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Fekete-Szego inequality for subclasses of analytic functions bounded by chebyshev polynomial

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Abstract

In this present work, we obtain certain coefficients of the subclasses of univalent functions and estimates the relevant connection to the famous classical Fekete-Szegö inequality of functions belonging to the class.

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Mathematics Subject Classification: 30C45, 30C50

1. INTRODUCTION

Chebyshev polynomials are used in many parts of numerical analysis and more generally in applications of mathematics, for more examples see Doha[2] and Mason[5]. The Chebyshev polynomials $T_n(x)$) and $U_n(x)$ of degree n of the first and second kinds, respectively, appropriate to the range [-1,1] of x, may be conveniently defined in terms of a transformation from x to θ as

$$T_n(x) = cosn\theta,$$

$$U_n(x) = \frac{sin(n+1)}{sin\theta}$$

where $x = cos\theta$.

In 1933, Fekete-Szegö[4] proved that

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3-4\mu & if \ \mu \leq 0 \\ 1+2exp\left(\dfrac{-2\mu}{1-\mu}\right) & if \ 0 \leq \mu \leq 1 \\ 4\mu - 3 & if \ \mu \geq 1 \end{cases}$$

holds for the functions $f \in \mathcal{S}$ and the result is sharp. The problem of finding the sharp bounds for the non-linear functional $|a_3 - \mu a_2^2|$ of any compact family of functions is popularly known as the Fekete-Szegö problem.

Let $\mathcal{A}$ denote the family of analytic functions in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ is of the form,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all functions in $\mathcal{A}$ which are also univalent in $\mathcal{U}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*$ of starlike functions in $\mathcal{U}$, if it satisfies the following inequality:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad (z \in \mathcal{U}).$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}$ of convex functions in $\mathcal{U}$, if it satisfies the following inequality:

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad (z \in \mathcal{U}).$$

With a view to recalling the principal of subordination between analytic functions, let the functions $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is *subordinate* to $g(z)$, if there exits a Schwarz function $\omega(z)$, analytic in $\mathcal{U}$ with

$$\omega(0) = 0 \quad and \quad |\omega(z)| < 1 \quad (z \in \mathcal{U}).$$

such that $f(z) = g(\omega(z)), z \in \mathcal{U}$. We denote this subordination by

$$f \prec g \quad or \quad f(z) \prec g(z).$$

In particular, if the function g is univalent in $\mathcal{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad and \quad f(\mathcal{U}) \subset g(\mathcal{U}).$$

Definition 1 *The function* $f \in \mathcal{A}$ *is in the class* $\mathcal{S}_s^*$, $t \in (\frac{1}{2}, 1]$ *if*

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec H(z,t) := \frac{1}{1 - 2tz + z^2} \qquad (z \in \mathcal{U}). \tag{1.2}$$

Definition 2 *The function* $f \in \mathcal{A}$ *is in the class* $\mathcal{C}_s$, $t \in (\frac{1}{2}, 1]$ *if*

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} \prec H(z,t) := \frac{1}{1 - 2tz + z^2} \qquad (z \in \mathcal{U}).$$

We note that if $t = cos\alpha$, $\alpha \in (-\pi/3, \pi/3)$, then

$$\begin{aligned} H(z,t) &= \frac{1}{1-2tz+z^2} \\ &= 1+\sum_{n=1}^{\infty} \frac{sin(n+1)\alpha}{sin\alpha} z^n \quad (z \in \mathrm{U}). \end{aligned}$$

Thus

$$H(z,t) = 1 + 2cos\alpha z + (3cos^2\alpha - sin^2\alpha)z^2 + \dots \quad (z \in \mathrm{U}).$$

Following[6], we write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \dots \quad (z \in \mathrm{U}, t \in (-1,1)),$$

Where

$$U_{n-1} = \frac{sin(\mathrm{narccost})}{\sqrt{1-t^2}} \quad (n \in \mathrm{N})$$

are the Chebyshev polynomials of the second kind. Also, it is known that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

and

$$U_1(t) = 2t, U_2(t) = 4t^2 - 1, U_3(t) = 8t^3 - 4t, \dots \tag{1.3}$$

The Chebychev polynomials $T_n(t)$, $t \in [-1,1]$, of the first kind have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1-tz}{1-2tz+z^2} \quad (z \in \mathrm{U}).$$

However, the Chebychev polynomials of the first kind $T_n(t)$ and of the second kind $U_n(t)$ are well connected by the following relationships

$$\begin{aligned} \frac{dT_n(t)}{dt} &= nU_{n-1}(t), \\ T_n(t) &= U_n(t) - tU_{n-1}(t), \\ 2T_n(t) &= U_n(t) - U_{n-2}(t). \end{aligned}$$

Motivated by the recent works of Dziok et al.,[3] and Altinkaya et al.,[1], We estimates the initial coefficients of univalent functions by using the Chebyshev polynomial expansions.

2. MAIN RESULTS

Theorem 1 *If* $f \in \mathrm{S}_s^*$*, then*

$$|a_2| \le t \tag{2.1}$$

and

$$|a_3| \le t + 2t^2 - \frac{1}{2}. \tag{2.2}$$

Proof. Let $f \in \mathrm{S}_s^*$. From (1.2), we have

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1 + U_1(t)\omega(z) + U_2(t)\omega^2(z) + \dots, \tag{2.3}$$

for some analytic function ω such that $\omega(0)=0$ and $|\omega(z)|<1$ for all $z\in\mathrm{U}$. It is well known that if,

$$|\omega(z)|=|c_1z+c_2z^2+c_3z^3+\ldots|<1, z\in\mathrm{U}, then \tag{2.4}$$

$$|c_j|\le 1, \text{for all} j\in\mathrm{N}$$

and

$$|c_2-\mu c_1^2|\le \max\{1;|\mu|\}, \qquad \text{for all} \mu\in\mathrm{R}. \tag{2.5}$$

By using (2.4) in (2.3) we get,

$$\frac{2zf'(z)}{f(z)-f(-z)}=1+U_1(t)c_1z+[U_1(t)c_2+U_2(t)c_1^2]z^2+\ldots \tag{2.6}$$

From (2.6), we can easily deduce that

$$\begin{aligned} 2a_2 &= U_1(t)c_1 \\ 2a_3 &= U_1(t)c_2+U_2(t)c_1^2 \end{aligned} \tag{2.7}$$

Using (1.3) in (2.7),

$$|a_2|\le t$$

and

$$|a_3|\le t+2t^2-\frac{1}{2}.$$

Theorem 2 *If* $f\in\mathrm{S}_s^*$*, then*

$$|a_3-\mu a_2^2|\le\begin{cases} t; & if\ \mu\in[\mu_1,\mu_2] \\ t\left|\frac{4t^2-1}{2t}-\mu t\right|; & if\ \mu\notin[\mu_1,\mu_2] \end{cases}$$

where

$$\mu_1=\frac{4t^2-(1+2t)}{2t^2}, \quad \mu_2=\frac{4t^2-(1-2t)}{2t^2}.$$

Proof. From (2.7)

$$|a_3-\mu a_2^2|=\frac{U_1(t)}{2}\left|c_2+\left(\frac{U_2(t)}{U_1(t)}-\mu\frac{U_1(t)}{2}\right)c_1^2\right|.$$

In view of (2.5), we conclude that

$$|a_3-\mu a_2^2|\le\frac{U_1(t)}{2}\max\left\{1,\left|\frac{U_2(t)}{U_1(t)}-\mu\frac{U_1(t)}{2}\right|\right\}. \tag{2.8}$$

Using (2.1) and (2.2) in (2.8)

$$|a_3-\mu a_2^2|\le \text{ t max }\left\{1,\left|\frac{4t^2-1}{2t}-\mu t\right|\right\}.$$

Since $t>0$, we have

$$\begin{aligned}\left|\frac{4t^2-1}{2t}-\mu t\right| &\le 1\\ &\Leftrightarrow \frac{4t^2-(1+2t)}{2t^2}\le\mu\le\frac{4t^2-(1-2t)}{2t^2}\\ &\Leftrightarrow \mu_1\le\mu\le\mu_2.\end{aligned}$$

Theorem 3 *If* $f\in\mathrm{C}_s$*, then*

$$|a_2|\le\frac{t}{2} \tag{2.9}$$

and

$$|a_3|\le\frac{t}{3}+\frac{2t^2}{3}-\frac{1}{6}. \tag{2.10}$$

Proof. From the definitions of the classes S_s^* and C_s, it is well known that the Alexander relation, the function $f\in\mathrm{C}_s$ if and only if $zf'\in\mathrm{S}_s^*$. Thus replacing a_n by na_n in (2.1) and (2.2) we obtain the above results.

Theorem 4 *If* $f\in\mathrm{C}_s$*, then*

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{t}{3}; & \text{if } \mu\in[\mu_3,\mu_4]\\ \dfrac{t}{3}\left|\dfrac{4t^2-1}{2t}-\mu\dfrac{3t}{4}\right|; & \text{if } \mu\notin[\mu_3,\mu_4]\end{cases}$$

where

$$\mu_3=\frac{2[4t^2-(1+2t)]}{3t^2},\quad \mu_4=\frac{2[4t^2-(1-2t)]}{3t^2}.$$

Proof. From (2.9) and (2.10)

$$|a_3-\mu a_2^2|=\frac{U_1(t)}{6}\left|c_2+\left(\frac{U_2(t)}{U_1(t)}-3\mu\frac{U_1(t)}{8}\right)c_1^2\right|.$$

In view of (2.5), we conclude that

$$|a_3-\mu a_2^2|\le\frac{U_1(t)}{6}\max\left\{1,\left|\frac{U_2(t)}{U_1(t)}-3\mu\frac{U_1(t)}{8}\right|\right\}. \qquad (2.11)$$

Using (2.9) and (2.10) in (2.11)

$$|a_3-\mu a_2^2|\le\frac{t}{3}\max\left\{1,\left|\frac{4t^2-1}{2t}-\mu\frac{3t}{4}\right|\right\}.$$

Since $t>0$, we have

$$\left|\frac{4t^2-1}{2t}-\mu\frac{3t}{4}\right|\le 1$$

$$\Leftrightarrow \frac{2[4t^2-(1+2t)]}{3t^2}\le\mu\le\frac{2[4t^2-(1-2t)]}{3t^2}$$

$$\Leftrightarrow \mu_3\le\mu\le\mu_4.$$

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