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The Generalized Hyers-Ulam-Rassias Stability of Cubic Functional Equations generalized by an Automorphism on Groups

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Abstract

We consider the generalized Hyers-Ulam-Rassias stability of cubic functional equations that are generalized with an automorphism on abelian groups.

Keywords and phrases: Hyers-Ulam-Rassias Stability, Cubic functional equations, Automorphism on groups.

1. INTRODUCTION

In a talk before the Mathematics Club of the University of Wisconsin in 1940, a Polish-American mathematician, S. M. Ulam [**19**] proposed the stability problem of the linear functional equation $f(x+y) = f(x)+f(y)$ where any solution $f(x)$ of this equation is called a linear function.

To make the statement of the problem precise, *let G_1 be a group and G_2 a metric group with the metric $d(\cdot,\cdot)$. Then given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G_1 \longrightarrow G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $F : G_1 \longrightarrow G_2$ with $d(f(x), F(x)) < \epsilon$ for all $x \in G_1$?*. In other words, the question would be generalized as "Under what conditions a mathematical object satisfying a certain property approximately must be close to an object satisfying the property exactly?".

In 1941, the first, affirmative, and partial solution to Ulam's question was provided by D. H. Hyers [**9**]. In his celebrated theorem Hyers explicitly constructed the linear function (or additive function) in Banach spaces directly from a given approximate function satisfying the well-known weak Hyers inequality with a positive constant. The Hyers stability result was first generalized in the stability involving p-powers of norm by Aoki [**1**]. In 1978 Th. M. Rassias [**14**] provided a generalization of Hyers' theorem that allows the Cauchy difference to become unbounded. For the last decades, stability problems of various functional equations, not only linear case, have been extensively investigated and generalized by many mathematicians (see [**3, 6, 8, 10, 15, 16, 17**]).

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called a quadratic functional equation and every solution of this functional equation is said to be a quadratic function or mapping (e.g. $f(x) = cx^2$).

The HyersUlam stability problem for the quadratic functional equation was first studied by Skof [**18**] in a normed space as the domain of a mapping of the equation. Cholewa [**5**] noticed that the results of Skof still hold in abelian groups.

In [**6**] Czerwik obtained the Hyers-Ulam-Rassias stability of the quadratic functional equation. See [**2, 13, 20**] for more results on the equation (1.1).

Jun and Kim [**10**] considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.2}$$

since it should be easy to see that a function $f(x) = cx^3$ is a solution of the equation (1.2) as the quadratic equation case. In a year they [**11**] proved the generalized Hyers-Ulam-Rassias stability of a different version of a cubic functional equation

$$f(x+2y)+f(x-2y)+6f(x)=f(x+y)+4f(x-y). \tag{1.3}$$

Since then the stability of cubic functional equations has been investigated by a number of authors (see [**4, 12**] for details).

As we notice there are different definitions for the stability of the cubic functional equations and we shall use the following definition of a cubic functional equation

$$f(x^2y)+f(x^2y^{-1})=2f(xy)+2f(xy^{-1})+12f(x) \tag{1.4}$$

for all x and y in a group G and investigate the generalized Hyers-Ulam-Rassias stability problem of the above equation (1.4) in a group structure. In order to give our results in Section 4 it is convenient to state the definition of a generalized metric on a set X and a result on a fixed point theorem of the alternative by Diaz and

Margolis [**7**].

Let X be a set. A function $d : X \times X \longrightarrow [0,\infty)$ is called a generalized metric on

X if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(3) $d(x,z) \le d(x,y) + d(y,z)$ for all $x,y,z \in X$.

Theorem 1.1. *Let* (X,d) *be a complete generalized metric space and let* $J : X \longrightarrow X$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$. *Then for each element* $x \in X$, *either* $d(J^n x, J^{n+1}x) = \infty$ *for all nonnegative integers n or there exists a positive* n_0 *such that*

(1) $d(J^n x, J^{n+1}x) < \infty$ *for all* $n \ge n_0$*;*

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of J;*

(3) y^* *is the unique fixed point of J in the set* $Y = \{y \in X | d(J^{n_0}x, y) < \infty\}$*;*

(4) $d(y,y^*) \le (1/(1-L))d(y,Jy)$ *for all* $y \in Y$.

In this paper, using the fixed point method we prove the generalized Hyers-Ulam-Rassias stability of a cubic functional equation that is generalized with an automorphism σ on a group G in an abelian group as follows;

$$f(x^2y) + f(x^2\sigma(y)) = 2f(xy) + 2f(x\sigma(y)) + 12f(x) \tag{1.5}$$

for all $x,y \in G$ where G is an abelian group with the identity e and σ is an automorphism on G with $\sigma \circ \sigma(x) = x$ for all $x \in G$. In Section 2 we establish the general solution of the cubic functional equation (1.4) corresponding to the symmetric biadditive function B for the quadratic functional equation (1.1) in an abelian group G. In Section 3 we prove the generalized Hyers-Ulam-Rassias stability of the cubic functional equation (1.4) in an abelian group G. Finally, we obtain, in Section 4, the Hyers-Ulam-Rassias stability of the generalized cubic functional equation (1.5).

2. THE GENERAL SOLUTION

In this section we show the general solution of the cubic functional equation (1.4) corresponding to the symmetric biadditve function B for the quadratic functional equation (1.1).

Theorem 2.1. *Let G be an abelian group with the identity e and Y a real vector space. Then a function $f : G \longrightarrow Y$ with $f(e) = 0$ satisfies the cubic functional equation (1.4) if and only if there is a function $T : G \times G \times G \longrightarrow Y$ such that $f(x) = T(x,x,x)$ for all $x \in G$, and T is symmetric for each variable and is additive for fixed two variables.*

Proof. Taking $x = y = e$ in the equation (1.4) it's not hard to have $f(e) = 0$. Substituting the identity e for x in (1.4) also gives $f(y)+f(y^{-1}) = 0$. Letting $y = e$ and $y = x$ in the equation (1.4) we have $f(x^2) = 8f(x)$ and $f(x^3) = 27f(x)$ for all $x \in G$, respectively. This observation leads us to $f(x^n) = n^3f(x)$ for all positive n by the induction argument for n. Using the commutative property of an abelian group G and replaying x and y with xy and xy^{-1}, respectively, in (1.4) we obtain the following equation for all $x,y \in G$

$$f(x^3y) + f(xy^3) = 12f(xy) + 16f(x) + 16f(y). \tag{2.1}$$

Similarly, plugging xy and y^2 for x and y, respectively, into (1.4) we obtain (2.2) $8f(xy^2) + 8f(x) = 12f(xy) + 2f(xy^3) + 2f(xy^{-1})$

for all $x,y \in G$. Then we interchanging x and y in the above equation (2.2) to get the relation

$$8f(yx^2) + 8f(y) = 12f(xy) + 2f(yx^3) - 2f(xy^{-1}). \tag{2.3}$$

Adding two equations (2.2) and (2.3) and applying the equation (2.1), we have

$$f(xy^2) + f(x^2y) = 6f(xy) + 3f(x) + 3f(y)$$

(2.4)

$$\text{or} \quad 3f(x) + 3f(y) = f(xy^2) + f(x^2y) - 6f(xy)$$

for all $x,y \in G$. It follows from (1.4) that

$$3f(x^2z) + 3f(x^2z^{-1}) + 3f(y^2z) + 3f(y^2z^{-1})$$

(2.5)

$$= 36f(x) + 36f(y) + 6f(xz) + 6f(xz^{-1}) + 6f(yz) + 6f(yz^{-1})$$

On the other hand, if we use, first, the second equation in (2.4) and then the cubic functional equation (1.4) again we have

$$\begin{aligned}
&[3f(x^2z) + 3f(y^2z)] + [3f(x^2z^{-1}) + 3f(y^2z^{-1})]\\
&=[f(x^4y^2z^3) + f(x^2y^4z^3) - 6f(x^2y^2z^2)]\\
&\quad + [f(x^4y^2z^{-3}) + f(x^2y^4z^{-3}) - 6f(x^2y^2z^{-2})]\\
&=[f((x^2y)^2z^3) + f((x^2y)^2z^{-3})] - 6f(x^2y^2z^2)\\
&\quad + [f((xy^2)^2z^3) + f((xy^2)^2z^{-3})] - 6f(x^2y^2z^{-2})\\
&=2f(x^2yz^3) + 2f(x^2yz^{-3}) + 12f(x^2y) - 48f(xyz)
\end{aligned} \tag{2.6}$$

$$+ 2f(xy^2z^3) + 2f(xy^2z^{-3}) + 12f(xy^2) - 48f(xyz^{-1})$$

Combining two equations (2.5) and (2.6) gives the relation

$$2f(x^2yz^3) + 2f(x^2yz^{-3}) + 12f(x^2y)$$
$$+ 2f(xy^2z^3) + 2f(xy^2z^{-3}) + 12f(xy^2)$$

(2.7)

$$=36f(x) + 36f(y) + 6f(xz) + 6f(xz^{-1}) + 6f(yz) + 6f(yz^{-1}) + 48f(xyz) + 48f(xyz^{-1}).$$

Additionally, if we express the left hand side of (2.5) by using the equations (2.4) and (1.4) then we should have

$$3f(x^2z) + 3f(y^2z^{-1}) + 3f(x^2z^{-1}) + 3f(y^2z)$$
$$=f(x^4y^2z) + f(x^2y^4z^{-1}) - 6f(x^2y^2)$$
(2.8) $$+ f(x^4y^2z^{-1}) + f(x^2y^4z) - 6f(x^2y^2)$$
$$=2f(x^2yz) + 2f(x^2yz^{-1}) + 12f(x^2y)$$
$$+ 2f(xy^2z) + 2f(xy^2z^{-1}) + 12f(xy^2) - 96f(xy)$$

where we applied the property of $f(x^n) = n^3f(x)$ for all positive n in the last step. Replacing z with z^3 in (2.8) and then using (2.7), we will get

$$3f(x^2z^3) + 3f(y^2z^{-3}) + 3f(x^2z^{-3}) + 3f(y^2z^3)$$
$$=2f(x^2yz^3) + 2f(x^2yz^{-3}) + 12f(x^2y)$$
(2.9) $$+ 2f(xy^2z^3) + 2f(xy^2z^{-3}) + 12f(xy^2) - 96f(xy) =36f(x) + 36f(y) + 6f(xz) + 6f(xz^{-1}) + 6f(yz) + 6f(yz^{-1}) + 48f(xyz) + 48f(xyz^{-1}) - 96f(xy).$$

Also, we use (2.4) and then (1.4) for the left hand side of (2.9) to find

$$3f(x^2z^3) + 3f(x^2z^{-3}) + 3f(y^2z^{-3}) + 3f(y^2z^3)$$
$$=f(x^6z^3) + f(x^6z^{-3}) - 6f(x^4) + f(y^6z^3) + f(y^6z^{-3}) - 6f(y^4)$$
(2.10) $$=27f(x^2z) + 27f(x^2z^{-1}) - 384f(x) + 27f(y^2z) + 27f(y^2z^{-1}) - 384f(y)$$
$$=27[12f(x) + 2f(xz) + 2f(xz^{-1}) + 12f(y) + 2f(yz) + 2f(yz^{-1})] - 384f(x) - 384f(y).$$

Therefore, we just have from (2.9) and (2.10) that

$$f(xyz) + f(xyz^{-1}) + 2f(x) + 2f(y)$$

(2.11)

$$=2f(xy) + 2f(xz) + f(xz^{-1}) + f(yz) + f(yz^{-1})$$

for all $x,y \in G$.

Now we define $T : G \times G \times G \longrightarrow Y$ by

$$T(x,y,z) = \frac{1}{24}[f(xyz) + f(xy^{-1}z^{-1}) - f(xyz^{-1}) - f(xy^{-1}z)] \tag{2.12}$$

for all $x,y,z \in G$. Then it follows that T is symmetric for each fixed variable since $f(x^{-1}) = -f(x)$ for all $x \in G$ and $T(x,x,x) = f(x)$ for all $x \in G$. In order to show that T is additive for two fixed variables we need to verify that for all $u,v,y,z \in G$

$$T(uv,y,z) = T(u,y,z) + T(v,y,z). \tag{2.13}$$

By the definition of T and the result from the equation (2.11) we are able to show that

$$\begin{aligned} &24T(uv,y,z)\\ =&f(uvyz) + f(uvy^{-1}z^{-1}) - f(uvyz^{-1}) - f(uvy^{-1}z)\\ =&2f(uv) + f(uyz) + f(uy^{-1}z^{-1}) + f(vyz) + f(vy^{-1}z^{-1})\\ &- 2f(u) - 2f(v)\\ &- 2f(uv) - f(uyz^{-1}) - f(uy^{-1}z) - f(vyz^{-1}) - f(vy^{-1}z)\\ &+ 2f(u) + 2f(v)\\ =&24T(u,y,z) + 24T(v,y,z). \end{aligned} \tag{2.14}$$

Conversely, assuming there is a function $T : G \times G \times G \longrightarrow Y$ such that $T(x,x,x) = f(x)$ for all $x \in G$, and T is symmetric for one variable and additive for two variables, it is easy to show that f is a cubic function satisfying the equation

(1.4).

3. GENERALIZED HYERS-ULAM-RASSIAS STABILITY

In this section we shall study the generalized Hyers-Ulam-Rassias stability for the cubic functional equation (1.4) on a group structure. Let R+ be the set of all nonnegative real numbers, G an abelian group with the identity e, and Y a Banach space with the norm $\|\cdot\|$ unless otherwise stated.

Theorem 3.1. *Let $\varphi : G \times G \longrightarrow$ R+ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\phi(x^{2^k}, e)}{8^k}$$

converges and

$$\lim_{n\to\infty} \frac{\phi(x^{2^n}, y^{2^n})}{8^n} = 0$$

for all x,y ∈ G. Assume that a function f : G −→ Y satisfies

(3.1) $\|f(x^2y) + f(x^2y^{-1}) - 2f(xy) - 2f(xy^{-1}) - 12f(x)\| \leq \varphi(x,y)$

for all x,y ∈ G. Then there is a unique cubic function C : G −→ Y satisfying both the equation (1.4) and the following inequality

(3.2) $$\|f(x) - C(x)\| \leq \frac{1}{16}\sum_{k=0}^{\infty}\frac{\phi(x^{2^k}, e)}{8^k}$$

for all x ∈ G. Moreover, the function C is given by

(3.3) $$C(x) = \lim_{n\to\infty}\frac{f(x^{2^n})}{8^n}$$

for all x ∈ G.

Proof. Starting with $y = e$ in (3.1) and dividing the resulting equation by 16 we have

(3.4) $$\left\|\frac{f(x^2)}{8} - f(x)\right\| \leq \frac{1}{16}\phi(x, e)$$

for all $x \in G$. Substituting x^2 for x in (3.4) and dividing by 8 gives

$$\left\|\frac{f(x^4)}{8^2} - f(x^2)\right\| \leq \frac{1}{16}\frac{\phi(x^2, e)}{8}$$

and we use the triangle inequality to obtain

(3.5) $$\left\|\frac{f(x^{2^2})}{8^2} - f(x)\right\| \leq \frac{1}{16}\left(\phi(x, e) + \frac{\phi(x^2, e)}{8}\right)$$

for all $x \in G$. Using the mathematical induction on n we conclude that

(3.6) $$\left\|\frac{f(x^{2^n})}{8^n} - f(x)\right\| \leq \frac{1}{16}\sum_{k=0}^{\infty}\frac{\phi(x^{2^k}, e)}{8^k}$$

for all $x \in G$. To show that the sequence $\{f(x^{2n})/8^n\}$ is a Cauchy sequence we replace x with x^{2m} and divide the inequality (3.6) by 8^m to have for $n,m > 0$,

$$\left\|\frac{f(x^{2^m 2^n})}{8^{n+m}} - \frac{f(x^{2^m})}{8^m}\right\| = \frac{1}{8^m}\left\|\frac{f(x^{2^m 2^n})}{8^n} - f(x^{2^m})\right\|$$

(3.7) $$\leq \frac{1}{16}\sum_{i=0}^{\infty}\frac{\phi(x^{2^i 2^m}, 0)}{8^{m+i}} < \infty.$$

Letting m tend to the infinity the limit of the left hand side of the above inequality goes to 0, which means that the sequence $\{f(x^{2n})/8^n\}$ is a Cauchy sequence. Hence we define $C(x)$ as the limit of the sequence $\{f(x^{2n})/8^n\}$, i.e,. $C(x) := \lim_{n\to\infty} f(x^{2n})/8^n$ for all $x \in G$. Taking the limit as n tends to the infinity in (3.6) shows the inequality (3.2) as desired.

In order to prove that C satisfies the equation (1.4) substitute x^{2n} and y^{2n} for x and y, respectively, in (3.1) and divide by 8^n. Then we have

$$\frac{1}{8^n}\|f((x^2y)^{2^n}) + f((x^2y^{-1})^{2^n}) - 2f((xy)^{2^n}) - 2f((xy^{-1})^{2^n}) - 12f(x^{2^n})\| \le \frac{\phi(x^{2^n}, y^{2^n})}{8^n}.$$

Taking the limit as $n \to \infty$ we show that C is a cubic function for all $x,y \in G$.

Lastly, we need to prove the uniqueness of the cubic function C involving the function f. Let us assume that there is a cubic function $D : G \longrightarrow Y$ satisfying (1.4) and the inequality (3.2). Then we easily have that $D(x^{2n}) = (2^n)^3D(x)$, not only $C(x^{2n}) = (2^n)^3C(x)$ since they are cubic functions. Therefore it follows from

(3.2) that

$$\begin{aligned}\|C(x) - D(x)\| &= \frac{1}{8^n}\|C(x^{2^n}) - D(x^{2^n})\| \\ &\le \frac{1}{8^n}\|C(x^{2^n}) - f(x^{2^n})\| + \frac{1}{8^n}\|f(x^{2^n}) - D(x^{2^n})\| \\ &\le \frac{1}{8^{n+1}}\sum_{k=0}^{\infty}\frac{\phi(x^{2^n2^k}, e)}{8^k}\end{aligned}$$

for all $x,y \in G$. Taking the limit of the above inequality as $n \to \infty$ we show the uniqueness of C and it completes the proof of the theorem.

4. GENERAL HYERS-ULAM-RASSIAS STABILITY ON GROUPS WITH AN AUTOMORPHISM: A FIXED POINT THEOREM OF THE ALTERNATIVE APPROACH

In this section we will investigate the Hyers-Ulam-Rassias stability of a cubic functional equation which is generalized with an automorphism σ on a group G as follows;

$$f(x^2y) + f(x^2\sigma(y)) = 2f(xy) + 2f(x\sigma(y)) + 12f(x)$$

for all $x,y \in G$ where G is an abelian group with the identity e and σ is an automorphism on G. Furthermore, we assume that $\sigma \circ \sigma(x) = x$ for all $x \in G$.

Theorem 4.1. *Let G be an abelian group with an identity e and B a Banach space with a norm* $\|\cdot\|$. *Suppose that a function* $\varphi : G \times G \longrightarrow \mathbb{R}+$ *or* $([0,\infty))$ *is given and there exists a constant L with* $0 < L < 1$ *such that* (4.1) $\varphi(x^2,y^2) \le 2L\varphi(x,y)$ *and* $\varphi(x\sigma(x),y\sigma(y)) \le 2L\varphi(x,y)$ *for all* $x,y \in G$. *Furthermore, let* $f : G \longrightarrow B$ *be a mapping such that* $f(e) = 0$ *and*

(4.2) $$\left\|f(x^2y) + f(x^2\sigma(y)) - 2\left[f(xy) + f(x\sigma(y))\right] - 12f(x)\right\| \leq \phi(x,y)$$

for all $x,y \in G$ where σ is an automorphism on G with $\sigma^2 = I$, i.e., $\sigma \circ \sigma(x) = x$ for all $x \in G$.

Then there exists a unique cubic function $C : G \longrightarrow B$ such that

(4.3) $$\|f(x) - C(x)\| \leq \left(\frac{1+2L}{2^4(1-L)}\right)\Phi(x)$$

for all $x \in G$ where $\Phi(x) = max\{\varphi(x,e),\varphi(e,x)\}$ for all $x \in G$.

Proof. First, we consider the set $\Omega = \{g : g : G \longrightarrow B, g(e) = 0\}$ and introduce a generalized metric d on Ω as follows:

$$d(g,h) = \inf\{\lambda \in [0,\infty] \mid \|g(x) - h(x)\| \leq \lambda\Phi(x) \text{for all} x \in G\}.$$

Let $\{f_n\}$ be a Cauchy sequence in (Ω,d). By the definition of the generalized metric d, we have that for any > 0, there exists a positive integer $N \in$ N such that

$$\|f_m(x) - f_n(x)\| \leq \epsilon\phi(x,x)$$

for all $x \in G$ and all $m,n \geq N$. Then for each point $x \in G$, the above inequality $\|f_m(x) - f_n(x)\| \leq \epsilon\phi(x,x)$) implies that $\{f_n(x)\}$ is a Cauchy sequence in the Banach space B. Hence $\{f_n(x)\}$ converges in B for each $x \in G$. We will now define a Cauchy sequence in (Ω,d).

Letting $y = e$ and $x = e$ in the inequality (4.2), respectively, we see that

(4.4) $$\|f(x^2) - 2^3f(x)\| \leq \frac{1}{2}\phi(x,e)$$

and

(4.5) $$\|f(y) + f(\sigma(y))\| \leq \varphi(e,y)$$

for all $x,y \in G$. Replacing y by $x\sigma(x)$ in the inequality (4.5), we obtain

(4.6) $$\|f(x\sigma(x))\| \leq \frac{1}{2}\phi(e, x\sigma(x))$$

for all $x \in G$. The inequalities (4.4) and (4.6) imply that

(4.7) $$\left\|\frac{1}{2^3}[f(x^2) + f(x\sigma(x))] - f(x)\right\| \leq \frac{1}{2^4}(1+2L)\Phi(x)$$

for all $x \in G$. Now we define

$$T^n(f)(x) = \frac{1}{2^{3n}}[f(x^{2^n}) + (2^n - 1)f((x\sigma(x))^{2^{n-1}})]$$

for all $x \in G$ and all $n \in$ N. We may let $T^0(f)(x) = f(x)$. The inequality (4.7) means that

$$\|T^1(f)(x) - T^0(f)(x)\| \le \frac{1}{2^4}(1+2L)\Phi(x)$$

for all $x \in G$. By using the mathematical induction, we will see that

$$\|T^n(f)(x) - T^{n-1}(f)(x)\| \le \frac{1+2L}{2^4}\left(\frac{L}{2}\right)^{n-1}\Phi(x) \tag{4.8}$$

for all $x \in G$ and $n \in \mathbb{N}$. We note that

$$\|T^n(f)(x) - T^m(f)(x)\| \le \frac{1+2L}{2^4}\sum_{j=m}^{n-1}\left(\frac{L}{2}\right)^{j}\Phi(x) \tag{4.9}$$

for all $x \in G$ and $n > m \in \mathbb{N}$.

Now we define a mapping $T : \Omega \longrightarrow \Omega$ by

$$T(g)(x) = \frac{1}{2^3}[g(x^2) + g(x\sigma(x))] \tag{4.10}$$

for all $x \in G$. We claim that T is strictly contractive on Ω.

Given $g, h \in \Omega$, let $\lambda \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le \lambda$. Then $\|g(x) - h(x)\| \le \lambda\Phi(x)$ for all $x \in G$. By the equation (4.1) we have

$$\begin{aligned}
&\|T(g)(x) - T(h)(x)\| = \frac{1}{2^3}\|g(x^2) - h(x^2) + g(x\sigma(x)) - h(x\sigma(x))\| \\
&\le \frac{1}{2^3}\|g(x^2) - h(x^2)\| + \frac{1}{2^3}\|g(x\sigma(x)) - h(x\sigma(x))\| \\
&\le \frac{\lambda}{2^3}\Phi(x^2) + \frac{\lambda}{2^3}\Phi(x\sigma(x)) \le \frac{1}{2}L\lambda \le L\lambda
\end{aligned}$$

for all $x \in G$. That is $d(T(g), T(h)) \le L\lambda$.

Hence we may conclude that

$d(T(g), T(h)) \le Ld(g,h)$

for any $g, h \in \Omega$. Since L is a constant with $0 < L < 1$, T is strictly contractive as claimed.

Also the inequality (4.7) implies that

$$d(T(f), f) \le \frac{1}{2^4}(1+2L) < \infty. \tag{4.11}$$

By the alternative of fixed point, there exists a mapping $C : G \longrightarrow B$ which is a fixed point of T such that $d(T^n(f), C) \to 0$ as $n \to \infty$. That is,

$$C(x) = \lim_{n\to\infty} T^n(f)(x)$$

for all $x \in G$. We claim that C is cubic.

$$
\begin{aligned}
&\|C(x^2y) + C(x^2\sigma(y)) - 2[C(xy) + C(x\sigma(y))] - 12C(x)\| \\
&\le \lim_{n\to\infty} \frac{1}{2^{3n}} \|f((x^2y)^{2^n}) + f((x^2\sigma(y))^{2^n}) - 2[f((xy)^{2^n}) + f((x\sigma(y))^{2^n})] - 12f(x^{2^n})\| \\
&+ \lim_{n\to\infty} \frac{2^n-1}{2^{3n}} \|f((x^2y\sigma(x^2y))^{2^{n-1}}) + f((x^2\sigma(y)\sigma(x^2\sigma(y)))^{2^{n-1}}) \\
&- 2[f((xy\sigma(xy))^{2^{n-1}}) + f((x\sigma(y)\sigma(x\sigma(y)))^{2^{n-1}})] - 12f((x\sigma(x))^{2^{n-1}})\| \\
&\le \lim_{n\to\infty} \frac{1}{2^{3n}} \phi(x^{2^n}, y^{2^n}) + \lim_{n\to\infty} \frac{2^n-1}{2^{3n}} \phi((x\sigma(x))^{2^{n-1}}, (y\sigma(y))^{2^{n-1}}) \\
&\le \lim_{n\to\infty} \frac{2^nL^n}{2^{3n}} \phi(x,y) + \lim_{n\to\infty} \frac{2^n-1}{2^{3n}} 2^nL^n\phi(x,y) \\
&= \lim_{n\to\infty} \left(\frac{L}{2}\right)^n \phi(x,y) = 0
\end{aligned}
$$

for all $x,y \in G$. This implies that C is cubic.

By the alternative of fixed point and the inequality (4.11) we get

$$d(f,C) \le \frac{1}{1-L} d(f,T(f)) \le \frac{1}{1-L} \frac{1+2L}{2^4}.$$

Hence the inequality (4.3) is true for all $x \in G$.

Assume that $\widetilde{C} : G \longrightarrow B$ is another cubic satisfying (4.3). By the definition of d and the inequality (4.3) we will see that

$$d(f,\widetilde{C}) \le \frac{1}{1-L} d(f,T(f)) \le \frac{1+2L}{2^4(1-L)}.$$

By the uniqueness of the fixed point of T, we may conclude that $C = \widetilde{C}$ as desired.

Remark 4.1. As you already notice the cubic functional equation with the automorphism σ on G (1.5) in this section is the generalization of the cubic functional equation (1.4). Hence we're able to obtain the Hyers-Ulam-Rassias stability of the cubic functional equation (1.4) from the perspective of the automorphism-cubic functional equation (1.5).

In Theorem 4.1 if we let $\sigma(x) = x^{-1}$ then we have the exactly same result in Theorem 3.1. Precisely, let G be an abelian group with an identity e and B a Banach space with a norm $\|\cdot\|$. Suppose that a function $\varphi : G\times G \longrightarrow [0,\infty)$ is given and there exists a constant L with $0 < L < 1$ such that

(4.12) $\qquad \varphi(e,e) \le 2L\varphi(x,y) \quad \text{and} \quad \varphi(x^2,y^2) \le 2L\varphi(x,y)$

for all $x,y \in G$. Furthermore, let $f : G \longrightarrow B$ be a mapping such that $f(e) = 0$ and

(4.13) $\qquad \left\|f(x^2y) + f(x^2y^{-1}) - 2\left[f(xy) + f(xy^{-1})\right] - 12f(x)\right\| \le \phi(x,y)$

for all $x,y \in G$. Then there is a unique cubic function $C : G \longrightarrow Y$ satisfying both the cubic functional equation (1.4) and the following inequality

$$\|f(x) - C(x)\| \le \frac{1}{16}\sum_{k=0}^{\infty}\frac{\phi(x^{2^k}, e)}{8^k} \tag{4.14}$$

for all $x \in G$. Then function C is given by

$$C(x) = \lim_{n\to\infty}\frac{f(x^{2^n})}{8^n} \tag{4.15}$$

for all $x \in G$.

Because we can see that $\varphi(x^2,e) \le 2L\varphi(x,e)$ implies, by the induction,

and so the $\sum_{k=0}^{\infty}\frac{\phi(x^{2^k}, e)}{8^k} \le \sum_{k=0}^{\infty}\frac{(2L)^k}{8^k}\phi(x,e) < \infty$ series converges and similarly $\varphi(x^2,y^2) \le 2L\varphi(x,y)$ gives us

$$\sum_{k=0}^{\infty}\frac{\phi(x^{2^k}, e)}{8^k} \quad \frac{\phi(x^{2^n}, y^{2^n})}{8^n} \le \frac{(2L)^n\phi(x,y)}{8^n}$$

for all $x,y \in G$ and all positive integer n and therefore we have

$$\lim_{n\to\infty}\frac{\phi(x^{2^n}, y^{2^n})}{8^n} = 0.$$

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