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On Fractional Time-Scale Differentiation

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Abstract

We consider the fractional differentiation on arbitrary time scales introduced by Benkhettou, Brito da Cruz and Torres. With this fractional differentiation we introduce an expansion of a function, we call the MAR expansion, by which we can also obtain Chu-Shih-Chieh's identity in combinatorics. We also give results on higher-order and chain rule of this fractional differentiation.

Keywords: Fractional differentiation, MAR expansion, time scales.

1. Introduction

Fractional differentiation refers to a noninteger order derivative of a function. This idea arises in 1695 when L'Hôpital asked Leibniz about $\frac{d^n f}{dt^n}$, $n \in \mathbb{N}$, if it also holds for arbitrary real number. In the letters to J. Wallis and J. Bernoulli (in 1697), Leibniz mentioned the possible approach to fractional differentiation in that sense. With this, the theory of fractional differentiation flourished as many mathematicians as well as researchers from many different fields work on the subject [6].

In 1990, Hilger introduced a unified approach to continuous and discrete calculus [4]. Also, Kolwankar and Gangal proposed a local fractional derivative operator to which it can be applied for nowhere differentiable functions [5].

The idea to join fractional differentiation with the calculus on time scales was developed by Bastos [2]. Recently, this theory receives attention from many researchers in pure and applied sciences. In [3], Benkhettou, Brito da Cruz and Torres introduced fractional calculus on time scales. It included fractional differentiation and fractional integration. Here, we consider their fractional differentiation. Using this fractional differentiation we introduce an expansion of a function, we call the MAR expansion. In the last part of the section on fractional expansion, we show how we can use this expansion to obtain Chu-Shih-Chieh's identity in combinatorics. We also give results on higher order fractional derivatives and fractional chain rule.

2. Significance of the Study

This research uses the fractional differentiation on time scales introduced by Benkhettou, Brito da Cruz, and Torres in [3]. Fractional differentiation is of increasing importance in other fields of science. The results may provide a mathematical framework to deal with functions which are not differentiable on $\mathbb{R}$ but on other time scales. It is also important to note that fractional time-scale differentiation can be used to unify discrete and continuous analysis.

3. Preliminaries

The following concepts are taken from [2] and [3].

Definition 3.1. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$.

Definition 3.2. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ as $\sigma(t) := \inf \{s \in \mathbb{T} : s > t\}$ and the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ as $\rho(t) := \sup \{s \in \mathbb{T} : s < t\}$.

Remark 3.3. Let $A = \{s \in \mathbb{T} : s > t\}$ and suppose $A = \varnothing$. Then, $\sigma(t) = M$ if $\mathbb{T}$ has a maximum M. Let $B = \{s \in \mathbb{T} : s < t\}$ and suppose $B = \varnothing$. Then, $\rho(t) = m$ if $\mathbb{T}$ has a minimum of m.

Definition 3.4. If $\sigma(t) > t$, then t is *right-scattered*. If $\rho(t) < t$ then t is said to be *left-scattered*. Points that are simultaneously right-scattered and left-scattered are called *isolated*. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*; if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*.

Definition 3.5. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.

Remark 3.6. Let $\mathbb{T}$ be a time scale. If $\mathbb{T}$ has a maximum M for which M is left-scattered, that is, $\rho(M) < M$, then make use of the notation $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.

The following is the fractional differentiation introduced by Benkhettou, Brito da Cruz, and Torres in [3], and then some of the results they obtained.

Definition 3.7. Let $f : \mathbb{T} \to \mathbb{R}$ be a function, $t \in \mathbb{T}^k$, and $\alpha \in (0, 1]$. For $\alpha \in (0, 1] \cap \big\{\frac{1}{q} : q \text{ is an odd number}\big\}$ (resp., $\alpha \in (0, 1] \setminus \big\{\frac{1}{q} : q \text{ is an odd number}\big\}$), define

$f^{(\alpha)}(t)$ to be the number (provided it exists) with the property that, given any $\epsilon > 0$, there exists $\delta > 0$ such that there is a δ*−neighborhood* U(where $U := (t-\delta, t+\delta) \cap \mathbb{T}$) of t (resp. *left* δ*−neighborhood* $U^- := (t-\delta, t) \cap \mathbb{T}$ of t), satisfying

$$\left| [f(\sigma(t)) - f(s)] - f^{(\alpha)}(t)\, [\sigma(t) - s]^{\alpha} \right| \leq \epsilon\, |\sigma(t) - s|^{\alpha},$$

for all $s \in U$ (resp., $s \in U^-$). Call $f^{(\alpha)}(t)$ to be the *fractional derivative* of f of order α at t. That is, f is *fractional differentiable* of order α at t.

The following are some of the results in fractional differentiation which are taken from [3].

Theorem 3.8. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. The following properties hold:

(i) Let $\alpha \in (0, 1] \cap \{ \frac{1}{q}$: q is an odd number $\}$. If t is right-dense and if f is fractional derivative of order α at t, then f is continuous at t.

(ii) Let $\alpha \in (0, 1] \setminus \{ \frac{1}{q}$: q is an odd number $\}$. If t is right-dense and if f is fractional derivative of order α at t, then f is left-continuous at t.

(iii) If f is continuous at t and t is right-scattered, then f is fractional differentiable of order α at t with

$$f^{(\alpha)}(t) = \frac{f(\sigma(t)) - f(t)}{(\mu(t))^{\alpha}}.$$

(iv) Let $\alpha \in (0, 1] \cap \{ \frac{1}{q}$: q is an odd number $\}$. If t is right-dense, then f is fractional differentiable of order α at t if and only if, the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{(t - s)^{\alpha}}$$

exists as a finite number. In this case,

$$f^{(\alpha)}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{(t - s)^{\alpha}}.$$

(v) Let $\alpha \in (0, 1] \setminus \{ \frac{1}{q}$: q is an odd number $\}$. If t is right-dense, then f is fractional differentiable of order α at t if and only if, the limit

$$\lim_{s \to t^-} \frac{f(t) - f(s)}{(t - s)^{\alpha}}$$

exists as a finite number. In this case,

$$f^{(\alpha)}(t) = \lim_{s \to t^-} \frac{f(t) - f(s)}{(t - s)^{\alpha}}.$$

(vi) If f is fractional differentiable of order α at t, then

$$f(\sigma(t)) = f(t) + (\mu(t))^{\alpha} f^{(\alpha)}(t).$$

Remark 3.9. In [2], $f^{(1)}$ is called the delta derivative. When the time scale is $\mathbb{R}$, then $f^{(1)} = f'$.

Theorem 3.10. Let c be a constant, $n \in \mathbb{N}$ and $\alpha \in (0, 1]$.

(i) If $f(t) = (t-c)^n$, then

$$f^{(\alpha)}(t) = (\mu(t))^{1-\alpha} \sum_{i=0}^{n-1} (\sigma(t) - c)^{n-1-i}(t-c)^i.$$

(ii) If $g(t) = \dfrac{1}{(t-c)^n}$, then

$$g^{(\alpha)}(t) = -(\mu(t))^{1-\alpha} \sum_{i=0}^{n-1} \frac{1}{(\sigma(t) - c)^{1+i}(t-c)^{n-i}},$$

provided $(t-c)(\sigma(t) - c) \neq 0$.

Definition 3.11. For a function $f : \mathbb{T} \to \mathbb{R}$, the *second fractional derivative* denoted by $f^{(2)}$ exists provided that $f^{(1)}$ is fractional differentiable on $\mathbb{T}^{k^2} = (\mathbb{T}^k)^k$ with derivative $f^{(2)} = \left(f^{(1)}\right)^{(1)} : \mathbb{T}^{k^2} \to \mathbb{R}$. For *higher order fractional derivative*, let β be a nonnegative real number with $\beta-$derivative then $f^{(\beta)} : \mathbb{T}^{k^{n+1}} \to \mathbb{R}$, then the *fractional derivative* of f of order β is given by

$$f^{(\beta)} := \left(f^{(n)}\right)^{(\alpha)}$$

where $n := \lfloor \beta \rfloor$, $n \in \mathbb{N}_0$ and $\alpha := \beta - n$ (i.e. $\alpha \in [0, 1]$).

Definition 3.12. Let f be a function, $h > 0$, $r, t \in h\mathbb{Z}$ with $r \leq t$. Define the notation

$$\left(\sum_{i=r}^{t}\right)_{h\mathbb{Z}} f(i) := f(r) + f(r+h) + \cdots + f(t-h) + f(t).$$

4. Main Result

4.1. Fractional Expansion

We present here an expansion of a fractional differentiable function.

Theorem 4.1. Let $h > 0$, $r \in h\mathbb{Z}$, $r \leq t$ and $\alpha \in (0, 1]$. If $f : h\mathbb{Z} \to \mathbb{R}$ is a continuous fractional differentiable function of order α at $t \in h\mathbb{Z}$ and $f(r-h)$ exists, then for $t \geq r$

$$f(t) = f(r-h) + h^{\alpha} \left(\sum_{i=r}^{t} \right)_{h\mathbb{Z}} f^{(\alpha)}(i-h). \tag{1}$$

Proof. Let $\alpha \in (0, 1]$ and $h > 0$. Suppose $f : h\mathbb{Z} \to \mathbb{R}$ is a continuous fractional differentiable function of order α. Note that for $r \in h\mathbb{Z}$, $r \leq t$,

$$f(t) = f(r-h) + f(r) - f(r-h) + f(r+h) - f(r) + \cdots + f(t) - f(t-h).$$

Since $\mu(t) = h, \forall t \in h\mathbb{Z}$,

$$\begin{aligned} f(t) =& f(r-h) + f(r + (-h + \mu(r-h))) - f(r-h) + f(r + \mu(r)) - f(r) \\ &+ \cdots + f(t + (-h + \mu(t-h))) - f(t-h) \\ =& f(r-h) + f(\sigma(r-h)) - f(r-h) + f(\sigma(r)) - f(r) \\ &+ \cdots + f(\sigma(t-h)) - f(t-h) \\ =& f(r-h) + (\mu(r-h))^{\alpha} \left[\frac{f(\sigma(r-h)) - f(r-h)}{(\mu(r-h))^{\alpha}} \right] + (\mu(r))^{\alpha} \\ &\cdot \left[\frac{f(\sigma(r)) - f(r)}{(\mu(r))^{\alpha}} \right] + \cdots + (\mu(t-h))^{\alpha} \left[\frac{f(\sigma(t-h)) - f(t-h)}{(\mu(t-h))^{\alpha}} \right]. \end{aligned}$$

Thus,

$$f(t) = f(r-h) + h^{\alpha} \left(\sum_{i=r}^{t} \right)_{h\mathbb{Z}} f^{(\alpha)}(i-h).$$

■

Theorem 4.2. [**MAR Expansion**] Let $r > 0$ and $f : r + \mathbb{Z} \to \mathbb{R}$ be a 1-order fractional differentiable function on the time scale $r + \mathbb{Z} = \{..., r-1, r, r+1, ...\}$. If $f^{(1)}$ is defined for each $t \geq r-1$ and $f(r-1)$ exists, then for $t \geq r$

$$f(t) = f(r-1) + \left(\sum_{i=r}^{t} \right)_{r+\mathbb{Z}} f^{(1)}(i-1). \tag{2}$$

Call (2) the *MAR expansion* of f at t.

Proof. The proof is similar to that of Theorem 4.1. ■

Remark 4.3. We call the right-hand side of equation (2) the **MAR expansion** of f at t.

Example 4.4. Consider the function $s(t) = \dfrac{1}{t}$. Then, for $i \geq 2$

$$s^{(1)}(i-1) = -\frac{1}{i(i-1)}$$

exists. Thus, by Theorem 4.2, the MAR expansion of $s(t)$, $t \geq 2$, is given by

$$s(t) = s(1) + \sum_{i=2}^{t} \left(-\frac{1}{i(i-1)} \right) = 1 - \sum_{i=2}^{t} \frac{1}{i(i-1)}.$$

The following are MAR expansions of some functions which the reader could solve:

(i) $t^n = \sum_{i=1}^{t} \left(\sum_{r=1}^{n} i^{(n-r)}(i-1)^r \right), \; t \geq 1, \; n \in \mathbb{N};$

(ii) $1 - \dfrac{1}{t^n} = \sum_{i=2}^{t} \left(\sum_{r=0}^{n-1} \dfrac{1}{i^{r+1}(i-1)^{n-r}} \right), \; t \geq 2, \; n \in \mathbb{N};$

(iii) $\ln t = \sum_{i=2}^{t} \ln \left(1 + \dfrac{1}{i-1} \right), \; t \geq 2;$

(iv) $\csc 1 \tan t = \sum_{i=1}^{t} \sec(i-1) \sec i, \; t \geq 1;$

Theorem 4.5. For $t \in \mathbb{T}$, let c be a constant, $n \in \mathbb{N}$ and $\alpha \in (0, 1]$.

(i) If $h(t) = (t-c)^{\frac{n}{2^r}}$ where $r \in \mathbb{Z}^+$, then

$$h^{(\alpha)}(t) = (\mu(t))^{1-\alpha} \; \frac{\sum_{i=0}^{n-1} (\sigma(t) - c)^{n-1-i}(t-c)^i}{\prod_{j=0}^{r-1} \left[(\sigma(t) - c)^{\frac{n}{2^{j+1}}} + (t-c)^{\frac{n}{2^{j+1}}} \right]}.$$

(ii) If $j(t) = (t-c)^{-\frac{n}{2^r}}$ where $r \in \mathbb{Z}^+$, then

$$j^{(\alpha)}(t) = -(\mu(t))^{1-\alpha} \; \frac{\sum_{i=0}^{n-1} (\sigma(t) - c)^{n(1-\frac{1}{2^r})-1-i}(t-c)^{i-\frac{n}{2^r}}}{\prod_{j=0}^{r-1} \left[(\sigma(t) - c)^{\frac{n}{2^{j+1}}} + (t-c)^{\frac{n}{2^{j+1}}} \right]}.$$

(iii) If $k(t) = (t-c)^{\frac{n}{q}}$ where q is an odd number, then

$$k^{(\alpha)}(t) = (\mu(t))^{1-\alpha} \frac{\displaystyle\sum_{i=0}^{n-1}(\sigma(t)-c)^{n-1-i}(t-c)^i}{\displaystyle\sum_{j=0}^{q-1}\left[(\sigma(t)-c)^{q-1-j}(t-c)^j\right]^{\frac{n}{q}}}.$$

(iv) If $m(t) = (t-c)^{-\frac{n}{q}}$ where q is an odd number, then

$$m^{(\alpha)}(t) = -(\mu(t))^{1-\alpha} \frac{\displaystyle\sum_{i=0}^{n-1}(\sigma(t)-c)^{n-1-i}(t-c)^i}{\displaystyle\sum_{j=0}^{q-1}\left[(\sigma(t)-c)^{q-j}(t-c)^{j+1}\right]^{\frac{n}{q}}}.$$

Proof. (i) Given $h(t) = (t-c)^{\frac{n}{2^r}}$, $n \in \mathbb{N}$, $r \in \mathbb{Z}^+$ then by Theorem 3.8 (iii) when t is right-scattered

$$\begin{aligned}
h^{(\alpha)}(t) =& \frac{h(\sigma(t)) - h(t)}{(\sigma(t)-t)^\alpha} \\
=& \frac{(\sigma(t)-c)^{\frac{n}{2^r}} - (t-c)^{\frac{n}{2^r}}}{(\sigma(t)-t)^\alpha} \cdot \frac{(\sigma(t)-c)^{\frac{n}{2^r}} + (t-c)^{\frac{n}{2^r}}}{(\sigma(t)-c)^{\frac{n}{2^r}} + (t-c)^{\frac{n}{2^r}}} \\
=& \frac{1}{(\sigma(t)-c)^{\frac{n}{2^r}} + (t-c)^{\frac{n}{2^r}}} \cdot \frac{(\sigma(t)-c)^{\frac{n}{2^{r-1}}} - (t-c)^{\frac{n}{2^{r-1}}}}{(\sigma(t)-t)^\alpha} \\
&\cdot \frac{(\sigma(t)-c)^{\frac{n}{2^{r-1}}} + (t-c)^{\frac{n}{2^{r-1}}}}{(\sigma(t)-c)^{\frac{n}{2^{r-1}}} + (t-c)^{\frac{n}{2^{r-1}}}}
\end{aligned}$$

Continuing this process r times, the equality becomes

$$\begin{aligned}
h^{(\alpha)}(t) =& \frac{1}{(\sigma(t)-c)^{\frac{n}{2^r}} + (t-c)^{\frac{n}{2^r}}} \cdot \frac{1}{(\sigma(t)-c)^{\frac{n}{2^{r-1}}} + (t-c)^{\frac{n}{2^{r-1}}}} \\
&\cdots \frac{1}{(\sigma(t)-c)^{\frac{n}{2}} + (t-c)^{\frac{n}{2}}} \cdot \frac{(\sigma(t)-c)^n - (t-c)^n}{(\sigma(t)-t)^\alpha}.
\end{aligned}$$

From Theorem 3.10 (i),

$$h^{(\alpha)}(t) = \frac{1}{\prod_{j=0}^{r-1}\left[(\sigma(t)-c)^{\frac{n}{2^{j+1}}}+(t-c)^{\frac{n}{2^{j+1}}}\right]} \cdot \left((\mu(t))^{1-\alpha}\sum_{i=0}^{n-1}(\sigma(t)-c)^{n-1-i}(t-c)^{i}\right)$$

$$=(\mu(t))^{1-\alpha}\frac{\sum_{i=0}^{n-1}(\sigma(t)-c)^{n-1-i}(t-c)^{i}}{\prod_{j=0}^{r-1}\left[(\sigma(t)-c)^{\frac{n}{2^{j+1}}}+(t-c)^{\frac{n}{2^{j+1}}}\right]}$$

For t that is right-dense, by Theorem 3.8 (iv) and (v),

$$h^{(\alpha)}(t) = \left(\lim_{s\to t}(t-s)^{\alpha}\right)\frac{\lim_{s\to t}\sum_{i=0}^{n-1}(\sigma(t)-c)^{n-1-i}(s-c)^{i}}{\lim_{s\to t}\prod_{j=0}^{r-1}\left[(\sigma(t)-c)^{\frac{n}{2^{j+1}}}+(s-c)^{\frac{n}{2^{j+1}}}\right]}.$$

Note that $\lim_{s\to t}(t-s)^{1-\alpha} = 0 = (\mu(t))^{1-\alpha}$ for $\alpha \in (0, 1)$ and for $\alpha = 1$, $\lim_{s\to t}(t-s)^{1-\alpha} = 1 = (\mu(t))^{1-\alpha}$. Hence,

$$h^{(\alpha)}(t) = (\mu(t))^{1-\alpha}\frac{\sum_{i=0}^{n-1}(\sigma(t)-c)^{n-1-i}(t-c)^{i}}{\prod_{j=0}^{r-1}\left[(\sigma(t)-c)^{\frac{n}{2^{j+1}}}+(t-c)^{\frac{n}{2^{j+1}}}\right]}.$$

(ii) Since $j(t) = \dfrac{1}{h(t)}$, using Theorem 3.8 (iv), one obtains

$$\begin{aligned}
j^{(\alpha)}(t) &= -\frac{h^{(\alpha)}(t)}{h(t)h(\sigma(t))} \\
&= -(\mu(t))^{1-\alpha} \frac{\displaystyle\sum_{i=0}^{n-1} (\sigma(t)-c)^{n-1-i}(t-c)^i}{\displaystyle\prod_{j=0}^{r-1}\left[(\sigma(t)-c)^{\frac{n}{2^{j+1}}} + (t-c)^{\frac{n}{2^{j+1}}}\right]} \cdot \frac{1}{(t-c)^{\frac{n}{2^r}}(\sigma(t)-c)^{\frac{n}{2^r}}} \\
&= -(\mu(t))^{1-\alpha} \frac{\displaystyle\sum_{i=0}^{n-1} (\sigma(t)-c)^{n(1-\frac{1}{2^r})-1-i}(t-c)^{i-\frac{n}{2^r}}}{\displaystyle\prod_{j=0}^{r-1}\left[(\sigma(t)-c)^{\frac{n}{2^{j+1}}} + (t-c)^{\frac{n}{2^{j+1}}}\right]}.
\end{aligned}$$

(iii) Suppose $k(t) = (t-c)^{\frac{n}{q}}$, where $n \in \mathbb{N}$, $c \in \mathbb{R}$ and q be an odd number. For the case when t is right-scattered, by Theorem 3.8 (iii)

$$\begin{aligned}
k^{(\alpha)}(t) =& \frac{(\sigma(t)-c)^{\frac{n}{q}} - (t-c)^{\frac{n}{q}}}{(\sigma(t)-t)^{\alpha}} \\
&\cdot \frac{\left[(\sigma(t)-c)^{\frac{n}{q}(q-1)} + (\sigma(t)-c)^{\frac{n}{q}(q-2)}(t-c)^{\frac{n}{q}(1)} + \cdots + (t-c)^{\frac{n}{q}(q-1)}\right]}{\left[(\sigma(t)-c)^{\frac{n}{q}(q-1)} + (\sigma(t)-c)^{\frac{n}{q}(q-2)}(t-c)^{\frac{n}{q}(1)} + \cdots + (t-c)^{\frac{n}{q}(q-1)}\right]} \\
=& \frac{1}{\left[(\sigma(t)-c)^{\frac{n}{q}(q-1)} + (\sigma(t)-c)^{\frac{n}{q}(q-2)}(t-c)^{\frac{n}{q}(1)} + \cdots + (t-c)^{\frac{n}{q}(q-1)}\right]} \\
&\cdot \frac{(\sigma(t)-c)^n - (t-c)^n}{(\sigma(t)-t)^{\alpha}}.
\end{aligned}$$

From part (i) of Theorem 3.10 it follows that

$$k^{(\alpha)}(t) = (\mu(t))^{1-\alpha} \frac{\displaystyle\sum_{i=0}^{n-1} (\sigma(t)-c)^{n-1-i}(t-c)^i}{\displaystyle\sum_{j=0}^{q-1}\left[(\sigma(t)-c)^{q-1-j}(t-c)^j\right]^{\frac{n}{q}}}.$$

For the case when t is right-dense and using a similar argument above

$$k^{(\alpha)}(t) = \lim_{s\to t}(t-s)^{1-\alpha}\frac{\displaystyle\sum_{i=0}^{n-1}(t-c)^{n-1-i}(s-c)^i}{\displaystyle\sum_{j=0}^{q-1}\left[(t-c)^{q-1-j}(s-c)^j\right]^{\frac{n}{q}}}.$$

Since $\sigma(t) = t$, $\lim_{s\to t}(t-s)^{1-\alpha} = 0 = (\mu(t))^{1-\alpha}$ for $0 < \alpha < 1$, $\lim_{s\to t}(t-s)^{1-\alpha} = 1 = (\mu(t))^{1-\alpha}$ when $\alpha = 1$ and where the quotient of the limits exist. Hence

$$\begin{aligned}
k^{(\alpha)}(t) &= \lim_{s\to t}(t-s)^{1-\alpha}\cdot\frac{\lim_{s\to t}\displaystyle\sum_{i=0}^{n-1}(\sigma(t)-c)^{n-1-i}(s-c)^i}{\lim_{s\to t}\displaystyle\sum_{j=0}^{q-1}\left[(\sigma(t)-c)^{q-1-j}(s-c)^j\right]^{\frac{n}{q}}}\\
&= (\mu(t))^{1-\alpha}\frac{\displaystyle\sum_{i=0}^{n-1}(\sigma(t)-c)^{n-1-i}(t-c)^i}{\displaystyle\sum_{j=0}^{q-1}\left[(\sigma(t)-c)^{q-1-j}(t-c)^j\right]^{\frac{n}{q}}}.
\end{aligned}$$

(iv) This can be proved by using the same argument applied in part (ii) of this theorem. ■

The next theorem extends Theorem 15 of [3].

Theorem 4.6. If $f_1, f_2, f_3, \ldots, f_{n-1}$ and f_n are fractional differentiable functions of order α at t, then for all $n \geq 2$,

$$(f_1f_2f_3\cdots f_{n-1}f_n)^{(\alpha)}(t) = \sum_{i=1}^{n} f_1(\sigma(t))\cdots f_{i-1}(\sigma(t))f_i^{(\alpha)}(t)f_{i+1}(t)\cdots f_n(t).$$

Proof. The proof immediately follows by the induction principle and by applying the product rule proved in [3]. ■

Corollary 4.7. If f is an $\alpha-$order fractional differentiable function at $t \in \mathbb{T}^k$. Then the $\alpha-$derivative of the $(n+1)th$ power of f, $n \in \mathbb{N}$ is

$$\left(f^{n+1}\right)^{(\alpha)}(t) = \left(\sum_{i=0}^{n} f^{n-i}(t)f^i(\sigma(t))\right)f^{(\alpha)}(t).$$

Proof. Let f be an $\alpha-$order fractional differentiable function at $t \in \mathbb{T}^k$. By the preceding theorem with $f = f_i$, $1 \leq i \leq n+1$, we have the result. ■

Corollary 4.8. Let t be a positive integer such that $t \geq 1$. For $r \in \mathbb{Z}^+$ with $r \leq t$,

$$\binom{t+r}{r+1} = \sum_{i=1}^{t} \binom{r+i-1}{r}. \tag{3}$$

Proof. To prove (3), consider the function $f(t) = \dfrac{t(t+1)(t+2)\cdots(t+r)}{r+1}$, $r \neq -1$. By applying the MAR expansion on f,

$$\begin{aligned}
\frac{t(t+1)(t+2)\cdots(t+r)}{r+1} &= \sum_{i=1}^{t} i(i+1)(i+2)\cdots(i-1+r) \\
\frac{(t+r)\cdots(t+2)(t+1)t(t-1)!}{(r+1)(t-1)!} &= \sum_{i=1}^{t} \frac{(i-1+r)\cdots(i+2)(i+1)i(i-1)!}{(i-1)!} \\
\frac{r!(t+r)!}{(r+1)![(t+r)-(r+1)]!} &= \sum_{i=1}^{t} \frac{r!(i-1+r)!}{r![(i-1+r)-r]!}.
\end{aligned}$$

Thus,

$$\binom{t+r}{r+1} = \sum_{i=1}^{t} \binom{r+i-1}{r}.$$

■

Remark 4.9. From (3), if $t = n - r + 1, n \geq r$, one obtains the *Chu Shih-Chieh's Identity*

$$\binom{n+1}{r+1} = \binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r}.$$

4.2. Higher Order Fractional Derivatives

We present here further results on higher-order fractional derivatives.

Theorem 4.10. Let $f : \mathbb{T} \to \mathbb{R}$ be defined by $f(t) = c, \forall c \in \mathbb{R}$ then $f^{(\beta)}(t) = 0$, for any $\beta \in \mathbb{R}^+$.

Proof. We first note that the α derivative of a constant is zero. From Definition 3.11,

$$f^{(\beta)}(t) = \left(c^{(n)}\right)^{(\alpha)}, \quad \text{where } \beta = n + \alpha,\ n \in \mathbb{N}_0 \text{ and } \alpha \in (0, 1].$$

For $n = 0$,

$$f^{(\beta)}(t) = (c)^{(\alpha)} = 0.$$

This time consider $n \geq 1$. Then,

$$\begin{aligned} f^{(\beta)}(t) &= \left((c)^{(1+n-1)}\right)^{(\alpha)} \\ &= \left(((c)^{(1)})^{(n-1)}\right)^{(\alpha)} \\ &= \left(0^{(n-1)}\right)^{(\alpha)} = 0. \end{aligned}$$

Hence, for any $\beta \in \mathbb{R}^+$ the conclusion holds. ■

Theorem 4.11. Let t be right dense, that is, $\forall t \in \mathbb{T}$, $\sigma(t) = t$ and let $f(t) = (t-c)^m$, $c \in \mathbb{R}$, $m \in \mathbb{N}$, then $f^{(\beta)}(t) = 0$, where $\beta \in \mathbb{R}^+$ and $\beta > m$.

Proof. Observe that for $\beta > m$,

$$f^{(\beta)}(t) = \left[\left((t-c)^m\right)^{(n)}\right]^{(\alpha)}, \qquad \text{where } n \geq m, \alpha \in (0, 1].$$

Since t is right-dense, it follows that for $\alpha = 1$ by Remark 3.9,$(t-c)^{(1)} = (t-c)'$. Thus,

$$\begin{aligned} f^{(\beta)}(t) =& \left[\left(((t-c)^m)^{(1)}\right)^{(n-1)}\right]^{(\alpha)} \\ =& \left[(m\left((t-c)^{m-1}\right)^{(1)})^{(n-2)}\right]^{(\alpha)} \\ &\vdots \\ f^{(\beta)}(t) =& \left[m(m-1)(m-2)\cdots(2)(1)(1)^{(n-m)}\right]^{(\alpha)}. \end{aligned}$$

If $n = m$, then $f^{(\beta)}(t) = m!(1)^{(\alpha)} = m!(0) = 0$. If $n > m$, then by Theorem 4.10, $f^{(\beta)}(t) = m!\left[(1)^{(n-m)}\right]^{(\alpha)} = 0$. ■

Theorem 4.12. Assume $f : \mathbb{T} \to \mathbb{R}$ and $g : \mathbb{T} \to \mathbb{R}$ be $\beta-$order fractional differentiable function on $\mathbb{T}^{k^{n+1}}$ where $n := \lfloor \beta \rfloor$, $\beta \in \mathbb{R}^+$. Then the following properties hold:

(i) $(f+g)^{(\beta)}(t) = f^{(\beta)}(t) + g^{(\beta)}(t)$;

(ii) $(\lambda f)^{(\beta)}(t) = \lambda f^{(\beta)}(t), \quad \lambda \in \mathbb{R}$.

Proof. This can be easily proved by applying the additive property of α-derivative n times.

■

Corollary 4.13. Let $h > 0$ and if $f : h\mathbb{Z} \to \mathbb{R}$ be a $\beta-$order fractional differentiable functions on $\mathbb{T}^{k^{n+1}}$. Then

$$f^{(\beta)}(t) = \frac{1}{h^{\beta}} \sum_{k=0}^{\lceil\beta\rceil} \binom{\lceil\beta\rceil}{k} (-1)^{\lceil\beta\rceil-k} f(t+kh). \tag{4}$$

Proof. Suppose $n = \lfloor\beta\rfloor$ such that $\alpha = \beta - n$, $n \in \mathbb{N}_0$. Let $\mathbb{T} = h\mathbb{Z}$ then, $\sigma(t) = \inf\{t + nh : n \in \mathbb{N}\} = t + h$ and $\mu(t) = \sigma(t) - t = t + h - t = h$, for all $t \in h\mathbb{Z}$. Let $\alpha \in (0, 1]$. By applying additive property of α derivative,

$$\begin{aligned}
f^{(\beta)}(t) =& (f^{(n)})^{(\alpha)}(t) \\
=& \left[\frac{1}{h^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} f(t+kh)\right]^{(\alpha)} \\
=& \frac{1}{h^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left[\frac{f(t+(k+1)h) - f(t+kh)}{h^{\alpha}}\right] \\
=& \frac{1}{h^{n+\alpha}}\left[\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} f(t+(k+1)h) - \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} f(t+kh)\right] \\
=& \frac{1}{h^{n+\alpha}}\left[\sum_{k=1}^{n+1}\binom{n}{k-1}(-1)^{n-(k-1)} f(t+kh) + \sum_{k=0}^{n}\binom{n}{k}(-1)^{n+1-k} f(t+kh)\right] \\
=& \frac{1}{h^{n+\alpha}}\left[\binom{n}{0}(-1)^{n+1} f(t) + \sum_{k=1}^{n}\left[\binom{n}{k-1} + \binom{n}{k}\right](-1)^{n+1-k} f(t+kh)\right. \\
& \left. + \binom{n}{n}(-1)^{0} f(t+(n+1)h)\right].
\end{aligned}$$

Consequently,

$$f^{(\beta)}(t) = \left(f^{(n)}\right)^{(\alpha)}(t) = \frac{1}{h^{\beta}} \sum_{k=0}^{\lceil\beta\rceil} \binom{\lceil\beta\rceil}{k} (-1)^{\lceil\beta\rceil-k} f(t+kh).$$

■

4.3. Fractional Chain Rule

We give here a chain rule formula.

Theorem 4.14. [Fractional Chain Rule] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous fractional differentiable function and suppose $g : \mathbb{T} \to \mathbb{R}$ be an $\alpha-$order fractional differentiable function. Then, the composition $f \circ g : \mathbb{T} \to \mathbb{R}$ is fractional differentiable of order α at t with the formula

$$(f \circ g)^{(\alpha)}(t) = \left\{\int_0^1 f'\left(g(t) + \varphi(\mu(t))^{\alpha} g^{(\alpha)}(t)\right) d\varphi\right\} g^{(\alpha)}(t). \tag{5}$$

Proof. To prove (5), first apply the concept of the ordinary substitution rule from calculus. Then

$$f(g(\sigma(t))) - f(g(s)) = \int_{g(s)}^{g(\sigma(t))} f'(\xi)d\xi.$$

Let $\xi = \varphi g(\sigma(t)) + (1-\varphi)g(s)$, $0 \leq \varphi \leq 1$. Consequently,

$$\begin{aligned} f(g(\sigma(t))) - f(g(s)) &= \int_0^1 f'(\varphi g(\sigma(t)) + (1-\varphi)g(s))[g(\sigma(t)) - g(s)]d\varphi \\ &= [g(\sigma(t)) - g(s)]\int_0^1 f'(\varphi g(\sigma(t)) + (1-\varphi)g(s))d\varphi. \end{aligned}$$

Suppose $t \in \mathbb{T}^k$ and let $\epsilon > 0$. Since g is an $\alpha-$order fractional differentiable function at t, then there exists a neighborhood U_1 of t (resp. U_1^-) such that

$$\left|[g(\sigma(t)) - g(s)] - g^{(\alpha)}(t)\,[\sigma(t) - s]^\alpha\right| \leq \epsilon_1\,|\sigma(t) - s|^\alpha\,, \tag{6}$$

for all $s \in U_1$ (resp $s \in U_1^-$), where

$$\epsilon_1 = \frac{\epsilon}{2\displaystyle\int_0^1 |f'(\varphi g(\sigma(t)) + (1-\varphi)g(t))|d\varphi}. \tag{7}$$

Moreover, f' is continuous on $\mathbb{R}$ and so, it is uniformly continuous on closed subset of $\mathbb{R}$. Note that g is also continuous since it is fractional differentiable of order α. Hence, there exists a neighborhood U_2 of t such that

$$|f'(\varphi g(\sigma(t)) + (1-\varphi)g(s)) - f'(\varphi g(\sigma(t)) + (1-\varphi)g(t))|.$$

$$\leq \frac{\epsilon}{2(\epsilon_1 + |g^{(\alpha)}(t)|)} \quad , \forall s \in U_2 \tag{8}$$

Note also that

$$\begin{aligned} &\left|[\varphi g(\sigma(t)) + (1-\varphi)g(s)] - [\varphi g(\sigma(t)) + (1-\varphi)g(t)]\right| \\ &\qquad = \left|\varphi g(\sigma(t)) + (1-\varphi)g(s)] - \varphi g(\sigma(t)) - (1-\varphi)g(t)]\right| \\ &\qquad \leq \left|g(s) - g(t)\right|, \end{aligned}$$

for all $0 \leq \varphi \leq 1$. Now, define a neighborhood $U = U_1 \cap U_2$ of t (resp. $U^- = U_1^- \cap U_2$) and let $s \in U$ (resp. $s \in U^-$). For convenience, let

$$\begin{aligned} \lambda &= \varphi g(\sigma(t)) + (1-\varphi)g(s) \quad \text{and} \\ \omega &= \varphi g(\sigma(t)) + (1-\varphi)g(t). \end{aligned}$$

Then,

$$\begin{aligned}
&\left|[(f\circ g)(\sigma(t))-(f\circ g)(t)]-\left(\int_0^1 f'(\omega)d\varphi\right)g^{(\alpha)}(t)\,[\sigma(t)-s]^\alpha\right|\\
&\quad=\left|[g(\sigma(t))-g(s)]\int_0^1 f'(\lambda)d\varphi-[\sigma(t)-s]^\alpha\,g^{(\alpha)}(t)\int_0^1 f'(\omega)d\varphi\right|\\
&\quad=\left|\left[(g(\sigma(t))-g(s))-[\sigma(t)-s]^\alpha g^{(\alpha)}(t)\right]\int_0^1 f'(\lambda)d\varphi\right.\\
&\qquad\left.+[\sigma(t)-s]^\alpha g^{(\alpha)}(t)\int_0^1 (f'(\lambda)-f'(\omega))d\varphi\right|\\
&\quad\le|[g(\sigma(t))-g(s)]-g^{(\alpha)}(t)[\sigma(t)-s]^\alpha|\int_0^1|f'(\lambda)|d\varphi\\
&\qquad+|\sigma(t)-s|^\alpha|g^{(\alpha)}(t)|\int_0^1|f'(\lambda)-f'(\omega)|d\varphi\\
&\quad\le\epsilon_1|\sigma(t)-s|^\alpha\int_0^1|f'(\lambda)|d\varphi+|\sigma(t)-s|^\alpha|g^{(\alpha)}(t)|\int_0^1|f'(\lambda)-f'(\omega)|d\varphi\\
&\quad=\epsilon_1|\sigma(t)-s|^\alpha\int_0^1(|f'(\lambda)|-|f'(\omega)|)d\varphi+\epsilon_1|\sigma(t)-s|^\alpha\int_0^1|f'(\omega)|d\varphi\\
&\qquad+|\sigma(t)-s|^\alpha|g^{(\alpha)}(t)|\int_0^1|f'(\lambda)-f'(\omega)|d\varphi\\
&\quad\le\epsilon_1|\sigma(t)-s|^\alpha\int_0^1|f'(\omega)|d\varphi\\
&\qquad+(\epsilon_1+|g^{(\alpha)}(t)|)|\sigma(t)-s|^\alpha\int_0^1|f'(\lambda)-f'(\omega)|d\varphi\\
&\quad\le\frac{\epsilon}{2}|\sigma(t)-s|^\alpha+(\epsilon_1+|g^{(\alpha)}(t)|)|\sigma(t)-s|^\alpha\int_0^1|f'(\lambda)-f'(\omega)|d\varphi\\
&\quad\le\frac{\epsilon}{2}|\sigma(t)-s|^\alpha+(\epsilon_1+|g^{(\alpha)}(t)|)|\sigma(t)-s|^\alpha\int_0^1\frac{\epsilon}{2(\epsilon_1+|g^{(\alpha)}(t)|)}d\varphi\\
&\quad\le\epsilon|\sigma(t)-s|^\alpha.
\end{aligned}$$

Therefore, $(f\circ g)$ is fractional differentiable of order α at t and its $\alpha-$fractional derivative is as claimed above. ■

Corollary 4.15. Assume $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be both fractional differentiable of order 1. Then, the 1-order fractional derivative of the composition $f\circ g:\mathbb{R}\to\mathbb{R}$ is the ordinary chain rule

$$(f\circ g)^{(1)}(t)=f'(g(t))g'(t).$$

Proof. It follows from Theorem 4.14 and by Remark 3.9. ■

Recommendation

It is recommended that one consider fractional integration on time scales in [3] and established the idea of partial fractional differentiation on time scales.

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