R(0) \subseteq \mathcal{B}$, so $0 \in f_R{}^{\subseteq \mathcal{B}}$ and $\emptyset \neq f_R{}^{\subseteq \mathcal{B}} \supseteq R$. Take $x, y \in f_R{}^{\mathcal{B}}$, $n, i \in R$ and $\alpha \in \Gamma$, which implies that $f_R(x) \subseteq \mathcal{B}$ and $f_R(y) \subseteq \mathcal{B}$. Now we need to prove that

(i) $(x - y) \in f_R{}^{\subseteq \mathcal{B}}$

(ii) $n + x - n \in f_R{}^{\subseteq \mathcal{B}}$

(iii) $x\alpha n \in f_R{}^{\subseteq \mathcal{B}}$ and $n\alpha(i + x) - n\alpha i \in f_R{}^{\subseteq \mathcal{B}}$

for all $x, y \in f_R{}^{\subseteq \mathcal{B}}$, $n, i \in R_f$ and $\alpha \in \Gamma$. Since f_R is a soft union Γ-ideal R of over U, so it follows that

(i) $f_R(x - y) \subseteq f_R(x) \cup f_R(y) \subseteq \mathcal{B} \cup B = \mathcal{B}$

(ii) $f_R(n + x - n) \subseteq f_R(x) \subseteq \mathcal{B}$

(iii) $f_R(x\alpha n) \subseteq f_R(x) \subseteq \mathcal{B}$ and

(iv) $f_R(n\alpha(i + x) - n\alpha i) \subseteq f_R(x) \subseteq \mathcal{B}$, Similarly we prove (ii).

This completes the proof. ■

Theorem 3.15. Let f_R and f_S are soft sets U and $\emptyset$ be a Γ-near ring isomorphism from R to S.

(i) If f_R is a double-framed soft Γ-ideal of R over U, then $\phi(f_R)$ is a double-framed soft Γ-ideal of $\mathcal{S}$ over U.

(ii) If f_S is a double-framed soft Γ-ideal of $\mathcal{S}$ over U, then $\phi^{-1}(f_S)$ is a double-framed soft Γ-ideal of R over U.

Proof. (i) let $x_1, x_2 \in \mathcal{S}$. since ϕ is surjective, so there exists $y_1, y_2 \in R$, such that $\phi(y_1) = x_1$, $\phi(y_2) = x_2$ we have

$$\begin{aligned}
\phi(f_R)(x_1 - x_2) &= \cup\{f_R(y)/\ y \in R,\ \phi(y) = x_1 - x_2\} \\
&= \cup\{f_R(y)/\ y \in R,\ y = \phi^{-1}(x_1 - x_2)\} \\
&= \cup\{f_R(y)/\ y \in R,\ y = \phi^{-1}(\phi(y_1 - y_2)) = y_1 - y_2\} \\
&= \cup\{f_R(y_1 - y_2)/\ y_i \in R,\ \phi(y_i) = x_1,\ i = 1, 2\} \\
&\subseteq \cup\{f_R(y_1) \cup f_R(y_2)/\ y_i \in R,\ \phi(y_i) = x_1,\ i = 1, 2\} \\
&= \cup\{f_R(y_1);\ y_1 \in R,\ \phi(y_1) = x_1\} \cup \{f_R(y_2);\ y_2 \in R,\ \phi(y_2) = x_2\} \\
&= \phi(f_R)(x_1) \cup \phi(f_R)(x_2)
\end{aligned}$$

Thus $\phi(f_R)(x_1 - x_2) \subseteq \phi(f_R)(x_1) \cup \phi(f_R)(x_2)$. Similarly, we can prove that $\phi(f_R)(x_1\alpha_1x_2) \subseteq \phi(f_R)(x_1) \cup \phi(f_R)(x_2)$. Now we prove that

$$\begin{aligned}
\phi(f_R)(x_1 + x_2 - x_1) &= \cup\{f_R(y)/\ y \in R,\ \ \phi(y) = (x_1 + x_2 - x_1)\} \\
&= \cup\{f_R(y)/\ y \in R,\ \ y = \phi^{-1}(x_1 + x_2 - x_1)\} \\
&= \cup\{f_R(y)/\ y \in R,\ \ y = \phi^{-1}(\phi(y_1 + y_2 - y_1)) \\
&= (y_1 + y_2 - y_1)\} \\
&= \cup\{f_R(y_1 - y_2)/\ y_1 \in R,\ \ \phi(y_i) = x_i,\ i = 1, 2)\} \\
&\subseteq \cup\{f_R(y_2)/\ y_2 \in R,\ \ \phi(y_2) = x_2)\} \\
&= \phi(f_R)(x_2)
\end{aligned}$$

Thus $\phi(f_R)(x_1 + x_2 - x_1) \subseteq \phi(f_R)(x_2)$. Now, let $x_1, x_2 \in \mathcal{S},\ y_1, y_2 \in R$, and $\alpha \in \Gamma$, then we have

$$\begin{aligned}
\phi(f_R)(x_1\alpha_1x_2) &= \cup\{f_R(y)/\ y \in R,\ \ \phi(y) = (x_1\alpha_1x_2)\} \\
&= \cup\{f_R(y)/\ y \in R,\ \ y = \phi^{-1}(x_1\alpha_1x_2)\} \\
&= \cup\{f_R(y)/\ y \in R,\ \ y = \phi^{-1}(\phi(y_1\alpha y_2)) = y_1\alpha y_2\} \\
&= \cup\{f_R(y_1\alpha y_2)/\ y_1 \in R,\ \ \phi(y_1) = x_i,\ i = 1, 2\} \\
&\subseteq \cup\{f_R(y_2)/\ y_2 \in R,\ \ \phi(y_2) = x_2\} \\
&= \phi(f_R)(x_2) \cup \phi(f_R)(x_2)
\end{aligned}$$

Now let $x_1, x_2, x_3 \in \mathcal{S},\ y_1, y_2, y_3 \in R$, and $\alpha \in \Gamma$. Then we have

$$\phi(f_R)(x_1\alpha_1(x_2 + x_3) - (x_1\alpha_1x_2))$$

$$\begin{aligned}
&= \cup\{f_R(y)/\ y \in R,\ \ \phi(y) = x_1\alpha_1(x_2 + x_3) - x_1\alpha_1x_2\} \\
&= \cup\{f_R(y)/\ y \in R,\ \ y = \phi^{-1}(x_1\alpha_1(x_2 + x_3) - x_1\alpha_1x_2)\} \\
&= \cup\big\{f_R(y)/\ y \in R,\ \ y = \phi^{-1}(\phi(y_1\alpha(y_2 + y_3) - y_1\alpha y_2) \\
&\qquad\qquad = (y_1\alpha(y_2 + y_3) - y_1\alpha y_2\big\} \\
&= \cup\{f_R(y_1\alpha(y_2 + y_3) - y_1\alpha y_2);\ y_i \in R,\ \ \phi(y_i) = x_i,\ i = 1, 2, 3\} \\
&\subseteq \cup\{f_R(y_3)/\ y_3 \in R,\ \ \phi(y_3) = x_3\} \\
&= \phi(f_R)(x_3).
\end{aligned}$$

Hence $\phi(f_R)$ is a double-framed soft Γ-ideal of $\mathcal{S}$ over U.
(ii) Let $x_1, x_2, x_3 \in \mathcal{S}$ and $\alpha_1 \in \Gamma$ and $\mathcal{B}_1 \in \Gamma_1$ Then,

$$\begin{aligned}
(\phi^{-1}(f_S))(x_1\alpha_1x_2) &= f_S(\phi(x_1\alpha_1x_2)) \\
&= f_S(\phi(x_1)\mathcal{B}\phi(x_2)) \\
&\subseteq f_S(\phi(x_1)) \cup f_S(\phi(x_2)) \\
&= (\phi^{-1}(f_S))(x_1) \cup (\phi^{-1}(f_S))(x_2).
\end{aligned}$$

Similarly,

$$(\phi^{-1}(f_S))(x_1 - x_2) \subseteq (\phi^{-1}(f_S))(x_1) \cup (\phi^{-1}(f_S))(x_2).$$

Also,

$$\begin{aligned}(\phi^{-1}(f_S))(x_1 + x_2 - x_1) &= f_S(\phi(x_1 + x_2 - x_1)) \\ &= f_S(\phi(x_1) + \phi(x_2) - \phi(x_1)) \\ &\subseteq f_S(\phi(x_2)) = (\phi^{-1}(f_S)(x_2)).\end{aligned}$$

Now, let $x_1, x_2 \in S$ and $\alpha_1 \in \Gamma$ and $\mathcal{B}_1 \in \Gamma_1$.
$(\phi^{-1}(f_S))(x_1\alpha_1x_2) = f_S(\phi(x_1\alpha_1x_2)) = f_S(\phi(x_1)\mathcal{B}_1\phi(x_2))$.
Finally, let $x_1, x_2, x_3 \in S$ and $\alpha_1 \in \Gamma$ and $\mathcal{B}_1 \in \Gamma_1$. Then,

$$\begin{aligned}(\phi^{-1}(f_S))(x_1\alpha_1(x_2 + x_3) - (x_1\alpha_1x_2)) &= f_S(\phi(x_1\alpha_1(x_2 + x_3) - (x_1\alpha_1x_2))) \\ &= f_S(\phi(x_1)\mathcal{B}_1\phi(x_2)) + \phi(x_3) - \phi(x_1)\mathcal{B}_1\phi(x_2)) \\ &\subseteq f_S(\phi(x_3)) = (\phi^{-1}(f_S))(x_3).\end{aligned}$$

Hence $\phi^{-1}(f_S)$ is double-framed soft Γ-ideal of R over U. ■

4. Conclusion

This paper is devoted to discussion of combination of soft set theory, set theory and Γ-near-ring. Based on the definition, we have introduced the concepts of double-framed soft sub Γ-near-rings and double-framed soft Γ-ideals of a Γ-near-ring with illustrative examples. We have then investigated these notions with respect to soft image, soft pre-image and β-inclusion of soft sets. Finally, we give some applications of double-framed soft Γ-near-rings to Γ-near-ring theory.

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