Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 6941–6962

http://www.ripublication.com/gjpam.htm

Common Coupled Fixed Point Results for two Hybrid Pairs of Mappings Under $\psi - \varphi$ Contraction on Modular Metric Spaces

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Abstract

We establish some common coupled fixed point theorems for two hybrid pairs of mappings under $\varphi - \psi$ contraction on modular metric spaces. An example is also given to validate our results. Our results generalize and extend various comparable results in the existing literature.

AMS subject classification: 47H10, 54H25.
Keywords: Modular metric spaces, Δ_2-condition, coupled fixed point, coupled coincidence point, $\varphi - \psi$ contraction, (EA) property, common (EA) property, occasionally w-compatibility.

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1. Introduction and Preliminaries

The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by Nakano in 1950 [27]. Further and the most complete development of these theories are due to Musielak, Orlicz, [24, 25] and their collaborators. By now the theory of modular linear (or close to linear) spaces is well known and well developed including several generalizations and a number of textbooks is devoted to the theory and applications. However, for certain problems from set-valued analysis, such as the definition of metric functional spaces and description of the action of multivalued superposition operators, the notion of a modular on a set X with an additional algebraic structure is too limited, and "linear" modular theory fails. In order to overcome this insufficiency, Chistyakov [13] introduced the notion of modular metric spaces generated by F-modular. In [10] Chistyakov define the notion of a modular on an arbitrary set, develop the theory of metric spaces generated by modulars, called modular metric spaces. A lot of mathematicians are interested fixed points of modular metric spaces, for example ([8] , [9] , [12] , [23]).

A brief recollection of basic concepts and facts in modular metric spaces; Let X be a nonempty set. Throughout this paper for a function $\omega : (0, \infty) \times X \times X \to (0, \infty)$, we will write

$$\omega_\lambda (x, y) = \omega (\lambda, x, y) ,$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 1.1. [10] A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a metric modular on X if it satisfies the following axioms:

(i) given $x, y \in X$, $x = y$ if and only if $\omega_\lambda (x, y) = 0$, for all $\lambda > 0$;

(ii) $\omega_\lambda (x, y) = \omega_\lambda (y, x)$, for all $\dot{\lambda} > 0$, and $x, y \in X$;

(iii) $\omega_{\lambda+\mu} (x, y) \leq \omega_\lambda (x, z) + \omega_\lambda (z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If, instead of (i), we have only the condition (i') $\omega_\lambda (x, x) = 0$ for all $\lambda > 0$ and $x \in X$, then ω is said to be a (metric) pseudomodular on X.

Further, ω is said to be convex if for $\lambda, \mu > 0$ and $x, y, z \in X$, it satisfies the inequality

$$\omega_{\lambda+\mu} (x, y) \leq \frac{\lambda}{\lambda + \mu} \omega_\lambda (x, z) + \frac{\mu}{\lambda + \mu} \omega_\mu (z, y) .$$

Note that for a metric pseudomodular ω on a set X, and any $x, y \in X$, the function $\lambda \to \omega_\lambda (x, y)$ is nonincreasing on $(0, \infty)$. Indeed, if $0 < \mu < \lambda$, then

$$\omega_\lambda (x, y) \leq \omega_{\lambda-\mu} (x, x) + \omega_\mu (x, y) = \omega_\mu (x, y) .$$

Definition 1.2. [10, 11] Let ω be a pseudomodular on X. Fix $x_0 \in X$. The two sets

$$X_\omega = X_\omega (x_0) = \{x \in X : \omega_\lambda (x, x_0) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X_\omega^* = X_\omega^*(x_0) = \{x \in X : \exists\lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x, x_0) < \infty\}$$

are said to be modular spaces (around x_0).

We obviously have $X_\omega \subset X_\omega^*$. In general this inclusion may be proper. It follows from [10, 11] that if ω is a modular on X, then the modular space X_ω can be equipped with a (nontrivial) metric, generated by ω and given by

$$d_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \le \lambda\},$$

for any $x, y \in X_\omega$.

If ω is a convex modular on X, according to [10, 11] the two modular spaces coincide, i.e. $X_\omega^* = X_\omega$, and this common set can be endowed with the metric d_ω^* given by

$$d_\omega^*(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \le 1\},$$

for any $x, y \in X_\omega$. These distances will be called Luxemburg distances.

Definition 1.3. [4] Let X_ω be a modular metric space.

(1) The sequence $\{x_n\}_{n\in N}$ in X_ω is said to be ω-convergent to $x \in X_\omega$ if and only if $\omega_1(x_n, x) \to 0$, as $n \to \infty$, x will be called the ω-limit of $\{x_n\}$.

(2) The sequence $\{x_n\}_{n\in N}$ in X_ω is said to be ω-Cauchy if $\omega_1(x_m, x_n) \to 0$ as $m, n \to \infty$.

(3) A subset M of X_ω is said to be ω-closed if the ω-limit of a ω-convergent sequence of M always belongs to M.

(4) A subset M of X_ω is said to be ω-complete if any ω-Cauchy sequence in M is a ω-convergent sequence and its ω-limit is in M.

(5) A subset M of X_ω is said to be ω-bounded if we have

$$\delta_\omega(M) = \sup\{\omega_1(x, y); x, y \in M\} < \infty.$$

(7) ω is said to satisfy the Fatou property if and only if for any sequence $\{x_n\}_{n\in N}$ in X_ω ω-convergent to x, we have

$$\omega_1(x, y) \le \lim_{n\to\infty} \inf \omega(x_n, y), \text{ for all } y \in X_\omega.$$

In general if $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$, for some $\lambda > 0$, then we may not have $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$, for all $\lambda > 0$. Therefore, as is done in modular function spaces, we will say that ω satisfies the $\triangle_2$−condition if this is the case, i.e. $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$, for some $\lambda > 0$ implies $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$, for all $\lambda > 0$. In [10], [11] one will find a discussion as regards the connection between $\omega-$convergence and metric convergence with respect to the Luxemburg distances. In particular, we have

$$\lim_{n\to\infty} d_\omega(x_n, x) = 0 \text{ if and only if } \lim_{n\to\infty} \omega_\lambda(x_n, x) = 0 \text{ for all } \lambda > 0,$$

for any $\{x_n\} \in X_\omega$ and $x \in X_\omega$. In particular we see that ω-convergence and d_ω-convergence are equivalent if and only if the modular ω satisfies the Δ_2-condition. Moreover, if the modular ω is convex, then we know that d^*_ω and d_ω are equivalent, which implies

$$\lim_{n\to\infty} d^*_\omega(x_n, x) = 0 \text{ if and only if } \lim_{n\to\infty} \omega_\lambda(x_n, x) = 0 \text{ for all } \lambda > 0,$$

for any $\{x_n\} \in X_\omega$ and $x \in X_\omega$.

The study of fixed points for multivalued contractions and non-expansive mappings using the Hausdorff metric was initiated by Markin [22]. The existence of fixed points for various multivalued contractive mappings has been studied by many authors under different conditions. The theory of multivalued mappings has application in control theory, convex optimization, differential inclusions and economics. In 1969, Nadler [26] extended the famous Banach Contraction Principle [5] from single-valued mapping to multivalued mapping and proved the fixed point theorem for the multivalued contraction.

In [4] Abdou and Khamsi define the Hausdorff modular metric and obtain a multivalued version of the results [3] in modular metric spaces.

The following notations and definitions, in [4] are suitable in order to setup our results:

For a subset M of modular metric space X_ω. Set

(i) $CB(M) = \{C : C$ is nonempty ω-closed and ω-bounded subset of $M\}$;

(ii) $K(M) = \{C : C$ is nonempty ω-compact subset of $M\}$;

(iii) the Hausdorff modular metric is defined on $CB(M)$ by

$$\begin{aligned} H_\omega(A, B) &= \max\left\{\sup_{x\in A} \omega_1(x, B), \sup \omega_1(y, A)\right\}, \\ \text{where } \omega_1(x, B) &= \inf_{y\in B} \omega(x, y). \end{aligned}$$

Definition 1.4. [4] Let (X, ω) be a modular metric space and M be a nonempty subset of X_ω.

A multivalued mapping $T : M \to CB(M)$ is called

(i) an $\omega-$contraction if there exists a constant $k \in [0, 1)$ such that for any $x, y \in M$,

$$H_\omega(T(x), T(y)) \leq k\omega_1(x, y);$$

(ii) a $(\varepsilon, k) - \omega-$uniformly locally contraction if there exists a constant $k \in [0, 1)$ such that for any $x, y \in M$,

$$H_\omega(T(x), T(y)) \leq k\omega_1(x, y), \text{ whenever } \omega_1(x, y) < \varepsilon.$$

Lemma 1.5. [4] Let (X, ω) be a modular metric space and M be a nonempty subset of X_ω. Let $A, B \in CB(M)$, then for each $\varepsilon > 0$ and $x \in A$, there exists $y \in B$ such that

$$\omega_1(x, y) \leq H_\omega(A, B) + \varepsilon.$$

Moreover, if B is $\omega-$compact and ω satisfies the Fatou property, then for any $x \in A$, there exists $y \in B$ such that

$$\omega_1(x, y) \leq H_\omega(A, B).$$

Lemma 1.6. [4] Let (X, ω) be a modular metric space. Assume that ω satisfies Δ_2-condition. Let M be a nonempty subset of X_ω. Let A_n be a sequence of sets in $CB(M)$, and suppose $\lim\limits_{n\to\infty} H_\omega(A_n, A_0) = 0$ where $A_0 \in CB(M)$. Then if $x_n \in A_n$ and $\lim\limits_{n\to\infty} x_n = x_0$, it follows that $x_0 \in A_0$.

In [7], Bhaskar and Lakshmikantham established some coupled fixed point theorems and applied these results to study the existence and uniqueness of solution for periodic boundary value problems. Lakshmikantham and Ciric [20] proved coupled coincidence and common coupled fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces and extended the results of Bhaskar and Lakshmikantham [7]. Deshpande and Handa [14] generalized and intuitionistically fuzzified the results of Bhaskar and Lakshmikantham [7], Lakshmikantham and Ciric [20]. In [16], Deshpande, Sharma and Handa proved a common coupled fixed point theorem for mappings under φ-contractive conditions on intuitionistic fuzzy metric spaces.

Samet et al. [28] claimed that most of the coupled fixed point theorems in the setting of single valued mappings on ordered metric spaces are consequences of well-known fixed point theorems. These concepts were extended by Abbas et al. [2] to multivalued mappings and who obtained coupled coincidence points and common coupled fixed point theorems involving hybrid pair of mappings satisfying generalized contractive conditions in complete metric spaces.

In [2], Abbas et al. introduced the following concept:

Definition 1.7. [2] Let X be a nonempty set, $F : X \times X \to 2^X$ (a collection of all nonempty subsets of X) and g be a self-mapping on X. An element $(x, y) \in X \times X$ is called

(1) a coupled coincidence point of hybrid pair $\{F, g\}$ if $g(x) \in F(x, y)$ and $g(y) \in F(y, x)$.

(2) a common coupled fixed point of hybrid pair $\{F, g\}$ if $x = g(x) \in F(x, y)$ and $y = g(y) \in F(y, x)$.

We denote the set of coupled coincidence points of mappings F and g by $C\{F, g\}$. Note that if $(x, y) \in C\{F, g\}$, then (y, x) is also in $C\{F, g\}$.

Definition 1.8. [2] Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The hybrid pair $\{F, g\}$ is called w-compatible if $g(F(x, y)) \subseteq F(gx, gy)$ whenever $(x, y) \in C\{F, g\}$.

Definition 1.9. [2] Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The mapping g is called F-weakly commuting at some point $(x, y) \in X \times X$ if $g^2x \in F(gx, gy)$ and $g^2y \in F(gy, gx)$.

Aamri and ElMoutawakil [1] defined (EA) property for self-mappings which contained the class of noncompatible mappings. Kamran [19] extended the (EA) property for hybrid pair $f : X \to X$ and $T : X \to 2^X$. Deshpande and Handa [15] introduce common (EA) property and occasional w-compatibility for hybrid pair $f : X \to X$, $F : X \times X \to 2^X$ and also introduce common (EA) property for two hybrid pairs $F, G : X \times X \to 2^X$ and $f, g : X \to X$.

Definition 1.10. [15] Mappings $f : X \to X$ and $F : X \times X \to CB(X)$ are said to satisfy the (EA) property if there exist sequences $\{x_n\}$, $\{y_n\}$ in X, some u, v in X and A, B in $CB(X)$ such that

$$\begin{aligned} \lim_{n\to\infty} fx_n &= u \in A = \lim_{n\to\infty} F(x_n, y_n), \\ \lim_{n\to\infty} fy_n &= v \in B = \lim_{n\to\infty} F(y_n, x_n). \end{aligned}$$

Definition 1.11. [15] Let $f, g : X \to X$ and $F, G : X \times X \to CB(X)$. The pairs $\{F, f\}$ and $\{G, g\}$ are said to satisfy the common (EA) property if there exist sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ in X, some u, v in X and A, B, C, D in $CB(X)$ such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n, y_n) &= A, \ \lim_{n\to\infty} G(u_n, v_n) = B, \\ \lim_{n\to\infty} fx_n &= \lim_{n\to\infty} gu_n = u \in A \cap B, \\ \lim_{n\to\infty} F(y_n, x_n) &= C, \lim_{n\to\infty} G(v_n, u_n) = D, \\ \lim_{n\to\infty} fy_n &= \lim_{n\to\infty} gv_n = v \in C \cap D. \end{aligned}$$

Definition 1.12. [15] Mappings $F : X \times X \to 2^X$ and $f : X \to X$ are said to be occasionally $w-$compatible if and only if there exists some point $(x, y) \in X \times X$ such that $fx \in F(x, y)$, $fy \in F(y, x)$ and $fF(x, y) \subseteq F(fx, fy)$.

In this paper, we establish some common coupled fixed point theorems for two hybrid pairs of mappings under $\varphi - \psi$ contraction on modular metric spaces. Our results will generalize and extend the results of Deshpande et al. [15] to modular metric spaces and also improve, extend, and generalize the results of Berinde [6], Gnana et al. [17], Jain et al. [18], Lakshmikantham et al [20] and Luong and Thuan [21]. Finally an example is also given to validate our results.

2. Main results

Let Φ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying

(i_φ) φ is continuous and strictly increasing,

(ii_φ) $\varphi(t) < t$ for all $t > 0$,

(iii_φ) $\varphi(t+s) \leq \varphi(t) + \varphi(s)$ for all $t, s > 0$.

and Ψ denote the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ which satisfies

(i_ψ) $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0+} \psi(t) = 0$,

(ii_ψ) $\psi(t) > 0$ for all $t > 0$ and $\psi(0) = 0$.

Note that, by (i_φ) and (ii_φ) we have that $\varphi(t) = 0$ if and only if $t = 0$. For example, functions $\varphi_1(t) = kt$ where $k > 0$, $\varphi_2(t) = \dfrac{t}{t+1}$, $\varphi_3(t) = \ln(t+1)$ and $\varphi_4(t) = \min\{t, 1\}$ are in Φ, $\psi_1(t) = kt$ where $k > 0$, $\psi_2(t) = \dfrac{\ln(2t+1)}{2}$ and

$$\psi_3(t) = \begin{cases} 1, & t = 0 \\ \dfrac{t}{t+1}, & 0 < t < 1 \\ 1, & t = 1 \\ \dfrac{t}{2}, & t > 1 \end{cases}$$

are in Ψ.

Now, we prove our main results.

Theorem 2.1. Let (X, ω) be a modular metric space such that ω satisfies the Δ_2-condition. Assume $F, G : X \times X \to CB(M)$, where M be nonempty ω-closed and ω-bounded subset of X_ω and $f, g : X \to X$ be mappings satisfying

(1.1) $\{F, f\}$ and $\{G, g\}$ satisfy the common (EA) property,

(1.2) for all $x, y, u, v \in X$, there exist some $\varphi \in \Phi$ and some $\psi \in \Psi$ such that

$$\begin{aligned} &\varphi\left(\frac{H_\omega(F(x, y), G(u, v)) + H_\omega(F(y, x), G(v, u))}{2}\right) \\ \leq\ &\varphi\left(\frac{\omega_1(fx, gu) + \omega_1(fy, gv)}{2}\right) - \psi\left(\frac{\omega_1(fx, gu) + \omega_1(fy, gv)}{2}\right). \end{aligned}$$

(1.3) $f(X)$ and $g(X)$ are closed subsets of X. Then

(a) F and f have a coupled coincidence point,

(b) G and g have a coupled coincidence point,

(c) F and f have a common coupled fixed point, if f is $F-$weakly commuting at (x, y) and $f^2x = fx$ and $f^2y = fy$ for $(x, y) \in C\{F, f\}$,

(d) G and g have a common coupled fixed point, if g is $G-$weakly commuting at $(\widetilde{x}, \widetilde{y})$ and $g^2\widetilde{x} = g\widetilde{x}$ and $g^2\widetilde{y} = g\widetilde{y}$ for $(\widetilde{x}, \widetilde{y}) \in C\{G, g\}$,

(e) F, G, f, g have common coupled fixed point provided that both (c) and (d) are true.

Proof. Since $\{F, f\}$ and $\{G, g\}$ satisfy the common (EA) property, there exist sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ in X, some u, v in X and A, B, C, D in $CB(X)$ such that

$$\begin{aligned}
\lim_{n\to\infty} F(x_n, y_n) &= A, \lim_{n\to\infty} G(u_n, v_n) = B, \\
\lim_{n\to\infty} fx_n &= \lim_{n\to\infty} gu_n = u \in A \cap B, \\
\lim_{n\to\infty} F(y_n, x_n) &= C, \lim_{n\to\infty} G(v_n, u_n) = D, \qquad (1.4)\\
\lim_{n\to\infty} fy_n &= \lim_{n\to\infty} gv_n = v \in C \cap D.
\end{aligned}$$

Since $f(X)$ and $g(X)$ are closed subsets of X, then there exist $x, y, \widetilde{x}, \widetilde{y} \in X$,

$$u = fx = g\widetilde{x} \text{ and } v = fy = g\widetilde{y}. \qquad (1.5)$$

Now, by using condition (1.2), we get

$$\begin{aligned}
&\varphi\left(\frac{H_\omega(F(x, y), G(u_n, v_n)) + H_\omega(F(y, x), G(v_n, u_n))}{2}\right) \\
\leq\ &\varphi\left(\frac{\omega_1(fx, gu_n) + \omega_1(fy, gv_n)}{2}\right) - \psi\left(\frac{\omega_1(fx, gu_n) + \omega_1(fy, gv_n)}{2}\right).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, by using (1.4), (1.5), (i_φ), (ii_φ) and (i_ψ), we obtain

$$\begin{aligned}
&\varphi\left(\frac{H_\omega(F(x, y), B) + H_\omega(F(y, x), D)}{2}\right) \\
\leq\ &\varphi(0) - 0 = 0 - 0 = 0,
\end{aligned}$$

which, by (i_φ) and (ii_φ), implies

$$H_\omega(F(x, y), B) = 0 \text{ and } H_\omega(F(y, x), D) = 0.$$

Since $fx \in B$ and $fy \in D$, it follows, by using Lemma 1.6 that

$$fx \in F(x, y) \text{ and } fy \in F(y, x),$$

that is, (x, y) is a coupled coincidence point of F and f. This proves (a). Again, by using condition (1.2), we get

$$\varphi\left(\frac{H_\omega(F(x_n, y_n), G(\widetilde{x}, \widetilde{y})) + H_\omega(F(y_n, x_n), G(\widetilde{y}, \widetilde{x}))}{2}\right)$$
$$\leq \varphi\left(\frac{\omega_1(fx_n, g\widetilde{x}) + \omega_1(fy_n, g\widetilde{y})}{2}\right) - \psi\left(\frac{\omega_1(fx_n, g\widetilde{x}) + \omega_1(fy_n, g\widetilde{y})}{2}\right).$$

Letting $n \to \infty$ in the above inequality, by using (1.4), (1.5), (i_φ), (ii_φ) and (i_ψ), we obtain

$$\varphi\left(\frac{H_\omega(A, G(\widetilde{x}, \widetilde{y})) + H_\omega(C, G(\widetilde{y}, \widetilde{x}))}{2}\right)$$
$$\leq \varphi(0) - 0 = 0 - 0 = 0,$$

which, by (i_φ) and (ii_φ), implies

$$H_\omega(A, G(\widetilde{x}, \widetilde{y})) = 0 \text{ and } H_\omega(C, G(\widetilde{y}, \widetilde{x})) = 0.$$

Since $g\widetilde{x} \in A$ and $g\widetilde{y} \in C$,it follows, by using Lemma 1.6 that

$$g\widetilde{x} \in G(\widetilde{x}, \widetilde{y}) \text{ and } g\widetilde{y} \in G(\widetilde{y}, \widetilde{x}),$$

that is, $(\widetilde{x}, \widetilde{y})$ is a coupled coincidence point of G and g. This proves (b).

Furthermore, from condition (c), we have f is $F-$weakly commuting at (x, y), that is, $f^2x \in F(fx, fy)$, $f^2y \in F(fy, fx)$ and $f^2x = fx$, $f^2y = fy$. Thus $fx = f^2x \in F(fx, fy)$ and $fy = f^2y \in F(fy, fx)$, that is, $u = fu \in F(u, v)$ and $v = fv \in F(v, u)$. This proves (c). A similar argument proves (d). Then (e) holds immediately. ■

Put $f = g$ in the Theorem 2.1, we get the following result:

Corollary 2.2. Let (X, ω) be a modular metric space such that ω satisfies the Δ_2-condition. Assume $F, G : X \times X \to CB(M)$, where M be nonempty ω-closed and ω-bounded subset of X_ω and $g : X \to X$ be mappings satisfying

(2.1) $\{F, g\}$ and $\{G, g\}$ satisfy the common (EA) property,

(2.2) for all $x, y, u, v \in X$, there exist some $\varphi \in \Phi$ and some $\psi \in \Psi$ such that

$$\varphi\left(\frac{H_\omega(F(x, y), G(u, v)) + H_\omega(F(y, x), G(v, u))}{2}\right)$$
$$\leq \varphi\left(\frac{\omega_1(gx, gu) + \omega_1(gy, gv)}{2}\right) - \psi\left(\frac{\omega_1(gx, gu) + \omega_1(gy, gv)}{2}\right).$$

(2.3) $g(X)$ is a closed subset of X. Then

(a) F and g have a coupled coincidence point,

(b) G and g have a coupled coincidence point,

(c) F and g have a common coupled fixed point, if g is $F-$weakly commuting at (x, y) and $g^2x = gx$ and $g^2y = gy$ for $(x, y) \in C\{F, g\}$,

(d) G and g have a common coupled fixed point, if g is $G-$weakly commuting at $(\widetilde{x}, \widetilde{y})$ and $g^2\widetilde{x} = g\widetilde{x}$ and $g^2\widetilde{y} = g\widetilde{y}$ for $(\widetilde{x}, \widetilde{y}) \in C\{G, g\}$,

(e) F, G, g have common coupled fixed point provided that both (c) and (d) are true.

Put $F = G$ and $f = g$ in the Theorem 2.1, we get the following result:

Corollary 2.3. Let (X, ω) be a modular metric space such that ω satisfies the Δ_2-condition. Assume $F : X \times X \to CB(M)$, where M be nonempty ω-closed and ω-bounded subset of X_ω and $g : X \to X$ be mappings satisfying

(3.1) $\{F, g\}$ satisfies the (EA) property,

(3.2) for all $x, y, u, v \in X$, there exist some $\varphi \in \Phi$ and some $\psi \in \Psi$ such that

$$\begin{aligned} &\varphi\left(\frac{H_\omega(F(x, y), F(u, v)) + H_\omega(F(y, x), F(v, u))}{2}\right) \\ \leq\ &\varphi\left(\frac{\omega_1(gx, gu) + \omega_1(gy, gv)}{2}\right) - \psi\left(\frac{\omega_1(gx, gu) + \omega_1(gy, gv)}{2}\right). \end{aligned}$$

If (2.3) holds. Then

(a) F and g have a coupled coincidence point,

(b) F and g have a common coupled fixed point, if g is $F-$weakly commuting at (x, y) and $g^2x = gx$ and $g^2y = gy$ for $(x, y) \in C\{F, g\}$.

Corollary 2.4. Let (X, ω) be a modular metric space such that ω satisfies the Δ_2-condition. Assume $F, G : X \times X \to CB(M)$, where M be nonempty ω-closed and ω-bounded subset of X_ω and $f, g : X \to X$ be mappings satisfying (1.1) and

(4.1) for all $x, y, u, v \in X$, there exists some $\psi \in \Psi$ such that

$$\begin{aligned} &H_\omega(F(x, y), G(u, v)) + H_\omega(F(y, x), G(v, u)) \\ \leq\ &\omega_1(fx, gu) + \omega_1(fy, gv) - 2\psi\left(\frac{\omega_1(fx, gu) + \omega_1(fy, gv)}{2}\right). \end{aligned}$$

If (1.3) holds. Then

(a) F and f have a coupled coincidence point,

(b) G and g have a coupled coincidence point,

(c) F and f have a common coupled fixed point, if f is $F-$weakly commuting at (x, y) and $f^2x = fx$ and $f^2y = fy$ for $(x, y) \in C\{F, f\}$,

(d) G and g have a common coupled fixed point, if g is $G-$weakly commuting at $(\widetilde{x}, \widetilde{y})$ and $g^2\widetilde{x} = g\widetilde{x}$ and $g^2\widetilde{y} = g\widetilde{y}$ for $(\widetilde{x}, \widetilde{y}) \in C\{G, g\}$,

(e) F, G, f, g have common coupled fixed point provided that both (c) and (d) are true.

Proof. If $\psi \in \Psi$, then for all $r > 0$, $r\psi \in \Psi$. Now divide the condition (4.1) by 4 and take $\varphi(t) = \frac{1}{2}t$, $t \in [0, +\infty)$, then the above condition reduces to the condition (1.2) with $\psi_1 = \frac{1}{2}\psi$ and hence by Theorem 2.1 we get Corollary 2.4 ■

Put $f = g$ in the Corollary 2.4, we get the following result:

Corollary 2.5. Let (X, ω) be a modular metric space such that ω satisfies the Δ_2-condition. Assume $F, G : X \times X \to CB(M)$, where M be nonempty ω-closed and ω-bounded subset of X_ω and $f, g : X \to X$ be mappings satisfying (2.1) and

(5.1) for all $x, y, u, v \in X$, there exists some $\psi \in \Psi$ such that

$$\begin{aligned} & H_\omega(F(x, y), G(u, v)) + H_\omega(F(y, x), G(v, u)) \\ \leq \; & \omega_1(gx, gu) + \omega_1(gy, gv) - 2\psi\left(\frac{\omega_1(gx, gu) + \omega_1(gy, gv)}{2}\right). \end{aligned}$$

If (2.3) holds. Then

(a) F and g have a coupled coincidence point,

(b) G and g have a coupled coincidence point,

(c) F and g have a common coupled fixed point, if g is F-weakly commuting at (x, y) and $g^2x = gx$ and $g^2y = gy$ for $(x, y) \in C\{F, g\}$,

(d) G and g have a common coupled fixed point, if g is $G-$weakly commuting at $(\widetilde{x}, \widetilde{y})$ and $g^2\widetilde{x} = g\widetilde{x}$ and $g^2\widetilde{y} = g\widetilde{y}$ for $(\widetilde{x}, \widetilde{y}) \in C\{G, g\}$,

(e) F, G, g have common coupled fixed point provided that both (c) and (d) are true.

Put $F = G$ and $f = g$ in the Corollary 2.4, we get the following result:

Corollary 2.6. Let (X, ω) be a modular metric space such that ω satisfies the Δ_2-condition. Assume $F, G : X \times X \to CB(M)$, where M be nonempty ω-closed and ω-bounded subset of X_ω and $g : X \to X$ be mappings satisfying (3.1) and

(6.1) for all $x, y, u, v \in X$, there exists some $\psi \in \Psi$ such that

$$\begin{aligned} & H_\omega(F(x, y), F(u, v)) + H_\omega(F(y, x), F(v, u)) \\ \leq \; & \omega_1(gx, gu) + \omega_1(gy, gv) - 2\psi\left(\frac{\omega_1(gx, gu) + \omega_1(gy, gv)}{2}\right). \end{aligned}$$

If (2.3) holds. Then

(a) F and g have a coupled coincidence point,

(b) F and g have a common coupled fixed point, if g is $F-$weakly commuting at (x, y) and $g^2x = gx$ and $g^2y = gy$ for $(x, y) \in C\{F, g\}$.

Theorem 2.7. Let (X, ω) be a modular metric space such that ω satisfies the Δ_2-condition and Fatou property. Assume $F, G : X \times X \to K(M)$ where M be nonempty ω-compact subset of M and $f, g : X \to X$ be mappings satisfying (1.1), (1.2) and

(7.1) $\{F, f\}$ and $\{G, g\}$ are w-compatible,

(7.2) Suppose that either

(a) $g(X)$ is a closed subset of X and $G(X \times X) \subseteq f(X)$ or

(b) $f(X)$ is a closed subset of X and $F(X \times X) \subseteq g(X)$,then F, G, f, g have a common coupled fixed point.

Proof. Since $\{F, f\}$ and $\{G, g\}$ satisfy the common (EA) property, there exist sequences $\{x_n\}, \{y_n\}, \{u_n\}$ and $\{v_n\}$ in X, some u, v in X and A, B, C, D in $CB(X)$ satisfying (1.4). Suppose (a) holds, that is, $g(X)$ is a closed subset of X, then there exist $\widetilde{x}, \widetilde{y} \in X$, we have

$$u = g\widetilde{x} \text{ and } v = g\widetilde{y}. \tag{7.3}$$

As in Theorem 2.1, we can prove that

$$g\widetilde{x} \in G(\widetilde{x}, \widetilde{y}) \text{ and } g\widetilde{y} \in G(\widetilde{y}, \widetilde{x}).$$

that is, $(\widetilde{x}, \widetilde{y})$ is a coupled coincidence point of G and g. Hence $(\widetilde{x}, \widetilde{y}) \in C\{G, g\}$. From w-compatibility of $\{G, g\}$, we have $gG(\widetilde{x}, \widetilde{y}) \subseteq G(g\widetilde{x}, g\widetilde{y})$, hence $g^2\widetilde{x} \in G(g\widetilde{x}, g\widetilde{y})$ and $g^2\widetilde{y} \in G(g\widetilde{y}, g\widetilde{x})$, that is, $gu \in G(u, v)$ and $gv \in G(v, u)$. Now, we shall show that $u = gu$ and $v = gv$. Suppose, not. Then, by condition (1.2), we get

$$\begin{aligned} &\varphi\left(\frac{H_\omega(F(x_n, y_n), G(u, v)) + H_\omega(F(y_n, x_n), G(v, u))}{2}\right) \\ \leq\ &\varphi\left(\frac{\omega_1(fx_n, gu) + \omega_1(fy_n, gv)}{2}\right) - \psi\left(\frac{\omega_1(fx_n, gu) + \omega_1(fy_n, gv)}{2}\right). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, by using (1.4) and (i_φ), we obtain

$$\begin{aligned} &\varphi\left(\frac{H_\omega(A, G(u, v)) + H_\omega(C, G(v, u))}{2}\right) \\ \leq\ &\varphi\left(\frac{\omega_1(u, gu) + \omega_1(v, gv)}{2}\right) - \lim_{n\to\infty} \psi\left(\frac{\omega_1(fx_n, gu) + \omega_1(fy_n, gv)}{2}\right). \end{aligned}$$

Since $u \in A$, $v \in C$, $gu \in G(u, v)$ and $gv \in G(v, u)$, therefore, by Lemma 1.5 and (i_ψ), we get

$$\begin{aligned}
& \varphi\left(\frac{\omega_1(u, gu) + \omega_1(v, gv)}{2}\right) \\
\leq\ & \varphi\left(\frac{H_\omega(A, G(u, v)) + H_\omega(C, G(v, u))}{2}\right) \\
\leq\ & \varphi\left(\frac{\omega_1(u, gu) + \omega_1(v, gv)}{2}\right) - \lim_{n\to\infty} \psi\left(\frac{\omega_1(fx_n, gu) + \omega_1(fy_n, gv)}{2}\right) \\
<\ & \varphi\left(\frac{\omega_1(u, gu) + \omega_1(v, gv)}{2}\right),
\end{aligned}$$

which is a contradiction. Thus $u = gu$ and $v = gv$. Hence, we have

$$u = gu \in G(u, v) \text{ and } v = gv \in G(v, u).$$

Since $G(X \times X) \subseteq f(X)$, then there exist $x, y \in X$ such that $fx = u = gu \in G(u, v)$ and $fy = v = gv \in G(v, u)$. Now, by condition (1.2), (i_φ), (ii_φ) and (ii_ψ), we get

$$\begin{aligned}
& \varphi\left(\frac{\omega_1(F(x, y), u) + \omega_1(F(y, x), v)}{2}\right) \\
\leq\ & \varphi\left(\frac{H_\omega(F(x, y), G(u, v)) + H_\omega(F(y, x), G(v, u))}{2}\right) \\
\leq\ & \varphi\left(\frac{\omega_1(fx, gu) + \omega_1(fy, gv)}{2}\right) - \psi\left(\frac{\omega_1(fx, gu) + \omega_1(fy, gv)}{2}\right) \\
\leq\ & 0 - 0 = 0,
\end{aligned}$$

which, by (i_φ) and (ii_φ), implies

$$\omega_1(F(x, y), u) = \omega_1(F(y, x), v) = 0.$$

Thus

$$u = fx \in F(x, y) \text{ and } v = fy \in F(y, x),$$

that is, (x, y) is a coupled coincidence point of F and f. Hence $(x, y) \in C\{F, f\}$. From w-compatibility of $\{F, f\}$, we have $fF(x, y) \subseteq F(fx, fy)$, hence $f^2x \in F(fx, fy)$ and $f^2y \in F(fy, fx)$, that is, $fu \in F(u, v)$ and $fv \in F(v, u)$. Now, we shall show that $fu = u$ and $fv = v$. Suppose, not. Then, by Lemma 1.5 and using condition (1.2), and (ii_ψ), we get

$$\begin{aligned}
&\varphi\left(\frac{\omega_1(fu,\ u)+\omega_1(fv,\ v)}{2}\right)\\
\leq\ &\varphi\left(\frac{H_\omega(F(u,\ v),\ G(u,\ v))+H_\omega(F(v,\ u),\ G(v,\ u))}{2}\right)\\
\leq\ &\varphi\left(\frac{\omega_1(fu,\ gu)+\omega_1(fv,\ gv)}{2}\right)-\psi\left(\frac{\omega_1(fu,\ gu)+\omega_1(fv,\ gv)}{2}\right)\\
\leq\ &\varphi\left(\frac{\omega_1(fu,\ u)+\omega_1(fv,\ v)}{2}\right)-\psi\left(\frac{\omega_1(fu,\ u)+\omega_1(fv,\ v)}{2}\right)\\
<\ &\varphi\left(\frac{\omega_1(fu,\ u)+\omega_1(fv,\ v)}{2}\right),
\end{aligned}$$

which is a contradiction. Thus $fu = u$ and $fv = v$. Hence, we have

$$u = fu \in F(u,\ v) \text{ and } v = fv \in F(v,\ u).$$

Therefore $(u,\ v)$ is a common coupled fixed point of the pairs $\{F,\ f\}$ and $\{G,\ g\}$. The proof is similar when (b) holds. ■

If we put $f = g$ in the Theorem 2.7, we get the following result:

Corollary 2.8. Let $(X,\ \omega)$ be a modular metric space such that ω satisfies the Δ_2-condition and Fatou property. Assume $F,\ G : X \times X \to K(M)$ where M be nonempty ω-compact subset of M and $g : X \to X$ be mappings satisfying (2.1), (2.2) and

(8.1) $\{F, g\}$ and $\{G, g\}$ are w-compatible,

(8.2) Suppose that either

- (a) $g(X)$ is a closed subset of X and $G(X \times X) \subseteq g(X)$ or
- (b) $g(X)$ is a closed subset of X and $F(X \times X) \subseteq g(X)$,then F, G, g have a common coupled fixed point.

If we put $F = G$ and $f = g$ in the Theorem 2.7, we get the following result:

Corollary 2.9. Let $(X,\ \omega)$ be a modular metric space such that ω satisfies the Δ_2-condition and Fatou property. Assume $F : X \times X \to K(M)$ where M be nonempty ω-compact subset of M and $g : X \to X$ be mappings satisfying (3.1), (3.2) and

(9.1) $\{F, g\}$ is w-compatible,

(9.2) $g(X)$ is a closed subset of X and $F(X \times X) \subseteq g(X)$, then F and g have a common coupled fixed point.

Corollary 2.10. Let (X, ω) be a modular metric space such that ω satisfies the Δ_2-condition and Fatou property. Assume $F, G : X \times X \to K(M)$ where M be nonempty ω-compact subset of M and $f, g : X \to X$ be mappings satisfying (1.1), (4.1), (7.1) and (7.2), then F, G, f, g have a common coupled fixed point.

Proof. If $\psi \in \Psi$, then for all $r > 0$, $r\psi \in \Psi$. If, we divide the condition (4.1) by 4 and take $\varphi(t) = \frac{1}{2}t$, $t \in [0, +\infty)$, then it reduces to the condition (1.2) with $\psi_1 = \frac{1}{2}\psi$ and hence by Theorem 2.7 we get Corollary 2.10 ∎

If we put $f = g$ in the Corollary 2.10, we get the following result:

Corollary 2.11. Let (X, ω) be a modular metric space such that ω satisfies the Δ_2-condition and Fatou property. Assume $F, G : X \times X \to K(M)$ where M be nonempty ω-compact subset of M and $g : X \to X$ be mappings satisfying (2.1), (5.1), (8.1) and (8.2), then F, G and g have a common coupled fixed point.

If we put $F = G$ and $f = g$ in the Corollary 2.10, we get the following result:

Corollary 2.12. Let (X, ω) be a modular metric space such that ω satisfies the Δ_2-condition and Fatou property. Assume $F : X \times X \to K(M)$ where M be nonempty ω-compact subset of M and $g : X \to X$ be mappings satisfying (3.1), (6.1), (9.1) and (9.2), then F and g have a common coupled fixed point.

Theorem 2.13. Let (X, ω) be a modular metric space such that ω satisfies the Δ_2-condition and Fatou property. Assume $F, G : X \times X \to K(M)$ where M be nonempty ω-compact subset of M and $f, g : X \to X$ be mappings satisfying (1.2) and

(13.1) $\{F, f\}$ and $\{G, g\}$ are occasionally w-compatible.

Then F, G, f, g have a common coupled fixed point.

Proof. Since the pairs $\{F, f\}$ and $\{G, g\}$ are occasionally w-compatible, therefore there exist some points $(x, y), (\widetilde{x}, \widetilde{y}) \in X \times X$ such that

$$\begin{aligned} &fx \in F(x, y),\ fy \in F(y, x),\ fF(x, y) \subseteq F(fx, fy), \\ &g\widetilde{x} \in G(\widetilde{x}, \widetilde{y}),\ g\widetilde{y} \in G(\widetilde{y}, \widetilde{x}),\ gG(\widetilde{x}, \widetilde{y}) \subseteq G(g\widetilde{x}, g\widetilde{y}). \end{aligned} \tag{13.2}$$

It follows that

$$\begin{aligned} f^2x &\in F(fx, fy),\ f^2y \in F(fy, fx), \\ g^2\widetilde{x} &\in G(g\widetilde{x}, g\widetilde{y}),\ g^2\widetilde{y} \in G(g\widetilde{y}, g\widetilde{x}). \end{aligned} \tag{13.3}$$

Now, we shall show that $u = fx = g\widetilde{x}$ and $v = fy = g\widetilde{y}$. Suppose, not. Then, by Lemma 1.5 and using condition (1.2) and (ii_ψ), we have

$$\begin{aligned} & \varphi\left(\frac{\omega_1(fx,\ g\widetilde{x}) + \omega_1(fy,\ g\widetilde{y})}{2}\right) \\ \leq\ & \varphi\left(\frac{H_\omega(F(x,\ y),\ G(\widetilde{x},\ \widetilde{y})) + H_\omega(F(y,\ x),\ G(\widetilde{y},\ \widetilde{x}))}{2}\right) \\ \leq\ & \varphi\left(\frac{\omega_1(fx,\ g\widetilde{x}) + \omega_1(fy,\ g\widetilde{y})}{2}\right) - \psi\left(\frac{\omega_1(fx,\ g\widetilde{x}) + \omega_1(fy,\ g\widetilde{y})}{2}\right) \\ <\ & \varphi\left(\frac{\omega_1(fx,\ g\widetilde{x}) + \omega_1(fy,\ g\widetilde{y})}{2}\right), \end{aligned}$$

which is a contradiction. Thus $fx = g\widetilde{x}$ and $fy = g\widetilde{y}$. Hence

$$u = fx = g\widetilde{x} \text{ and } v = fy = g\widetilde{y}. \tag{13.4}$$

Thus, by (13.3), we get

$$\begin{aligned} & fu \in F(u,\ v),\ fv \in F(v,\ u), \\ & gu \in G(u,\ v),\ gv \in G(v,\ u). \end{aligned} \tag{13.5}$$

Now, we shall show that $u = fu = gu$ and $v = fv = gv$. Suppose, not. Then, by Lemma 1.5 and using condition (1.2) and (ii_ψ), we have

$$\begin{aligned} & \varphi\left(\frac{\omega_1(fu,\ u) + \omega_1(fv,\ v)}{2}\right) \\ \leq\ & \varphi\left(\frac{H_\omega(F(u,\ v),\ G(\widetilde{x},\ \widetilde{y})) + H_\omega(F(v,\ u),\ G(\widetilde{y},\ \widetilde{x}))}{2}\right) \\ \leq\ & \varphi\left(\frac{\omega_1(fu,\ g\widetilde{x}) + \omega_1(fv,\ g\widetilde{y})}{2}\right) - \psi\left(\frac{\omega_1(fu,\ g\widetilde{x}) + \omega_1(fv,\ g\widetilde{y})}{2}\right) \\ \leq\ & \varphi\left(\frac{\omega_1(fu,\ u) + \omega_1(fv,\ v)}{2}\right) - \psi\left(\frac{\omega_1(fu,\ u) + \omega_1(fv,\ v)}{2}\right) \\ <\ & \varphi\left(\frac{\omega_1(fu,\ u) + \omega_1(fv,\ v)}{2}\right), \end{aligned}$$

which is a contradiction. Thus,

$$u = fu \text{ and } v = fv. \tag{13.6}$$

Similarly, we can show that

$$u = gu \text{ and } v = gv. \tag{13.7}$$

Thus, by (13.5), (13.6) and (13.7), we get

$$\begin{aligned} u &= fu \in F(u,\ v),\ v = fv \in F(v,\ u), \\ u &= gu \in G(u,\ v),\ v = gv \in G(v,\ u), \end{aligned}$$

that is, $(u,\ v)$ is a common coupled fixed point of $F,\ G,\ f,\ g$. ■

Put $f = g$ in the Theorem 2.13, we get the following result:

Corollary 2.14. Let $(X,\ \omega)$ be a modular metric space such that ω satisfies the Δ_2-condition and Fatou property. Assume $F,\ G : X \times X \to K(M)$ where M be nonempty ω-compact subset of M and $g : X \to X$ be mappings satisfying (2.2) and

(14.1) $\{F, g\}$ and $\{G, g\}$ are occasionally w-compatible.

Then $F,\ G,\ g$ have a common coupled fixed point.

Put $F = G$ and $f = g$ in the Theorem 2.13, we get the following result:

Corollary 2.15. Let (X, ω) be a modular metric space such that ω satisfies the Δ_2-condition and Fatou property. Assume $F : X \times X \to K(M)$ where M be nonempty ω-compact subset of M and $g : X \to X$ be mappings satisfying (3.2) and

(15.1) $\{F, g\}$ is occasionally w-compatible.

Then F and g have a common coupled fixed point.

Corollary 2.16. Let (X, ω) be a modular metric space such that ω satisfies the Δ_2-condition and Fatou property. Assume $F,\ G : X \times X \to K(M)$ where M be nonempty ω-compact subset of M and $f,\ g : X \to X$ be mappings satisfying (4.1) and (13.1), then $F,\ G,\ f,\ g$ have a common coupled fixed point.

Proof. If $\psi \in \Psi$, then for all $r > 0$, $r\psi \in \Psi$. If, we divide the condition (4.1) by 4 and take $\varphi(t) = \frac{1}{2}t$, $t \in [0, +\infty)$, then it reduces to the condition (1.2) with $\psi_1 = \frac{1}{2}\psi$ and hence by Theorem 2.13 we get Corollary 2.16. ■

Put $f = g$ in the Corollary 2.16, we get the following result:

Corollary 2.17. Let (X, ω) be a modular metric space such that ω satisfies the Δ_2-condition and Fatou property. Assume $F,\ G : X \times X \to K(M)$ where M be nonempty ω-compact subset of M and $g : X \to X$ be mappings satisfying (5.1) and (14.1), then $F,\ G$ and g have a common coupled fixed point.

Put $F = G$ and $f = g$ in the Corollary 16, we get the following result:

Corollary 2.18. Let (X, ω) be a modular metric space such that ω satisfies the Δ_2-condition and Fatou property. Assume $F : X \times X \to K(M)$ where M be nonempty

ω-compact subset of M and $g : X \to X$ be mappings satisfying (6.1) and (15.1), then F and g have a common coupled fixed point.

Example 2.19. Suppose that $X = [0, 1]$, and $\omega : (0, \infty) \times X \times X \to [0, +\infty)$ defined as $\omega_1(x, y) = \max\{x, y\}$ and $\omega_1(x, x) = 0$ for all $x, y \in X$. Let $F, G : X \times X \to CB(M)$, where M be nonempty ω-closed and ω-bounded subset of X_ω, be defined as

$$F(x,\ y) = \begin{cases} \{0\},\ \text{for } x,\ y = 1 \\ \left[0,\ \dfrac{x^2 + y^2}{4}\right],\ \text{for } x,\ y \in [0,\ 1) \end{cases}$$

and

$$G(x,\ y) = \begin{cases} \{0\},\ \text{for } x,\ y = 1 \\ \left[0,\ \dfrac{x + y}{8}\right],\ \text{for } x,\ y \in [0,\ 1). \end{cases}$$

Suppose $f, g : X \to X$ be defined as

$$fx = \begin{cases} x^2,\ x \neq 1, \\ \dfrac{3}{2},\ x = 1, \end{cases} \quad \text{for all } x \in X$$

and

$$gx = \begin{cases} \dfrac{x}{2},\ x \neq 1, \\ 1,\ x = 1, \end{cases} \quad \text{for all } x \in X.$$

Define $\varphi : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(t) = \frac{t}{2},\ \text{for all } t > 0,$$

and $\psi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = \begin{cases} \dfrac{t}{4},\ \text{for } t \neq 1 \\ 1,\ \text{for } t = 1. \end{cases}$$

Now, for all $x, y, u, v \in X$ with $x, y, u, v \in [0, 1)$, we have

Case (a). If $\frac{x^2+y^2}{4} = \frac{u+v}{8}$, then

$$\begin{aligned}
&\varphi\left(\frac{H_\omega(F(x,\ y),\ G(u,\ v)) + H_\omega(F(y,\ x),\ G(v,\ u))}{2}\right)\\
=\ & \frac{1}{4}\left[H_\omega(F(x,\ y),\ G(u,\ v)) + H_\omega(F(y,\ x),\ G(v,\ u))\right]\\
=\ & \frac{1}{4}\left[\frac{u+v}{8} + \frac{v+u}{8}\right]\\
\le\ & \frac{1}{8}\left[\max\left\{x^2,\ \frac{u}{2}\right\} + \max\left\{y^2,\ \frac{v}{2}\right\}\right]\\
\le\ & \frac{1}{8}\left[\omega_1(fx,\ gu) + \omega_1(fy,\ gv)\right]\\
\le\ & \frac{1}{4}\left(\frac{\omega_1(fx,\ gu) + \omega_1(fy,\ gv)}{2}\right)\\
\le\ & \varphi\left(\frac{\omega_1(fx,\ gu) + \omega_1(fy,\ gv)}{2}\right) - \psi\left(\frac{\omega_1(fx,\ gu) + \omega_1(fy,\ gv)}{2}\right).
\end{aligned}$$

Case (b). If $\frac{x^2+y^2}{4} \neq \frac{u+v}{8}$ with $\frac{x^2+y^2}{4} < \frac{u+v}{8}$, then

$$\begin{aligned}
&\varphi\left(\frac{H_\omega(F(x,\ y),\ G(u,\ v)) + H_\omega(F(y,\ x),\ G(v,\ u))}{2}\right)\\
=\ & \frac{1}{4}\left[H_\omega(F(x,\ y),\ G(u,\ v)) + H_\omega(F(y,\ x),\ G(v,\ u))\right]\\
=\ & \frac{1}{4}\left[\frac{u+v}{8} + \frac{v+u}{8}\right]\\
\le\ & \frac{1}{8}\left[\max\left\{x^2,\ \frac{u}{2}\right\} + \max\left\{y^2,\ \frac{v}{2}\right\}\right]\\
\le\ & \frac{1}{8}\left[\omega_1(fx,\ gu) + \omega_1(fy,\ gv)\right]\\
\le\ & \frac{1}{4}\left(\frac{\omega_1(fx,\ gu) + \omega_1(fy,\ gv)}{2}\right)\\
\le\ & \varphi\left(\frac{\omega_1(fx,\ gu) + \omega_1(fy,\ gv)}{2}\right) - \psi\left(\frac{\omega_1(fx,\ gu) + \omega_1(fy,\ gv)}{2}\right).
\end{aligned}$$

Similarly, we obtain the same result for $\frac{u+v}{8} < \frac{x^2+y^2}{4}$. Thus the contractive condition (1.2) is satisfied for all $x,\ y,\ u,\ v \in X$ with $x,\ y,\ u,\ v \in [0,\ 1)$. Again, for all $x,\ y,\ u,$

$v \in X$ with $x, y \in [0, 1)$ and $u, v = 1$, we have

$$\begin{aligned}
&\varphi\left(\frac{H_\omega(F(x, y), G(u, v)) + H_\omega(F(y, x), G(v, u))}{2}\right) \\
= &\frac{1}{4}[H_\omega(F(x, y), G(u, v)) + H_\omega(F(y, x), G(v, u))] \\
= &\frac{1}{4}\left[\frac{x^2+y^2}{4} + \frac{y^2+x^2}{4}\right] \\
\leq &\frac{1}{8}\left[\max\left\{x^2, \frac{u}{2}\right\} + \max\left\{y^2, \frac{v}{2}\right\}\right] \\
\leq &\frac{1}{8}[\omega_1(fx, gu) + \omega_1(fy, gv)] \\
\leq &\frac{1}{4}\left(\frac{\omega_1(fx, gu) + \omega_1(fy, gv)}{2}\right) \\
\leq &\varphi\left(\frac{\omega_1(fx, gu) + \omega_1(fy, gv)}{2}\right) - \psi\left(\frac{\omega_1(fx, gu) + \omega_1(fy, gv)}{2}\right).
\end{aligned}$$

Thus the contractive condition (1.2) is satisfied for all $x, y, u, v \in X$ with $x, y \in [0, 1)$ and $u, v = 1$. Similarly, we can see that the contractive condition (1.2) is satisfied for all $x, y, u, v \in X$ with $x, y, u, v = 1$. Hence, the hybrid pairs $\{F, f\}$ and $\{G, g\}$ satisfy the condition (1.2), for all $x, y, u, v \in X$. In addition, all the other conditions of Theorem 2.1, Theorem 2.7 and Theorem 2.13 are satisfied and $z = (0, 0)$ is a common coupled fixed point of F, G, f, g. ∎

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