Some Properties on Connected Accurate Domination in Fuzzy Graphs

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Abstract

In this paper, the concept of connected accurate domination in fuzzy graphs discussed. A dominating set $D$ of a fuzzy graph $G$ is said to be an accurate dominating set if $V - D$ has no dominating set of same cardinality $|D|$. Let $G$ be a connected fuzzy graph and an accurate dominating set $D$ of $G$ is said to be a connected accurate dominating set if an induced fuzzy subgraph $(D)$ of $G$ is connected. The connected accurate domination number of a fuzzy graph $G$ is the minimum cardinality taken over all connected accurate dominating sets of $G$, and it is denoted by $\gamma_{fca}(G)$. We prove some results on connected accurate dominating set and exact values of connected accurate domination number, $\gamma_{fca}(G)$, for some standard fuzzy graphs are found.

Keywords: Accurate dominating set, Connected Accurate dominating set, Accurate domination number, Connected Accurate domination number, Strong arc and Strong neighbors.

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1. INTRODUCTION

The study of dominating sets in graphs was started by Ore [12] and Berge [1]. Then, Cockayne and Hedetniemi [3] has introduced the domination number and the independent domination number. An accurate domination and connected accurate domination was introduced by Kulli and Kattimani [4, 5].


2. PRELIMINARIES

A fuzzy graph \(G = \langle \sigma, \mu \rangle\) is a pair of functions \(\sigma: V \rightarrow [0,1]\) and \(\mu: V \times V \rightarrow [0,1]\), where for all \(x, y \in V\) we have \(\mu(x, y) \leq \sigma(x) \land \sigma(y)\).

A fuzzy graph \(H = \langle \tau, \rho \rangle\) is called a fuzzy subgraph of \(G\) if \(\tau(v_i) \leq \sigma(v_i)\) for all \(v_i \in V\) and \(\rho(v_i, v_j) \leq \mu(v_i, v_j)\) for all \(v_i, v_j \in V\).

The underlying crisp graph of a fuzzy graph \(G = \langle \sigma, \mu \rangle\) is denoted by \(G^* = \langle \sigma^*, \mu^* \rangle\), where \(\sigma^* = \{v_i \in V/\sigma(v_i) > 0\}\) and \(\mu^* = \{(v_i, v_j) \in V \times V/\mu(v_i, v_j)\}\) is a strong arc.

A vertex in \(G\) is called an isolated vertex if it is not adjacent to any vertices of \(G\). An edge in \(G\) is called an isolated edge if it is not adjacent to any edge in \(G\).

A path with \(n\) vertices in a fuzzy graph is denoted by \(P_n\). A cycle with \(n\) vertices in a fuzzy graph is denoted by \(C_n\).

A fuzzy graph \(G = \langle \sigma, \mu \rangle\) is said to be a complete fuzzy graph, if \(\mu(v_i, v_j) = \sigma(v_i) \land \sigma(v_j)\) for all \(v_i, v_j \in \sigma^*\).

An arc \((x, y)\) in a fuzzy graph \(G = \langle \sigma, \mu \rangle\) is said to be strong if \(\mu^{\sigma}(x, y) = \mu(x, y)\) then \(x, y\) are called strong neighbors. The strong neighborhood of the node \(u\) is defined as \(N_s(u) = \{v \in V: (u, v)\text{is a strong arc}\}\).

The strong neighborhood degree of a vertex \(u\) is defined as \(dN_s(u) = \sum_{v \in N_s(u)} \sigma(v)\). The minimum strong neighborhood degree of a fuzzy graph \(G\) is defined as \(\delta_{N_s}(G) = \min \{dN_s(u)/u \in V(G)\}\) and the maximum strong
neighborhood degree of a fuzzy graph $G$ is defined as $\Delta_{N_S}(G) = \max \{ dN_S(u)/u \in V(G) \}$.

Let $G = (\sigma, \mu)$ be a fuzzy graph and $\tau$ be any fuzzy subset of $\sigma$, (i.e) $\tau(u) \leq \sigma(u)$ for all $u$. Then the fuzzy subgraph of $G$ induced strong neighborhood by $\tau$ is the maximal fuzzy subgraph of $G$ that has fuzzy node set $\tau$. Evidently, this is just the fuzzy graph $(\tau, \rho)$, where $\rho(u, v) = \{(u, v)/\{u, v\}$ is a strong arc} for all $u, v \in \tau$.

A subset $D$ of $V$ is called a dominating set of a fuzzy graph $G$ if for every $v \in V - D$, there exist $u \in D$ such that $u$ dominates $v$. The domination number, $\gamma_f(G)$ of a fuzzy graph $G$ is the minimum cardinality taken over all dominating sets of fuzzy graph $G$.

A dominating set $D$ is said to be an accurate dominating set of a fuzzy graph $G$ if $V - D$ has no dominating set of cardinality $|D|$. The accurate domination number of a fuzzy graph $G$, is denoted by $\gamma_{fa}(G)$, is the minimum fuzzy cardinality taken over all accurate dominating sets of a fuzzy graph $G$.

3. CONNECTED ACCURATE DOMINATION IN FUZZY GRAPHS

In this section, we define connected accurate dominating set and connected accurate domination number of a fuzzy graph with suitable examples. We also discuss some results on connected accurate domination number of the fuzzy graphs.

**Definition 3.1:**

Let $G$ be a fuzzy graph and $D$ be a subset of $V$. Then, $D$ is said to be an accurate dominating set of a fuzzy graph $G$, if $V - D$ has no dominating set of cardinality $|D|$. The accurate domination number of a fuzzy graph $G$, is denoted by $\gamma_{fa}(G)$, is the minimum cardinality taken over all accurate dominating sets of a fuzzy graph $G$.

**Remark:** An accurate dominating set of a fuzzy graph $G$ could or could not be a minimal dominating set.

**Definition 3.2:**

Let $G$ be a connected fuzzy graph and an accurate dominating set $D$, where $D$ is a subset of $V$, of a fuzzy graph $G$ is said to be a connected accurate dominating set if a fuzzy induced subgraph of $\langle D \rangle$ is connected. The connected accurate domination number of a fuzzy graph $G$ is the minimum cardinality taken over all connected accurate dominating sets of a fuzzy graph $G$, and it is denoted by $\gamma_{fca}(G)$.
Example 3.3:

In Fig 3.1, \{b, d\}, \{b, e\}, \{c, e\}, \{a, c, d\}, \{a, b, d\} are some of the dominating sets of fuzzy graph \(G\). And \{b, e\}, \{a, c, d\}, \{a, b, d\}, \{a, b, e\} are some accurate dominating sets of \(G\) in fig 3.1.

Then, \{b, e\}, \{a, b, e\}, \{b, d, e\}, \{b, c, e\} are connected accurate dominating sets of \(G\) in fig 3.1. and its connected accurate domination number is \(\gamma_{fca}(G)\).

\[
\gamma_{fca}(G) = |\{b, e\}| = 0.8 + 0.7 = 1.5
\]

\[
\Rightarrow \gamma_{fca}(G) = 1.5
\]

Theorem 3.4:

If \(G\) be a connected fuzzy graph, where \(P \geq 3\) nodes, then \(\gamma_f(G) \leq \gamma_{fa}(G) \leq \gamma_{fca}(G)\).

Proof:

Let \(G\) be a connected fuzzy graph, with \(P \geq 3\) nodes.

A subset \(D\) of \(V\), be a minimum dominating set of a fuzzy graph \(G\) and its domination number is \(\gamma_f(G) = |D|\).

Case (i): \(\gamma_f(G) \leq \gamma_{fa}(G)\)

Let \(D\) be a dominating set of fuzzy graph \(G\). Suppose that, \(V - D\) has no dominating set with same cardinality \(|D|\), then \(D\) will be an accurate dominating set of fuzzy graph \(G\), (i.e.) \(\gamma_f(G) = |D| = \gamma_{fa}(G)\).

We know that, every accurate dominating set of a fuzzy graph \(G\) is also a dominating set of \(G\).
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\[ \gamma_f(G) \leq \gamma_{fa}(G) \quad \rightarrow \quad (3.1) \]

**Case (ii):** \( \gamma_{fa}(G) \leq \gamma_{fca}(G) \)

Similarly by case (i), Let \( G \) be a connected fuzzy graph and \( D \) be an accurate dominating set of a fuzzy graph \( G \).

If an induced subgraph \( D \) is connected, then \( D \) be a connected accurate dominating set of a fuzzy graph \( G \).

\[ \therefore \gamma_{fa}(G) \leq \gamma_{fca}(G) \quad \rightarrow \quad (3.2) \]

From equations (3.1) & (3.2), we get, \( \gamma_f(G) \leq \gamma_{fa}(G) \leq \gamma_{fca}(G) \).

**Theorem 3.5:**

If \( G \) be a connected fuzzy graph, with \( P \geq 3 \) nodes, then \( \gamma_{fc}(G) \leq \gamma_{fca}(G) \).

**Proof:**

Let \( G \) be any connected fuzzy graph, with \( P \geq 3 \) nodes, and let \( D \) be a minimum connected dominating set of a fuzzy graph \( G \), then the connected domination number of \( G \) is denoted by \( \gamma_{fc}(G) = |D| \).

Suppose, \( V - D \) has no connected dominating set with fuzzy cardinality \( |D| \), then \( D \) is said to be a connected accurate dominating set of a fuzzy graph \( G \).

We also know that, every connected accurate dominating set of a fuzzy graph \( G \) is also a connected dominating set of \( G \).

Therefore, \( \gamma_{fc}(G) \leq \gamma_{fca}(G) \).

**Theorem 3.6:**

Let \( G \) be any connected fuzzy graph, with \( P \geq 5 \) nodes, then, \( \frac{P}{\Delta_{N_3}+1} \leq \gamma_{fca}(G) \leq P - 2 \).

**Proof:**

Let \( G \) be a connected fuzzy graph, with \( P \geq 5 \) nodes. A subset \( D \) of \( V \), be a dominating set of a fuzzy graph \( G \), and \( v \in D \), be a vertex of fuzzy graph \( G \) whose, \( |N_s(v)| = P - 1 = \Delta_{N_3}(G) \).

Then, \( D = \{v\} \) dominates all \((P - 1)\) vertices of \( G \) and also dominated by itself.

\[ \therefore \gamma_f(G) \geq \frac{P}{\Delta_{N_3}+1} \geq \frac{P}{P-1+1} \]
\[ \gamma_f(G) \geq 1. \]

By theorem 3.4, we know that, for any connected fuzzy graph \( G \), with \( P \geq 5 \),
\[ \gamma_f(G) \leq \gamma_{fa}(G) \leq \gamma_{fca}(G). \]
\[ \therefore \frac{p}{\Delta_{N_s+1}} \leq \gamma_f(G) \]
\[ \frac{p}{\Delta_{N_s+1}} \leq \gamma_{fca}(G). \]

Now, let us prove that, \( \gamma_{fca}(G) \leq P - 2 \).

Since, \( G \) is a connected fuzzy graph, then there exists a strong path between each pair of vertices of the fuzzy graph \( G \).

Now, let \( v_1 \) and \( v_n \) be the initial and terminal vertices of a strong path \( P_n \).

Then, the connected dominating set of \( P_n \) is \( D = V - \{v_1, v_n\} \).

That is the connected domination number of a fuzzy graph \( G \) is
\[ \gamma_{fca}(G) = |D| \]
\[ \gamma_{fca}(G) \leq |V - \{v_1, v_n\}| \]
\[ \gamma_{fca}(G) \leq P - 2. \]

**Theorem 3.7:**

If \( G \) be a connected fuzzy graph and \( S \) be a set of all strong arcs in \( G \), then \( \gamma_{fca}(G) \leq 2S - P \).

**Proof:**

Let \( G \) be a connected fuzzy graph.

Let \( S \) be a set of all strong arcs in \( G \).

Since, \( G \) is a connected fuzzy graph.
\[ |S| \geq P - 1 \quad \rightarrow \quad (3.3) \]

By theorem 3.6,
\[ \gamma_{fca}(G) \leq P - 2 \]
\[ \leq 2P - P - 2 \]
\[ \leq 2(P - 1) - P \]
\[ \leq 2S - P \quad \text{(By equation (3.3))} \]
\[ \therefore \gamma_{fca}(G) \leq 2S - P. \]
Theorem 3.8:
Let $G$ be a connected fuzzy graph and $T$ be a spanning fuzzy tree of $G$, then
\[ \gamma_{fca}(G) = P - \mathcal{E}_T \] where, $\mathcal{E}_T$ be a number of pendent vertices of a fuzzy graph $G$.

Proof:
Let $G$ be a connected fuzzy graph and $T$ be a spanning fuzzy tree of a fuzzy graph $G$ and also $T$ is connected.

Let $\mathcal{E}_T$ be a number of pendent vertices of a spanning fuzzy tree $T$.

(ie) $\mathcal{E}_T = \{ v \in V | |N_s(v)| = 1 \}$.

Since, every $u \in \mathcal{E}_T$ has a neighborhood degree atmost one, (ie) $dN_s(u) = 1$, and the dominating set $D$ is a connected accurate dominating set of $G$, therefore, there exist $v \in D$ for every $u \in \mathcal{E}_T$.

Then, the connected accurate dominating set $D = V - \{ v \in V | |N_s(v)| = 1 \}$.

\[ \therefore \text{The connected accurate domination number of a fuzzy graph } G \text{ is} \]
\[ \gamma_{fca}(G) = |D| \]
\[ \leq |V - \{ v \in V | |N_s(v)| = 1 \}| \]
\[ \gamma_{fca}(G) \leq P - \mathcal{E}_T. \]

Corollary:
In a connected fuzzy graph $G$, with $P \geq 3$ nodes, let $H$ be any connected spanning fuzzy subgraph then, $\gamma_{fca}(G) \leq \gamma_{fca}(H)$.

Proof:
Since, $H$ is a spanning fuzzy subgraph of a connected fuzzy graph $G$.

It is evident that every connected accurate dominating set of a spanning fuzzy subgraph $H$ is also a connected accurate dominating set of a connected fuzzy graph $G$. Hence $\gamma_{fca}(G) \leq \gamma_{fca}(H)$.

4. CONNECTED ACCURATE DOMINATION NUMBER OF SOME STANDARD CONNECTED FUZZY GRAPHS

In this section we discuss about the connected accurate dominating set and connected accurate domination number of some standard fuzzy graphs.
Observation 4.1:
Let $G$ be a complete fuzzy graph, $G^* = K_p$, with $p \geq 3$, then $\gamma_{fca}(K_p) \leq \left\lfloor \frac{p}{2} \right\rfloor + 1$.

Proof:
Let $G$ be a complete fuzzy graph and $D$, subset of $V$, be a connected accurate dominating set of $G$, therefore $V - D$ should not have any connected dominating set with same fuzzy cardinality $|D|$.

Since, $D$ is a connected accurate dominating set of $G$, then $D$ must have at most $\left\lfloor \frac{p}{2} \right\rfloor + 1$ vertices and the induced fuzzy subgraph $(D)$ is connected.

Therefore, connected accurate domination number of $G$ is

$$|D| \leq \left\lfloor \frac{p}{2} \right\rfloor + 1,$$

$$\gamma_{fca}(K_p) \leq \left\lfloor \frac{p}{2} \right\rfloor + 1.$$

Observation 4.2:
For any connected fuzzy graph $G$ and $G^* = P_n$, is a path $P_n$, with $n \geq 3$ nodes,

$$\gamma_{fca}(P_n) \leq n - 2.$$

Proof:
Let $G$ be a connected fuzzy graph and the underlying crisp graph $G^*$ of a fuzzy graph $G$ is a path with $n \geq 3$ vertices, $P_n$.

Since, $D$ is a connected accurate dominating set of a fuzzy graph $G$ and $G$ is a path, then $D$ must have exactly $n - 2$ vertices, i.e., $D = V - \{v_1, v_n\}$, where, $v_1$ and $v_n$ are the initial and terminal vertices of a path $P_n$.

Therefore, connected accurate domination number of a fuzzy graph $P_n$ is

$$\gamma_{fca}(P_n) = |D|$$

$$\leq |V - \{v_1, v_n\}| = n - 2$$

ie) $\gamma_{fca}(P_n) \leq n - 2$.

Observation 4.3:
For any connected fuzzy graph $G$, and $G^* = C_n$, with $n \geq 5$ vertices, $\gamma_{fca}(C_n) \leq n - 2$. 
Proof:

Let $G$ be a connected fuzzy graph and the underlying crisp graph $G^*$ of a fuzzy graph $G$ is a cycle $C_n$, with $n \geq 5$ vertices.

By observation 4.2, since, $D$ is a connected accurate dominating set of a fuzzy graph $G$ and $G$ is a cycle, then $D$ must have exactly $n - 2$ vertices, i.e., $D = V - \{u, v\}$, where, $u$ and $v$ are the strong neighbors to each other in a fuzzy graph $C_n$.

Therefore, each $v_i$ vertex will dominate two vertices $v_{i-1}$ and $v_{i+1}$ of a fuzzy graph $C_n$.

Hence, the connected accurate domination number of a fuzzy graph $C_n$ is

$$\gamma_{fca}(C_n) = |D|$$

$$\leq |V - \{u, v\}| = n - 2$$

ie) $\gamma_{fca}(C_n) \leq n - 2$.

Observation 4.4:

For any connected fuzzy graph $G$ and $G^* = K_{m,n}$, where $m \leq n$, $\gamma_{fca}(K_{m,n}) \leq m + 1$.

Proof:

Let $G$ be a connected fuzzy graph and its underlying crisp graph $G^* = K_{m,n}$ where $m \leq n$.

Since, $K_{m,n}$ is a complete bipartite fuzzy graph then, the vertex set is partitioned into two sets $V_1$ and $V_2$ where, $|V_1| = m$ and $|V_2| = n$.

Since, $D$ be a connected accurate dominating set, then $D$ must have at most $m + 1$ vertices, i.e., $D = V_1 \cup \{v\}$ where $v \in V_2$.

Therefore, the connected accurate domination number of a complete bipartite fuzzy graph is

$$\gamma_{fca}(K_{m,n}) = |D|$$

$$\leq |V_1 \cup \{v\}|$$

$$\leq m + 1.$$
**Proof:**

Let $G$ be a connected fuzzy graph and its underlying crisp graph $G^*$ is a star $S_n$, with $n + 1$ nodes, $n \geq 2$.

We know that, fuzzy graph $S_n$ is also a complete bipartite fuzzy graph $K_{1,n}$.

Therefore, by observation 4.4, $\gamma_{fca}(G) = |D| \leq |\{u\} \cup \{v\}| \leq 2$.

**5. CONCLUSION:**

Some important results of connected accurate domination of a fuzzy graph discussed. The exact values of accurate domination number, $\gamma_{fca}(G)$, for some standard fuzzy graphs are found. Further works are to find the relation between connected accurate domination numbers with other accurate domination parameters of fuzzy graphs.

**REFERENCES**


