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A Fixed Point Result in D*-Metric Spaces

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Abstract

In this paper a fixed point theorem for generalized quasi-contractions of D*-metric spaces have been proved and also some of its consequences are given.

Keywords: D*-metric spaces, complete D*-metric space, quasi-contraction, generalized quasi-contraction

I. INTRODUCTION

Generally fixed point theorems were established for self-maps of metric spaces. Certain fixed point theorems were proved for self-maps of metrizable topological spaces also since such spaces, for all practical purposes, can be considered as metric spaces.

Recently in 1992 B. C. Dhage [1] has initiated a study of general metric spaces called *D*-metric spaces. Later several researchers have made a significant contribution to the fixed point theorems of D-metric spaces in [2], [3], [4], [5] and [6].As a probable modification of D-metric spaces, very recently, Shaban Sedghi, Nabi Shobe and Haiyun Zhou [7] have introducedThe notion of quasi-contractions on *D**-metric spaces has been generalized by Brian Fisher [8]. Analogously we define generalized quasi-contractions among the selfmaps of *D**-metric spaces.

II. PRELIMINARIES

2.1 Definition: Let X be a non-empty set. A function $D^*: X^3 \to [0,\infty)$ is said to be a ***generalized metric*** or ***D*-metric*** on X, if it satisfies the following conditions:

(i) $D^*(x,y,z) \geq 0$ for all $x,y,z \in X$

(ii) $D^*(x,y,z) = 0$ if and only if $x = y = z$

(iii) $D^*(x,y,z) = D^*(\sigma(x,y,z))$ for all $x,y,z \in X$,

where $\sigma(x,y,z)$ is a permutation of the set $\{x,y,z\}$

(iv) $D^*(x,y,z) \leq D^*(x,y,w) + D^*(w,z,z)$ for all $x,y,z,w \in X$.

The pair (X, D*), where D* is a generalized metric on X is called a ***D*-metric space*** or a ***generalized metric space.***

2.2 Definition: A D*-metric space (X, D*) is said to be ***complete***, if every Cauchy sequence in it converges in it.

2.3 Definition: A selfmap f of a D*-metric space (X, D*) is called a ***quasi-contraction***, if there is a number q with $0 \leq q < 1$ such that

$$D^*(fx, fy, fy) \leq q \max\{ D^*(x,y,y), D^*(x,fx,fx), D^*(y,fy,fy), D^*(x,fy,fy), D^*(y,fx,fx) \}$$

for all $x,y \in X$

2.4 Definition: A selfmap f of a D*-metric space (X, D*) is called a ***generalized quasi-contraction***, if for some fixed positive integers k and l, there is a number q with $0 \leq q < 1$ such that

$$D^*\left(f^k x, f^l y, f^l y\right) \leq q.\max\{ D^*\left(f^r x, f^s y, f^s y\right), D^*\left(f^r x, f^{r'} x, f^{r'} x\right), D^*\left(f^s y, f^{s'} y, f^{s'} y\right) : 0 \leq r, r' \leq k,\ 0 \leq s, s' \leq l \}$$

III. MAIN RESULT

3.1 Theorem: Suppose f is a generalized quasi-contraction on a complete D*-metric spaces (X , D*) and let f be continuous. Then f has a unique fixed point.

Proof: By increasing the value of q if necessary, we may assume that $\frac{1}{2} \leq q < 1$ and the inequality in the generalized quasi-contraction will still hold. But we will then have that $\frac{q}{1-q} \geq 1$. Assume that $k \geq l$

Let x be an arbitrary point in X, we claim that the sequence $\{f^n x : n = 1, 2, 3, \ldots\}$ is bounded.

If possible assume that the sequence $\{f^n x : n = 1, 2, 3, \ldots\}$ is unbounded. Then the sequence $\{D^*(f^n x, f^l x, f^l x) : n = 1, 2, 3, \ldots\}$ is unbounded.

Let $K = \frac{q}{1-q}.\max\{D^*(f^i x, f^l x, f^l x) : 0 \le i \le k\}$. Since $\{D^*(f^n x, f^l x, f^l x) : n = 1, 2, 3, \ldots\}$ is unbounded, there exists an integer n such that $D^*(f^n x, f^l x, f^l x) > K$ and let n_0 be smallest such n and since $\frac{q}{1-q} \ge 1$, $n_0 > k \ge l$. Thus

$$D^*(f^{n_0} x, f^l x, f^l x) > K = \frac{q}{1-q}.\max\{D^*(f^i x, f^l x, f^l x) : 0 \le i \le k\}$$

That is,

(3.1.1) $$D^*(f^{n_0} x, f^l x, f^l x) > \frac{q}{1-q}.\max\{D^*(f^i x, f^l x, f^l x) : 0 \le i \le k\}$$

Now it follows that

$$\begin{aligned}(1-q)D^*(f^{n_0} x, f^l x, f^l x) &> q.\max\{D^*(f^i x, f^l x, f^l x) : 0 \le i \le k\}\\ &\ge q.\max\{\, D^*(f^i x, f^r x, f^r x) - D^*(f^r x, f^l x, f^l x) :\\ &\qquad\qquad 0 \le i \le k;\ 0 \le r < n_0 \,\}\\ &\ge q.\max\{\, D^*(f^i x, f^r x, f^r x) - D^*(f^{n_0} x, f^l x, f^l x) :\\ &\qquad\qquad 0 \le i \le k;\ 0 \le r < n_0 \,\}\\ &= q.\max\{\, D^*(f^i x, f^r x, f^r x) : 0 \le i \le k;\ 0 \le r < n_0 \,\}\\ &\qquad - q.D^*(f^{n_0} x, f^l x, f^l x)\end{aligned}$$

And hence

(3.1.2)

$$D^*\left(f^{n_0}x, f^l x, f^l x\right) > q.\max\left\{ D^*\left(f^i x, f^r x, f^r x\right): 0 \le i \le k;\ 0 \le r < n_0 \right\}$$

We shall now prove that

(3.1.3)

$$D^*\left(f^{n_0}x, f^l x, f^l x\right) > q.\max\left\{ D^*\left(f^i x, f^r x, f^r x\right): 0 \le i, r < n_0 \right\}$$

If not,

(3.1.4)

$$D^*\left(f^{n_0}x, f^l x, f^l x\right) \le q.\max\left\{ D^*\left(f^i x, f^r x, f^r x\right): k < i, r < n_0 \right\}$$

Now applying the inequality in the definition of the generalized quasi-contraction repeatedly to the inequality (3.1.4), we get

$$D^*\left(f^i x, f^r x, f^r x\right) = D^*\left(f^k f^{i-k}x, f^l f^{r-l}x, f^l f^{r-l}x\right)$$

$$\le q.\max\left\{ D^*\left(f^{r_1+i-k}x, f^{s_1+r-l}x, f^{s_1+r-l}x\right),\right.$$

$$D^*\left(f^{r_1+i-k}x, f^{r_1'+i-k}x, f^{r_1'+i-k}x\right),$$

$$D^*\left(f^{s_1+r-l}x, f^{s_1'+r-l}x, f^{s_1'+r-l}x\right):$$

$$\left. 0 \le r_1, r_1' \le k; 0 \le s_1, s_1' \le l \right\}$$

$$= q.\max\left\{ D^*\left(f^{r_1+i-k}x, f^{s_1+r-l}x, f^{s_1+r-l}x\right),\right.$$

$$D^*\left(f^{r_1+i-k}x, f^{r_1'+i-k}x, f^{r_1'+i-k}x\right),$$

$$D^*\left(f^{s_1+r-l}x, f^{s_1'+r-l}x, f^{s_1'+r-l}x\right):$$

$$0 \le r_1 + i - k \le n_0, k - l \le s_1 + r - l \le n_0,$$

$$\left. 0 \le r_1' + i - k \le n_0, k - l \le s_1' + r - l \le n_0 \right\}$$

$$= q.\max\left\{ D^*\left(f^p x, f^t x, f^t x \right), D^*\left(f^p x, f^{p'} x, f^{p'} x \right), \right.$$

$$\left. D^*\left(f^t x, f^{t'} x, f^{t'} x \right) : k \le p, p', t, t' \le n_0 \right\},$$

omitting the terms of the form $D^*\left(f^i x, f^r x, f^r x \right)$ with $0 \le i \le k$, because of inequality (3.1.2).

Thus

$$D^*\left(f^{n_0} x, f^l x, f^l x \right) \le q.\max\left\{ D^*\left(f^i x, f^r x, f^r x \right) : k < i, r < n_0 \right\}$$

$$\le q^2.\max\left\{ D^*\left(f^i x, f^r x, f^r x \right) : k < i, r < n_0 \right\}$$

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$$\le q^m.\max\left\{ D^*\left(f^i x, f^r x, f^r x \right) : k < i, r < n_0 \right\}$$

for $m = 1, 2, 3, \ldots$ and on letting $m \to \infty$, it follows that

$D^*\left(f^{n_0} x, f^l x, f^l x \right) = 0$, which is a contradiction. Therefore the inequality (4.2.4) holds. However, we now have

$D^*\left(f^{n_0} x, f^l x, f^l x \right) = D^*\left(f^k f^{n_0 - k} x, f^l x, f^l x \right)$

and on using inequality in the definition of the generalized quasi-contraction, we have

$$D^*\left(f^{n_0} x, f^l x, f^l x \right) \le q.\max\left\{ D^*\left(f^{r+n_0-k} x, f^s x, f^s x \right), \right.$$

$$D^*\left(f^{r+n_0-k} x, f^{r'+n_0-k} x, f^{r'+n_0-k} x \right),$$

$$\left. D^*\left(f^s x, f^{s'} x, f^{s'} x \right) : 0 \le r, r' \le k; 0 \le s, s' \le l \right\}$$

$$= q.\max\{\ D^*\left(f^p x, f^s x, f^s x\right),$$

$$D^*\left(f^p x, f^{p'} x, f^{p'} x\right),$$

$$D^*\left(f^s x, f^{s'} x, f^{s'} x\right): n_0 - k \le p, p' \le n_0; 0 \le s, s' \le l\ \}$$

(since, $0 \le r \le k$ implies $n_0 - k \le r + n_0 - k \le n_0$)

Therefore

$$D^*\left(f^{n_0} x, f^l x, f^l x\right) \le q.\max\{\ D^*\left(f^p x, f^s x, f^s x\right),$$

$$D^*\left(f^p x, f^{p'} x, f^{p'} x\right), D^*\left(f^s x, f^{s'} x, f^{s'} x\right):$$

$$n_0 - k \le p, p' \le n_0; 0 \le s, s' \le l\ \}$$

$$\le q.\max\{\ D^*\left(f^r x, f^s x, f^s x\right): 0 \le r, s \le n_0\ \}$$

And this is impossible because of the inequality (3.1.3). Hence we get that the sequence $\{f^n x : n = 1, 2, 3, \ldots\}$ is bounded for any $x \in X$. This implies that $\{D^*\left(f^r x, f^s x, f^s x\right): r \ge 0, s \ge 0\}$ is bounded. Therefore $M = Sup\{D^*\left(f^r x, f^s x, f^s x\right): r, s = 0, 1, 2, 3, \ldots\} < \infty$. Since, $q < 1$, $q^n \to 0$ as $n \to \infty$, so that for every $\varepsilon > 0$, there is a natural number N such that $q^n < \dfrac{\varepsilon}{M}$ for all $n \ge N$. In particular, $q^N.M < \varepsilon$. Let $N_0 = N.\max\{k, l\}$. If $m \ge N_0$, $n \ge N_0$, $D^*\left(f^m x, f^n x, f^n x\right) < q^{N_0}.M < q^N.M < \varepsilon$.

Thus $\{f^n x : n = 1, 2, 3, \ldots\}$ is a Cauchy sequence in the complete D^*-metric space (X, D^*) and hence has a limit, say, u. That is, $u = \lim_{n \to \infty} f^n x$. As f is continuous, this implies that $fu = f\left(\lim_{n \to \infty} f^n x\right) = \lim_{n \to \infty} f^{n+1} x = u$, and so u is a fixed point of f.

To prove the uniqueness of u, let $u' \in X$ be such that $fu' = u'$. From inequality in the generalized quasi-contraction, we get

$$D*(u,u',u') = D*\left(f^k u, f^l u', f^l u'\right)$$

$$\leq q.\max\left\{ D*\left(f^r u, f^s u', f^s u'\right), D*\left(f^r u, f^{r'} u, f^{r'} u\right), D*\left(f^s u', f^{s'} u', f^{s'} u'\right) : 0 \leq r, r' \leq k; 0 \leq s, s' \leq l \right\}$$

$$= q.D*(u,u',u')$$

and since $0 \leq q < 1$, we get that $D*(u,u',u') = 0$ which implies that $u = u'$. Thus the fixed point of f is unique in *X*.

3.2 CONSEQUENCES OF THEOREM 3.1

We present some consequences of Theorem 3.1 in this section.

In the case one of k and l is equal to 1, and f is a generalized quasi-contraction, the condition of continuity of f can be relaxed in Theorem 3.1. We see the same in the following:

3.2.1 Theorem: Suppose f is a selfmap of a complete *D**-metric space (*X*, *D**) such that the inequality

(3.2.2)

$$D*\left(f^k x, fy, fy\right) \leq q.\max\left\{ D*\left(f^r x, f^s y, f^s y\right), D*\left(f^r x, f^{r'} x, f^{r'} x\right), D*\left(f^s y, f^{s'} y, f^{s'} y\right) : 0 \leq r, r' \leq k;\ s = 0,1 \right\}$$

holds for all $x, y \in X$, where $0 \leq q < 1$ and for some fixed integer $k > 0$. Then f has a unique fixed point.

Proof: Let x be an arbitrary point of X. Then as in the proof of Theorem 3.1, the sequence $\left\{D*\left(f^n x, f^l x, f^l x\right) : n = 1, 2, 3, \ldots\right\}$ is a Cauchy sequence in the complete *D**-metric space (*X*, *D**), and hence has a limit $u \in X$. For $n \geq k$, we now have

$$D^*\left(f^n x, fu, fu\right) = D^*\left(f^k f^{n-k} x, fu, fu\right)$$

$$\leq q.\max\left\{ D^*\left(f^{r+n-k} x, f^s u, f^s u\right), D^*\left(f^{r+n-k} x, f^{r'+n-k} x, f^{r'+n-k} x\right), D^*\left(u, fu, fu\right) : 0 \leq r, r' \leq k; s = 0,1 \right\}$$

$$\leq q.\max\left\{ D^*\left(f^p x, f^s u, f^s u\right), D^*\left(f^p x, f^{p'} x, f^{p'} x\right), D^*\left(u, fu, fu\right) : n-k \leq p, p' \leq n; s = 0,1 \right\}$$

and on letting $n \to \infty$, we get that

$$D^*\left(u, fu, fu\right) \leq q.\max\left\{ D^*\left(u, f^s u, f^s u\right), D^*\left(u,u,u\right), D^*\left(u, fu, fu\right) : 0 \leq r, r' \leq k; s = 0,1 \right\}$$

$$= q.\max\left\{ D^*\left(u, fu, fu\right), D^*\left(u, fu, fu\right) \right\}$$

$$= q.D^*\left(u, fu, fu\right)$$

This implies that $D^*\left(u, fu, fu\right) = 0$, since $0 \leq q < 1$ and hence $fu = u$, showing that u is fixed point of f.

To prove the uniqueness of fixed point of f, suppose that $fu = u$ and $fv = v$ for some $u, v \in X$. Then we have

$$D^*(u,v,v) = D^*\left(f^{n}u,\, fv,\, fv\right) = D^*\left(f^{k} f^{n-k}u,\, fv,\, fv\right)$$

$$\leq q.\max\left\{ \; D^*\left(f^{r+n-k}u,\, f^{s}v,\, f^{s}v\right), D^*\left(f^{r+n-k}u,\, f^{r'+n-k}u,\, f^{r'+n-k}u\right),\right.$$

$$\left. D^*(v,\, fv,\, fv) : 0 \leq r, r' \leq k; s = 0,1 \; \right\}$$

$$= q.\max\left\{ \; D^*\left(f^{p}u,\, f^{s}v,\, f^{s}v\right), D^*\left(f^{p}u,\, f^{p'}u,\, f^{p'}u\right),\right.$$

$$\left. D^*(v,\, fv,\, fv) : n-k \leq p, p' \leq n; s = 0,1 \; \right\}$$

On letting $n \to \infty$, we get

$$D^*(u,v,v) \leq q.\max\left\{ \; D^*(u,v,v), D^*(u,u,u), D^*(v,v,v) \; \right\}$$

$$= D^*(u,v,v)$$

This implies that $D^*(u,v,v) = 0$, since $0 \leq q < 1$, and hence $u = v$. Thus f has a unique fixed point.

3.2.3 Corollary: Suppose f is a selfmap of a complete D^*-metric space (X , D^*) such that the inequality

(3.2.4)
$$D^*\left(fx,\, fy,\, fy\right) \leq q.\max\left\{ \; D^*(x,y,y), D^*(x,\, fx,\, fx), D^*(y,\, fy,\, fy),\, D^*(x,\, fy,\, fy), D^*(y,\, fx,\, fx) \; \right\}$$

holds for all $x, y \in X$, where $0 \leq q < 1$. Then f has a unique fixed point.

Proof: Taking k = 1 in the Theorem 4.3.1, we get the corollary.

3.2.5 Remark: For generalized quasi-contraction on complete D^*-metric space with integers k and l both greater than or equal to 2, the condition of continuity of f is necessary to have a fixed point.

The following is an example justifying the above remark.

3.2.5 Example: Let $X=[0,1]$ and $D^*: X^3 \to [0,\infty)$ defined by $D^*(x,y,z)=\max\{|x-y|,|y-z|,|z-x|\}$ for $x,y,z\in X$. Then (*X*, *D**) is a *D**-metric space. Define $f: X\to X$ by

$$fx=\begin{cases}1 & if\ x=0\\ \dfrac{x}{2} & if\ x\neq 0\end{cases}$$

To show that *f* is a generalized quasi-contraction. Consider

$$D^*\left(f^k x, f^l y, f^l y\right)=D^*\left(f^{k-1}fx, f^{l-1}fy, f^{l-1}fy\right)$$

$$=\begin{cases}D^*\left(f^{k-1}(1), f^{l-1}(1), f^{l-1}(1)\right) & if\ x=0=y\\ D^*\left(f^{k-1}(1), f^{l-1}\left(\dfrac{y}{2}\right), f^{l-1}\left(\dfrac{y}{2}\right)\right) & if\ x=0, y\neq 0\\ D^*\left(f^{k-1}\left(\dfrac{x}{2}\right), f^{l-1}(1), f^{l-1}(1)\right) & if\ x\neq 0, y=0\\ D^*\left(f^{k-1}\left(\dfrac{x}{2}\right), f^{l-1}\left(\dfrac{y}{2}\right), f^{l-1}\left(\dfrac{y}{2}\right)\right) & if\ x\neq 0, y\neq 0\end{cases}$$

$$=\begin{cases}D^*\left(\dfrac{1}{2^{k-1}},\dfrac{1}{2^{l-1}},\dfrac{1}{2^{l-1}}\right) & if\ x=0=y\\ D^*\left(\dfrac{1}{2^{k-1}},\dfrac{y}{2^{l}},\dfrac{y}{2^{l}}\right) & if\ x=0, y\neq 0\\ D^*\left(\dfrac{x}{2^{k}},\dfrac{1}{2^{l-1}},\dfrac{1}{2^{l-1}}\right) & if\ x\neq 0, y=0\\ D^*\left(\dfrac{x}{2^{k}},\dfrac{y}{2^{l}},\dfrac{y}{2^{l}}\right) & if\ x\neq 0, y\neq 0\end{cases}$$

$$= \begin{cases} \left| \frac{1}{2^{k-1}} - \frac{1}{2^{l-1}} \right| & if \;\; x = 0 = y \\ \left| \frac{1}{2^{k-1}} - \frac{y}{2^{l}} \right| & if \;\; x = 0, y \neq 0 \\ \left| \frac{x}{2^{k}} - \frac{1}{2^{l-1}} \right| & if \;\; x \neq 0, y = 0 \\ \left| \frac{x}{2^{k}} - \frac{y}{2^{l}} \right| & if \;\; x \neq 0, y \neq 0 \end{cases}$$

Also

$$D^*\left(f^{k-1}x, f^{l-1}y, f^{l-1}y \right) = \begin{cases} \left| \frac{1}{2^{k-2}} - \frac{1}{2^{l-2}} \right| & if \;\; x = 0 = y \\ \left| \frac{1}{2^{k-2}} - \frac{y}{2^{l-1}} \right| & if \;\; x = 0, y \neq 0 \\ \left| \frac{x}{2^{k-1}} - \frac{1}{2^{l-2}} \right| & if \;\; x \neq 0, y = 0 \\ \left| \frac{x}{2^{k-1}} - \frac{y}{2^{l-1}} \right| & if \;\; x \neq 0, y \neq 0 \end{cases}$$

And hence

$$D^*\left(f^{k-1}x, f^{l-1}y, f^{l-1}y \right) = 2. \begin{cases} \left| \frac{1}{2^{k-1}} - \frac{1}{2^{l-1}} \right| & if \;\; x = 0 = y \\ \left| \frac{1}{2^{k-1}} - \frac{y}{2^{l}} \right| & if \;\; x = 0, y \neq 0 \\ \left| \frac{x}{2^{k}} - \frac{1}{2^{l-1}} \right| & if \;\; x \neq 0, y = 0 \\ \left| \frac{x}{2^{k}} - \frac{y}{2^{l}} \right| & if \;\; x \neq 0, y \neq 0 \end{cases}$$

That is,

$D^*\left(f^{k-1}x, f^{l-1}y, f^{l-1}y \right) = 2.D^*\left(f^{k}x, f^{l}y, f^{l}y \right)$ for all $x, y \in X$, which implies that

$$D^*\left(f^{k}x, f^{l}y, f^{l}y \right) \leq \frac{1}{2}.D^*\left(f^{k-1}x, f^{l-1}y, f^{l-1}y \right) \text{ for all } x, y \in X .$$

Thus f is a contraction on the D^*-metric space (X, D^*) and hence a generalized quasi-contraction. But f has no fixed point in X.

REFERENCES

[1]. B. C. Dhage, "Generalized metric spaces and mappings with fixed point", Bulletin of Calcutta Math. Soc. Vol-84, No. 4, pp 329-336(1992)

[2]. B. Ahmad, M. Ashraf and B.E. Rhoades, "Fixed point theorems for expansive mappings in D-metric spaces", Indian J. Pure and Applied Mathematics Vol-30, No.10, pp.1513-1518 (2001).

[3]. B. C. Dhage, "Common fixed point principle in D-metric spaces", Bulletin of Calcutta Math. Soc. Vol-91, No. 6, pp 475-480 (1999)

[4]. B. C. Dhage, A. M. Pathan and B. E. Rhoades, "A general extension principle for fixed point theorems in D-metric spaces", Int. J. Math. & Math. Sci., Vol.23, pp.441-448 (2000)

[5]. B.E. Rhoades, "A fixed point theorem for generalized metric spaces", Int. J. Math. Sci. 1 (2005).

[6] B. Singh and R.K. Sharma, "Common fixed points via compatible maps in D-metric space", Radovi Matemeticki 11 (2002); pp 145-153.

[7] Shaban Sedghi, Nabi Shobe and Haiyun Zhou, "A common fixed point theorem in D*-metric spaces", Journal of fixed point theory and applications, Haindavi Publishing corporation,

[8]. B. Fisher, "Quasi-contractions on metric spaces", Proc. Amer. Math. Soc. 75 (1979), pp 51-54