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Jacobi Spectral Collocation Methods for Abel-Volterra Integral Equations of Second Kind

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Abstract

Y. Chen and T. Tang [6] have applied a Spectral method called Jacobi collocation method to solve Volterra integral equations with singular kernel $K(t,s)=(t-s)^{-\mu}$, the error analysis for this method is carried out for $0<\mu<\frac{1}{2}$ under the assumption that the underlying solution is smooth. In this paper we extend the result of Y. Chen and T. Tang [6] to the Abel-Volterra integral equations with $\mu=\frac{1}{2}$. We also provide error estimate under a more general regularity assumption on the exact solution.

Keywords: Jacobi Spectral collocation method, Abel-Volterra integral equation, Convergence analysis.

1. INTRODUCTION.

Consider linear Abel-Volterra integral equations of the second kind:

$$u(t)=g(t)+\int_0^t (t-s)^{\frac{-1}{2}} K(t,s)u(s)ds, \qquad t\in I \tag{1.1}$$

where $I=[0,\ T]$, $g\in C(I)$ is given, $u(t)\in C(I)$ is unknown and $K\in C(I\times I)$ is given with $K(t,\ t)\neq 0$ for $t\in I$.

The numerical treatment of Abel-Volterra integral equation (1.1) is not simple, mainly due to the fact that the solutions of (1.1) usually have a weak singularity at $t=0$, even when the inhomogeneous term $g(t)$ is regular. For any positive integer *m*, if both *g* and *K* have continuous derivatives of order *m*, as discussed in [4], there exists a function $Z = Z(t, v)$ possessing continuous derivatives of order *m*, such that the solution of (1.1) can be written as $u(t) = Z\left(t, \sqrt{t}\right)$. As this will be the standing point of this paper, the detailed regularity result of (1.1) is given below.

Lemma 1. Assume that $g \in C^m(I)$ and $K \in C^m(I \times I)$ with $K(t, t) \neq 0$ on $I = [0, T]$. Then, the regularity of the unique solution of the weakly singular Abel-Volterra integral equation (1.1) can be described by

$u \in C^m(0,T] \cap C(I)$ with $|u'(t)| \leq Ct^{\frac{-1}{2}}$ for $t \in (0,T]$;

$u(t) = \sum_{(j,k)} \gamma_{j,k} t^{j+\frac{k}{2}} + U_m(t), \quad t \in I$

where $(j, k) := \left\{(j, k) : j, k \text{ are non-negative integers, } j + \frac{k}{2} < m\right\}$, $\gamma_{j,k}$ are some constants, and $U_m(\cdot) \in C^m(I)$.

The above result implies that near $t = 0$ the m-th derivative of the solution $u(t)$ behaves like $u^{(m)}(t) \sim t^{\frac{1}{2}-m}$, which indicates that $u \notin C^m(I)$. Several methods have been proposed to recover high order convergence properties for (1.1) using collocation type methods, see, e.g., [3,8,22,23] and using multi-step method, see, e.g.,[13]. We point out that for (1.1) without the singular kernel, spectral methods and the corresponding error analysis have been provided recently [24]; see also [1,2,25] for spectral methods to pantograph type delay differential equations. In both cases, the underlying solutions are smooth.

We will first mention the difference between this work and the one in [7] where a Jacobi spectral collocation method is proposed for the weakly singular Volterra integral equations. In [7], to handle the non-smoothness of the underlying solutions, both function transformation and variable transformation are used. However, it is found that the function transformation (see also [8]) generally makes the resulting equations and the corresponding approximations more complicated. The present approach only makes one transformation, i.e., coordinate transformation, but not the transformation for the unknown function. In [6], Chen Y and Tang T, have studied convergence analysis of the Jacobi Spectral-Collocation method for Volterra integral equations with the singular kernel $K(t, s) = (t - s)^{-\mu}$ and $0 < \mu < 1/2$ under the assumption that the underlying solution is smooth. Note that $0<\mu<1/2$ means that the Abel type kernel is not included.

In this work, we will consider the case, that the exact solutions of (1.1) are non-smooth. This case may occur when the source function *g* in (1.1) is smooth; see, e.g., [4, Theorem 6.1.11]. In this case, although the Jacobi-collocation spectral method can be implemented in a straightforward manner, the relevant polynomial approximation theory cannot be employed directly to obtain the desired convergence results, see. e.g.. [21]. However, for the case of Abel kernel in (1.1) we can overcome this difficulty by taking a simple variable transformation so that the resulting equation possesses a smooth solution. With a more elegant proof, we will not only extend the convergence analysis in [6] to the Abel kernel type but also establish the error estimates under a more general regularity assumption on the exact solution of (1.1).

2. JACOBI-COLLOCATION METHOD

Let $\Lambda=[-1,\,1]$. In order to discretize problem (1.1), we denote the polynomial space of degree less than or equal to *N by* $\mathscr{P}_N$. As defined in [5], let us denote $J_N^{\alpha,\,\beta}(x)$, the Jacobi polynomial of degree *N* with respect to the weight function

$$w^{\alpha,\,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta},\quad -1<\alpha,\beta<1.$$

Let $x_i^{\alpha,\,\beta}$ be the points of the Gauss-Jacobi quadrature formula, defined by

$$J_{N+1}^{\alpha,\,\beta}\left(x_i^{\alpha,\,\beta}\right)=0,\quad i=0,\,...,\,N,$$

arranged in increasing order: $x_0^{\alpha,\,\beta}<x_1^{\alpha,\,\beta}<...<x_N^{\alpha,\,\beta}$. The associated weights of the Gauss-Jacobi quadrature formula are denoted by $w_i^{\alpha,\,\beta}$, $0\le i\le N$. By variable transformation

$$t=z^2,\;\; z=\sqrt{t},\;\; s=w^2,\;\; w=\sqrt{s}$$

we change the weakly singular problem (1.1) as follows:

$$\bar{y}(z)=\bar{f}(z)+\int_0^z(z-w)^{-1/2}\,\bar{K}(z,\,w)\bar{y}(w)\,dw,\quad z\in\left[0,\,\sqrt{T}\right], \tag{2.1}$$

where

$$\bar{f}(z)=g\left(z^2\right),\quad \bar{K}(z,\,w)=2(z+w)^{-1/2}\,wK\left(z^2,\,w^2\right)\quad\text{and}\quad \bar{y}(z)=u\left(z^2\right)$$

is the smooth solution of equation (2.1). For the sake of applying the theory of orthogonal polynomial conveniently, by the linear transformation

$$z=\frac{\sqrt{T}\,(1+x)}{2},\quad w=\frac{\sqrt{T}\,(1+t)}{2},$$

letting

$$y(x)=\bar{y}\left(\frac{\sqrt{T}\,(1+x)}{2}\right),\ f(x)=\bar{f}\left(\frac{\sqrt{T}\,(1+x)}{2}\right),$$

the weakly singular problem (2.1) can be rewritten as follows:

$$y(x)=f(x)+\int_{-1}^{x}(x-t)^{-1/2}\,\tilde{K}(x,\ \mathrm{t})\,y(t)\,dt,\quad x\in\Lambda \tag{2.2}$$

where

$$\tilde{K}(x,\ \mathrm{t})=\left(\frac{\sqrt{T}}{2}\right)^{1/2}\bar{K}\left(\frac{\sqrt{T}}{2}(1+x),\ \frac{\sqrt{T}}{2}(1+t)\right).$$

The Jacobi-Collocation method for equation (2.2) is to find $y_N\in\mathscr{P}_N$. such that for all $0\le i\le N,$

$$y_N\left(x_i^{\alpha,\,\beta}\right)=f\left(x_i^{\alpha,\,\beta}\right)+\int_{-1}^{x_i^{\alpha,\,\beta}}\left(x_i^{\alpha,\,\beta}-t\right)^{-1/2}\tilde{K}\left(x_i^{\alpha,\,\beta},t\right)y_N(t)\,dt,\ 0\le i\le N. \tag{2.3}$$

In order to obtain the higher-order accuracy for the Volterra integral equations problem, the main difficulty is to compute the integral term in (2.3). We rewrite the integral term in (2.3) into the form:

$$\int_{-1}^{x_i^{\alpha,\,\beta}}\left(x_i^{\alpha,\,\beta}-t\right)^{-1/2}\tilde{K}\left(x_i^{\alpha,\,\beta},t\right)y_N(t)\,dt$$
$$=\int_{-1}^{1}(1-\theta)^{-1/2}K_1\left(x_i^{\alpha,\,\beta},t_i(\theta)\right)y_N\left(t_i(\theta)\right)d\theta$$

by the linear variable transformation

$$t=t_i(\theta)=\frac{1+x_i^{\alpha,\,\beta}}{2}\theta+\frac{x_i^{\alpha,\,\beta}-1}{2},\quad \theta\in\Lambda$$

where

$$K_1\left(x_i^{\alpha,\,\beta},\ \mathrm{t}_i(\theta)\right)=\left(\frac{1+x_i^{\alpha,\,\beta}}{2}\right)^{1/2}\tilde{K}\left(x_i^{\alpha,\,\beta},\ \mathrm{t}_i(\theta)\right)$$

We discretize the integral term by the Gauss quadrature formula relative to the Jacobi weight function $\rho(x) = w^{-\frac{1}{2},\, 0}(x)$, and therefore, the full collocation scheme (with numerical integration) becomes

$$y_N\left(x_i^{\alpha,\,\beta}\right) = f\left(x_i^{\alpha,\,\beta}\right) + \sum_{k=0}^{N} K_1\left(x_i^{\alpha,\,\beta},\, t_i(\theta_k)\right) y_N\left(t_i(\theta_k)\right)\rho_k,\ 0 \le i \le N, \tag{2.4}$$

where

$$\theta_k = x_k^{-\frac{1}{2},\, 0},\ \ \rho_k = w_k^{-\frac{1}{2},\, 0},\quad k = 0,\, 1,\, ...,\, N.$$

Since the exact solution of problem (1.1) can be written as $u(t) = y(x)$, we can define $u_N(t) = y_N(x),\ t \in I,\ x \in \Lambda,$ as the approximated solution of problem(1.1).

Throughout this paper, C will denote a positive constant that is independent of N but which will depend on the length T and on bounds for the given functions $f,\ \tilde{K}$.

3. SOME USEFUL LEMMAS

Let $L^2_{w^{\alpha,\,\beta}}(\Lambda)$ be the space of measurable functions whose square is Lebesgue integrable in Λ relative to the weight function $w^{\alpha,\,\beta}(x)$. The inner product and norm of $L^2_{w^{\alpha,\,\beta}}(\Lambda)$ are defined by

$$(u,\, v)_{w^{\alpha,\,\beta},\,\Lambda} = \int_\Lambda u(x)v(x)w^{\alpha,\,\beta}(x)dx,\ \ \forall\ u, v \in L^2_{w^{\alpha,\,\beta}}(\Lambda)$$

$$\|u\|_{w^{\alpha,\,\beta},\,\Lambda} = \sqrt{(u,\, u)_{w^{\alpha,\,\beta},\,\Lambda}}$$

For a non-negative integer m, define

$$H^m_{w^{\alpha,\beta}}(\Lambda) := \left\{v:\ \|v\|_{m,w^{\alpha,\,\beta}} < \infty\right\}$$

with

$$|v|_{m,\,w^{\alpha,\,\beta}} = \left\|\partial_x^m v\right\|_{w^{\alpha,\,\beta}},\quad \|v\|_{m,\,w^{\alpha,\,\beta}} = \left(\sum_{k=0}^{m} |v|^2_{k,\,w^{\alpha,\,\beta}}\right)^{1/2}.$$

Particularly, let $w^c(x) = w^{-\frac{1}{2},\,-\frac{1}{2}}(x)$ be the Chebyshev weight function.

In bounding from the above approximation error, only some of the $L^2_{w^{\alpha,\beta}}$- norms appearing on the right-hand side of above norm enter into play. Thus, it is convenient to introduce the semi- norms

$$|v|_{H^{m;N}_{w^{\alpha,\beta}}(\Lambda)} = \left(\sum_{k=\min(m,\,N+1)}^{m} \left\| \partial_x^k v \right\|^2_{L^2_{w^{\alpha,\beta}}(\Lambda)} \right)^{\frac{1}{2}}.$$

Hereafter, in cases where no confusion would arise, the domain symbol Λ may be dropped from the notations.

In the purpose of carrying out an error analysis to the spectral collocation method, we introduce two approximation operators as follows. First, we define the Lagrange interpolation operator $I_N^{\alpha,\beta} : \mathscr{C}(\Lambda) \to \mathscr{P}_N(\Lambda)$, by $\forall v \in \mathscr{C}(\Lambda), I_N^{\alpha,\beta} v \in \mathscr{P}_N(\Lambda)$, such that

$$I_N^{\alpha,\beta} v\left(x_i^{\alpha,\beta}\right) = v\left(x_i^{\alpha,\beta}\right), \quad 0 \le i \le N,$$

see, e.g., [5,20]. The Lagrange interpolating polynomial can be written in the form

$$I_N^{\alpha,\beta} v(x) = \sum_{i=0}^{N} v\left(x_i^{\alpha,\beta}\right) h_i(x),$$

where $h_i(x)$ is the Lagrange interpolation basis function associated with $x_i^{\alpha,\beta}$.

Then, for all $v \in H^m_{w^{\alpha,\beta}}(\Lambda),\ m \ge 1,$ the following optimal error estimates hold (see [5]):

$$\left\| v - I_N^{\alpha,\beta} v \right\|_{w^{\alpha,\beta}} \le CN^{-m} |v|_{H^{m;N}_{w^{\alpha,\beta}}}. \tag{3.1}$$

Next, we introduce a discrete inner product. For any $u,\ v \in \mathscr{C}(\Lambda)$, define

$$(u,\ v)_N = \sum_{i=0}^{N} u\left(x_i^{\alpha,\beta}\right) v\left(x_i^{\alpha,\beta}\right) w_i^{\alpha,\beta}. \tag{3.2}$$

For Gauss-Jacobi quadrature formula, the error estimate is well known [5]: $\forall\ \phi \in \mathscr{P}_N$,

$$\left|(v,\ \phi)_{w^{\alpha,\ \beta}} - (v,\ \phi)_N\right| \le CN^{-m} |v|_{H^{m;\ N}_{w^{\alpha,\ \beta}}} \|\phi\|_{w^{\alpha,\ \beta}},\ v \in H^m_{w^{\alpha,\ \beta}}(\Lambda),\ m \ge 1. \quad (3.3)$$

From [14], we have the following result on the Lebesgue constant for the Lagrange interpolation polynomials associated with the zeros of the Jacobi polynomials.

Lemma 2. Let $\{h_j(x)\}_{j=0}^N$ be the N-th Lagrange interpolation polynomials associated with the Gauss points of the Jacobi polynomials. Then

$$\left\|I_N^{\alpha,\ \beta}\right\|_\infty := \max_{x\in\Lambda} \sum_{j=0}^N \left|h_j(x)\right| = \begin{cases} O(\log N), & -1 < \alpha,\ \beta \le -\frac{1}{2} \\ O\left(N^{\gamma+\frac{1}{2}}\right), & \gamma = \max(\alpha,\ \beta),\ \text{otherwise.} \end{cases} \quad (3.4)$$

As demonstrated in [7], we have the following result.

Lemma 3. Suppose that $L \ge 0,\ 0 < \mu < 1,$ and $v(x)$ is a non-negative, locally integrable function defined on Λ satisfying

$$u(x) \le v(x) + L\int_{-1}^x (x-t)^{-\mu} u(t)\,dt.$$

Then there exists a constant $C = C(\mu)$ such that

$$u(x) \le v(x) + CL\int_{-1}^x (x-t)^{-\mu} v(t)\,dt, \quad -1 \le x < 1.$$

From now on, for $r \ge 0$ and $k \in [0,\ 1]$, $\mathscr{C}^{r,\ k}(\Lambda)$ will denote the space of functions whose r-th derivates are Holder continuous with exponent k, endowed with the usual norm:

$$\|v\|_{r,\ k} = \max_{0\le k\le r} \max_{x\in\Lambda} \left|\partial_x^k v(x)\right| + \max_{0\le k\le r} \sup_{\substack{x,\ y\in\Lambda \\ x\ne y}} \frac{\left|\partial_x^k v(x) - \partial_x^k v(y)\right|}{|x-y|^k}.$$

When $k = 0$, $\mathscr{C}^{r,\ 0}(\Lambda)$ denotes the space of functions with r continuous derivatives on Λ, which is also commonly denoted by $\mathscr{C}^r(\Lambda)$, and with nor$\|.\|_r$.

We shall make use of a result of Ragozin [17, 18] (see also [10]), which states that, for non-negative integer r and $k \in (0,\ 1)$, there exists a constant $C_{r,\ k} > 0$ such that for any function $v \in \mathscr{C}^{r,\,k}(\Lambda)$, there exists a polynomial function $\mathscr{T}_N v \in \mathscr{P}_N$. such that $\|v - \mathscr{T}_N v\|_\infty \le C_{r,\,k} N^{-(r+k)} \|v\|_{r,\,k}$. (3.5)

Actually, as stated in [17,18], $\mathscr{T}_N$ is a linear operator from $\mathscr{C}^{r,\,k}(\Lambda)$ into $\mathscr{P}_N$.

We further define a linear, weakly singular integral operator $\mathscr{M}$:

$$\mathscr{M} v = \int_{-1}^{x} (x-t)^{-1/2} \tilde{K}(x,\, \mathrm{t}) v(t)\, dt. \tag{3.6}$$

Below we will show that $\mathscr{M}$ is compact as an operator from $\mathscr{C}(\Lambda)$ to $\mathscr{C}^{0,\,k}(\Lambda)$ provided that the index *k* satisfies $0 < k < 1/2$. The proof of the following lemma can be found in [7].

Lemma 4. Let $\mathscr{M}$ be defined by (3.6). Then, for any function $v \in \mathscr{C}(\Lambda)$, there exists a positive constant *C*, which is dependent on $\|\tilde{K}\|_{0,\,k}$, such that

$$\|\mathscr{M} v\|_{0,\,k} \le C\|v\|_\infty, \quad 0 < k < \frac{1}{2}, \tag{3.7}$$

where $\|\cdot\|_\infty$ the standard norms in $\mathscr{C}(\Lambda)$.

4. CONVERGENCE ANALYSIS

The objective of this section is to analyze the approximation scheme (2.4). First, we derive the error estimate in L^∞ norm of the Jacobi collocation method.

4.1 Error Estimate in L^∞

***Theorem* 1.** Let u be the exact solution to the Volterra integral equation (2.2), which is assumed to be sufficiently smooth. Let the approximated solution u_N be obtained by using the spectral collocation scheme (2.4). If $u \in H^m_{w^{\alpha,\,\beta}}(\Lambda) \cap H^m_{w^c}(\Lambda), (m \ge 1)$

then

$\|u - u_N\|_\infty$

$$\leq \begin{cases} CN^{\frac{1}{2}-m}\log N\left(|u|_{H^{m;\,N}_{w^c}} + N^{-1/2}K^{*}\|u\|_{\infty}\right), & -1<\alpha,\ \beta\leq -\frac{1}{2}, \\ CN^{1+\gamma-m}\left(|u|_{H^{m;\,N}_{w^c}} + N^{-1/2}K^{*}\|u\|_{\infty}\right), & \gamma=\max\{\alpha,\ \beta\}<0, \end{cases} \tag{4.1}$$

for N sufficiently large, where

$$K^{*} = \max_{0\leq i\leq N}\left|K_1\left(x_i^{\alpha,\,\beta},\ \mathrm{t}_i(\cdot)\right)\right|_{H^{m;\,N}_{\rho}}. \tag{4.2}$$

Proof **:** First, we use the weighted inner product to rewrite (2.2) as

$$u\left(x_i^{\alpha,\,\beta}\right)=f\left(x_i^{\alpha,\,\beta}\right)+\left(K_1\left(x_i^{\alpha,\beta},t(.)\right),u\left(t_i(.)\right)\right)_{\rho},\ 0\leq i\leq N. \tag{4.3}$$

By using the discrete inner product (3.2), the numerical scheme (2.4) can be written as

$$u_N\left(x_i^{\alpha,\,\beta}\right)=f\left(x_i^{\alpha,\,\beta}\right)+\left(K_1\left(x_i^{\alpha,\,\beta},\ \mathrm{t}_i(\cdot)\right),\ u_N\left(t_i(\cdot)\right)\right)_N,\ 0\leq i\leq N. \tag{4.4}$$

Subtracting (4.4) from (4.3) gives

$$u\left(x_i^{\alpha,\,\beta}\right)-u_N\left(x_i^{\alpha,\,\beta}\right)=\left(K_1\left(x_i^{\alpha,\,\beta},\ \mathrm{t}_i(\cdot)\right),\ e\left(t_i(\cdot)\right)\right)_{\rho}+I_{i,\,2}$$

$$=\int_{-1}^{x_i^{\alpha,\,\beta}}\left(x_i^{\alpha,\,\beta}-t\right)^{-1/2}\tilde{K}\left(x_i^{\alpha,\,\beta},\ \mathrm{t}\right)e(t)\,dt+I_{i,\,2} \tag{4.5}$$

for $0\leq i\leq N$, where $e(x)=u(x)-u_N(x)$ is the error function, and

$$I_{i,2}=\left(K_1\left(x_i^{\alpha,\,\beta},\ \mathrm{t}_i(\cdot)\right),\ u_N\left(t_i(\cdot)\right)\right)_{\rho}-\left(K_1\left(x_i^{\alpha,\,\beta},\ \mathrm{t}_i(\cdot)\right),\ u_N\left(t_i(\cdot)\right)\right)_N.$$

Multiplying $h_i(x)$ on both sides of the error equation (4.5) and summing up from $i=0$ to $i=N$
yield

$$I_N^{\alpha,\,\beta}u-u_N=I_N^{\alpha,\,\beta}\left(\int_{-1}^{x}(x-t)^{-1/2}\tilde{K}(x,\ \mathrm{t})e(t)\,dt\right)+\sum_{i=0}^{N}I_{i,\,2}h_i(x). \tag{4.6}$$

Consequently

$$e(x)=\int_{-1}^{x}(x-t)^{-1/2}\tilde{K}(x,\ \mathrm{t})e(t)\,dt+I_1+I_2+I_3, \tag{4.7}$$

where

$$I_1 = u - I_N^{\alpha,\,\beta}u,\; I_2 = \sum_{i=0}^{N} I_{i,\,2}h_i\left(x\right),$$

$$I_3 = I_N^{\alpha,\,\beta}\left(\int_{-1}^{x}\left(x-t\right)^{-1/2}\tilde{K}\left(x,\,\mathrm{t}\right)e\left(t\right)dt\right) - \int_{-1}^{x}\left(x-t\right)^{-1/2}\tilde{K}\left(x,\,\mathrm{t}\right)e\left(t\right)dt.$$

It follows from the Gronwall inequality in Lemma 3 that

$$\left\|e\right\|_\infty \le C\left(\left\|I_1\right\|_\infty + \left\|I_2\right\|_\infty + \left\|I_3\right\|_\infty\right). \tag{4.8}$$

Let $I_N^c u \in \mathscr{P}_N$ denote the interpolant of u at the Chebyshev Gauss points. From [5, (5.5.28)], the interpolation error estimate in the maximum norm is given by

$$\left\|u - I_N^c u\right\|_\infty \le CN^{\frac{1}{2}-m}\left|u\right|_{H_{w^c}^{m;\,N}}. \tag{4.9}$$

By using (4.9), Lemma 2, and noting that

$$I_N^{\alpha,\,\beta}p\left(x\right) = p\left(x\right),\;\; \forall\, p\left(x\right) \in \mathscr{P}_N,$$

we obtain

$$\begin{aligned}
\left\|I_1\right\|_\infty &= \left\|u - I_N^{\alpha,\,\beta}u\right\|_\infty \\
&= \left\|u - I_N^c u + I_N^{\alpha,\,\beta}\left(I_N^c u\right) - I_N^{\alpha,\,\beta}u\right\|_\infty \\
&\le \left\|u - I_N^c u\right\|_\infty + \left\|I_N^{\alpha,\,\beta}\left(I_N^c u - u\right)\right\|_\infty \\
&\le \left(1 + \left\|I_N^{\alpha,\beta}\right\|_\infty\right)\left\|u - I_N^c u\right\|_\infty \\
&\le \begin{cases} CN^{\frac{1}{2}-m}\log N\left|u\right|_{H_{w^c}^{m;\,N}}, & -1 < \alpha,\; \beta \le -\frac{1}{2}, \\ CN^{1+\gamma-m}\left|u\right|_{H_{w^c}^{m;\,N}}, & \gamma = \max\left\{\alpha,\; \beta\right\},\ \text{otherwise}. \end{cases}
\end{aligned} \tag{4.10}$$

Next, using the integration error estimates (3.3) for Jacobi-Gauss polynomials quadrature, we have

$$\max_{0\le i\le N}\left|I_{i,\,2}\right|\le CN^{-m}\max_{0\le i\le N}\left|K_1\left(x_i^{\alpha,\,\beta},\ \mathrm{t}_i(\cdot)\right)\right|_{H_\rho^{m;\,N}}\max_{0\le i\le N}\left\|u_N\left(t_i(\cdot)\right)\right\|_\rho$$

$$\le CN^{-m}K^*\left(\|e\|_\infty+\|u\|_\infty\right), \tag{4.11}$$

where K^* is defined as in (4.2). Hence , by combining with Lemma 2,

yields

$$\|I_2\|_\infty=\left\|\sum_{i=0}^{N}I_{i,\,2}h_i(x)\right\|_\infty$$

$$\le C\max_{0\le i\le N}\left|I_{i,\,2}\right|\max_{x\in\Lambda}\sum_{j=0}^{N}\left|h_j(x)\right|$$

$$\le\begin{cases}CN^{-m}\log NK^*\left(\|e\|_\infty+\|u\|_\infty\right), & -1<\alpha,\ \beta\le-\frac{1}{2},\\ CN^{\frac{1}{2}+\gamma-m}K^*\left(\|e\|_\infty+\|u\|_\infty\right), & \gamma=\max\{\alpha,\ \beta\},\ \text{otherwise},\end{cases} \tag{4.12}$$

for sufficiently large *N*.

We now estimate the third term I_3. It follows from (3.5), Lemmas 2 and 4 that

$$\|I_3\|_\infty\ =\ \left\|\left(I_N^{\alpha,\,\beta}-I\right)\mathscr{M}e\right\|_\infty$$

$$=\ \left\|\left(I_N^{\alpha,\,\beta}-I\right)\left(\mathscr{M}e-\mathscr{T}_N\mathscr{M}e\right)\right\|_\infty$$

$$\le\left(1+\left\|I_N^{\alpha,\beta}\right\|_\infty\right)\left\|\mathscr{M}e-\mathscr{T}_N\mathscr{M}e\right\|_\infty$$

$$\le\ C\left(1+\left\|I_N^{\alpha,\,\beta}\right\|_\infty\right)N^{-k}\ \left\|\mathscr{M}e\right\|_{0,\,k}$$

$$\le\ \begin{cases}CN^{-k}\log N\|e\|_\infty, & -1<\alpha,\ \beta\le-\frac{1}{2},\\ CN^{\frac{1}{2}+\gamma-k}\|e\|_\infty, & \gamma=\max\{\alpha,\ \beta\}\ \text{,otherwise},\end{cases} \tag{4.13}$$

where in the last step, we have used Lemma 4 under the condition $0<k<1/2$.

It is clear that

$$\|I_3\|_\infty \le \frac{1}{3}\|e\|_\infty \tag{4.14}$$

under the following assumption

$$\begin{cases} 0 < k < \frac{1}{2}, & -1 < \alpha,\ \beta \le \frac{1}{2}, \\ \frac{1}{2} + \gamma < k < \frac{1}{2}, & \gamma = \max\{\alpha,\ \beta\} < 0, \end{cases} \tag{4.15}$$

provided that N is sufficiently large. Combining (4.8), (4.10), (4.12) and (4.14) gives the desired estimate (4.1).

4.2. Error Estimate in Weighted L^2 Norm

To prove the error estimate in weighted L^2 norm, we need the generalized Hardy's inequality with weights (see, e.g., [9, 12, 19]).

Lemma 5. For all measurable function $f \ge 0,$ the generalized Hardy's inequality

$$\left(\int_a^b |(Tf)(x)|^q u(x)dx\right)^{1/q} \le C\left(\int_a^b |f(x)|^p v(x)dx\right)^{1/p}$$

holds if and only if

$$\sup_{a<x<b}\left(\int_x^b u(t)dt\right)^{1/q}\left(\int_a^x v^{1-p'}(t)dt\right)^{1/p'} < \infty, \quad p' = \frac{p}{p-1},$$

for the case $1 < p \le q < \infty.$ Here, T is an operator of the form $(Tf)(x) = \int_a^x k(x,\ t) f(t)dt$

with $k(x,\ t)$ a given kernel, u and v weight functions and $-\infty \le a < b \le \infty.$

From [14, Theorem 1], we have the following weighted mean convergence result of Lagrange interpolation based at the zeros of Jacobi polynomials.

Lemma 6. For every bounded function $v(x),$ there exists a constant C independent of v such that

$$\sup_N \left\| \sum_{j=0}^N v(x_j) h_j(x) \right\|_{L^2_{w^{\alpha,\ \beta}}(\Lambda)} \le C\|v\|_\infty,$$

where $h_i(x)$ is the Lagrange interpolation basis function associated with the Jacobi collocation points $x_i^{\alpha,\beta}$, $i = 0, 1, ..., N$.

***Theorem* 2.** Let u be the exact solution to the Volterra integral equation(2.2), which is assumed to be sufficiently smooth. Let the approximated solution u_N be obtained by using the spectral collocation scheme (2.4). Assume that

$u \in H^m_{w^{\alpha,\beta}}(\wedge) H^m_{w^c}(\wedge), m \geq 1.$

Then for *N* sufficiently large,

$$\|u-u_N\|_{w^{\alpha,\beta}} \leq \begin{cases} CN^{-m}\left(U_2 + N^{\frac{1}{2}-k}\log NU_1\right), & -1<\alpha,\ \beta \leq -\frac{1}{2}, \\ CN^{-m}\left(U_2+N^{1+\gamma-k}U_1\right), & \gamma = \max\{\alpha,\ \beta\}<0, \end{cases} \tag{4.16}$$

for any *k* satisfying (4.15), where

$$U_1 = K^*|u|_{H^{m;\,N}_{w^c}},\quad U_2 = |u|_{H^{m;\,N}_{w^{\alpha,\beta}}} + K^*\|u\|_\infty, \tag{4.17}$$

and K^* is defined by (4.2).

Proof **:** By using the generalization of Gronwall's inequality in Lemma 3. It follows from(4.7)that

$$e(x) \leq C\int_{-1}^{x}(x-t)^{-1/2}\tilde{K}(x,\ \text{t})(I_1+I_2+I_3)(t)dt + I_1+I_2+I_3. \tag{4.18}$$

By the generalized Hardy's inequality in Lemma 5, we obtain that

$$\|e\|_{w^{\alpha,\beta}} \leq C\left\|\int_{-1}^{x}(x-t)^{-1/2}\tilde{K}(x,\ \text{t})(I_1+I_2+I_3)(t)dt\right\|_{w^{\alpha,\beta}}$$

$$+C\left(\|I_1\|_{w^{\alpha,\beta}} + \|I_2\|_{w^{\alpha,\beta}} + \|I_3\|_{w^{\alpha,\beta}}\right)$$

$$\leq C\left(\|I_1\|_{w^{\alpha,\beta}} + \|I_2\|_{w^{\alpha,\beta}} + \|I_3\|_{w^{\alpha,\beta}}\right).$$

Now , by applying (3.1), we obtain that

$$\|I_1\|_{w^{\alpha,\beta}} = \left\|u - I_N^{\alpha,\beta}u\right\|_{w^{\alpha,\beta}} \leq CN^{-m}|u|_{H^{m;\,N}_{w^{\alpha,\beta}}}. \tag{4.19}$$

By using Lemma 6 and (4.11), we have

$$\|I_2\|_{w^{\alpha,\beta}} = \left\|\sum_{i=0}^{N} I_{i,\,2} h_i(x)\right\|_{w^{\alpha,\beta}} \le C \max_{0\le i\le N}\left|I_{i,\,2}\right| \le CN^{-m}K^*\left(\|e\|_\infty + \|u\|_\infty\right). \quad (4.20)$$

Finally , it follows from Lemmas 6, 4, and (3.5) that

$$\begin{aligned}\|I_3\|_{w^{\alpha,\beta}} &= \left\|\left(I_N^{\alpha,\beta} - I\right)\mathscr{M}e\right\|_{w^{\alpha,\beta}} \\ &= \left\|\left(I_N^{\alpha,\beta} - I\right)\left(\mathscr{M}e - \mathscr{T}_N\mathscr{M}e\right)\right\|_{w^{\alpha,\beta}} \\ &\le C\left\|\mathscr{M}e - \mathscr{T}_N\mathscr{M}e\right\|_\infty \\ &\le CN^{-k}\ \|e\|_\infty \qquad (4.21)\end{aligned}$$

where, in the last step, we used Lemma 4 for any $k \in (0,\ 1/2)$.

By the convergence result in Theorem 1, we obtain that

$$\|I_3\|_{w^{\alpha,\beta}}$$

$$\le \begin{cases} CN^{\frac{1}{2}-m-k}\log N\left(|u|_{H^{m;\,N}_{w^c}} + N^{-1/2}K^*\|u\|_\infty\right), & -1<\alpha,\ \beta\le -\frac{1}{2}, \\ CN^{1+\gamma-m-k}\left(|u|_{H^{m;\,N}_{w^c}} + N^{-1/2}K^*\|u\|_\infty\right), & \gamma = \max\{\alpha,\ \beta\} < 0, \end{cases} \quad (4.22)$$

for N sufficiently large and any k satisfying (4.15). The desired estimate (4.16) is obtained by combining (4.19), (4.20), and (4.22).

Finally, we can derive the main result of this chapter, i.e., the error estimates for the numerical solutions to the Abel-Volterra integral equation (1.1).

***Theorem* 3.** Let y and y_N be the exact solution and approximated solution of the Volterra integral equation (1.1), respectively. If the given data $g(t)$ and $K(t,\ s)$ in(1.1) belong to $C^m(I)$, then

$$\|y - y_N\|_{L^\infty(I)}$$

$$\leq \begin{cases} CN^{\frac{1}{2}-m} \log N \left(\left| y(t(\cdot)) \right|_{H_{w^c}^{m;N}} + N^{-1/2} K^* \left\| y(t(\cdot)) \right\|_\infty \right), & -1 < \alpha,\ \beta \leq -\frac{1}{2}, \\ CN^{1+\gamma-m} \left(\left| y(t(\cdot)) \right|_{H_{w^c}^{m;N}} + N^{-1/2} K^* \left\| y(t(\cdot)) \right\|_\infty \right), & \gamma = \max\{\alpha,\ \beta\} < 0, \end{cases}$$

and

$$\left\| y - y_N \right\|_{w^{\alpha,\beta}} \leq \begin{cases} CN^{-m} \left(U_2 + N^{\frac{1}{2}-k} \log N U_1 \right), & -1 < \alpha,\ \beta \leq -\frac{1}{2}, \\ CN^{-m} \left(U_2 + N^{1+\gamma-k} U_1 \right), & \gamma = \max\{\alpha,\ \beta\} < 0, \end{cases}$$

for any k satisfying (4.15), where

$$U_1 = K^* \left| y(t(\cdot)) \right|_{H_{w^c}^{m;N}}, \quad U_2 = \left| y(t(\cdot)) \right|_{H_{w^{\alpha,\beta}}^{m;N}} + K^* \left\| y(t(\cdot)) \right\|_\infty,$$

and K^* is defined by (4.2).

Remark 1. In this chapter, we consider the spectral collocation methods based on the Jacobi-Gauss points corresponding to the weights $w^{\alpha,\beta}$, $-1 < \alpha,\ \beta < 0$. The reason for this consideration is that, we employ Lemmas 2 and 4 to estimate the bound for $\|I_3\|_\infty$ in equation (4.13). However, this is not the only case for the convergence results to hold. For example, in [11] for the special Chebyshev

weight

$$\alpha = \beta = -\frac{1}{2},$$

Lemma 2 holds not only for the Gauss points but also for the Gauss-Lobatto points, i.e., the bound for Lebesgue constant $\left\| I_N^{CGL} \right\|_\infty = O(\log N)$, where I_N^{CGL} based on the Chevbyshev-Gauss-Lobatto points. This means that similar convergence results in Theorems 1 and 2 also hold for the Chebyshev-Gauss- Lobatto points.

5 . NUMERICAL EXPERIMENTS

5.1. Implementation

Here, we choose to use the Lagrangian polynomials as a basis of the approximation spaces. Let $\{h_i : i = 0, \dots, N\}$ be the Lagrangian polynomials associated with Gauss-Jacobi points $\left\{x_i^{\alpha,\beta} : i = 0, \dots, N\right\}$. That is $h_i(x) \in \mathscr{P}_N(\Lambda)$, such that $h_i\left(x_k^{\alpha,\beta}\right) = \delta_{ik}$, where δ denotes the Kronecker function. It is seen that the set $\{h_i : i = 0, \dots, N\}$ forms a basis of $\mathscr{P}_N(\Lambda)$:

$\mathscr{P}_N(\Lambda) = \operatorname{span}\{h_i(x) : i = 0, \dots, N\}.$

Let $\left\{u_j = u_N\left(x_j^{\alpha,\beta}\right)\right\}_{j=0}^{N}$. By expressing u_N in this basis

$$u_N(x) = \sum_{j=0}^{N} u_j h_j(x) \Rightarrow u_N\left(t\left(x_i^{\alpha,\beta}, \theta_k\right)\right) = \sum_{j=0}^{N} u_j h_j\left(t\left(x_i^{\alpha,\beta}, \theta_k\right)\right), \tag{5.1}$$

the scheme (2.5) is equivalent to

$$u_i = f\left(x_i^{\alpha,\beta}\right) + \sum_{j=0}^{N} u_j \left(\sum_{k=0}^{N} K_1\left(x_i^{\alpha,\beta}, t\left(x_i^{\alpha,\beta}, \theta_k\right)\right) h_j\left(t\left(x_i^{\alpha,\beta}, \theta_k\right)\right)\rho_k\right),$$
$$0 \le i \le N, \tag{5.2 }$$

and we arrive at the matrix statement of (5.2):

$$(I - A)U_N = F, \tag{5.3}$$

where $I = I_{N\times N}$ denotes the identity matrix,

$$F = \left[f\left(x_0^{\alpha,\beta}\right), \dots, f\left(x_N^{\alpha,\beta}\right)\right]^T, \; U_N = [u_0, \dots, u_N]^T,$$

and $A = \left(A_{ij}\right)_{N\times N}$ with

$$A_{ij} = \sum_{k=0}^{N} K_1\left(x_i^{\alpha,\beta}, \mathrm{t}\left(x_i^{\alpha,\beta}, \theta_k\right)\right) h_j\left(t\left(x_i^{\alpha,\beta}, \theta_k\right)\right)\rho_k.$$

Noting that (5.3) is a non symmetric system. We use Bicgstab[16] iterative methods to solve it.

5.2. Numerical Results

we present the numerical results obtained by the proposed collocation spectral method. The estimates in Theorem 3 indicates that the convergence of numerical solutions is exponential if the exact solution is smooth. To confirm the theoretical prediction, a numerical experiment is carried out by considering the following example.

Example 1. Consider the linear Abel-Volterra integral equations of the second kind

$$y(t)=b(t)-\int_0^t (t-s)^{-1/2} y(s)ds, \quad 0\le t\le T,$$

with the exact solution $y(t)=\dfrac{\sin t}{\sqrt{t}}$. By calculation, $b(t)=\dfrac{\sin t}{\sqrt{t}}+\pi \sin\dfrac{t}{2} J_0\left(\dfrac{t}{2}\right)$,

where $J_0(z)$ is the Bessel function defined by

$$J_0(z)=\sum_{k=0}^{\infty}\frac{\left(-z^2\right)^k}{(k!)^2 4^k}.$$

This problem has the property stated at the beginning of this paper, i.e., $y'(t)=\dfrac{\cos t}{\sqrt{t}}+\dfrac{\sin t}{\sqrt{t^3}} \sim \dfrac{1}{\sqrt{t}}\left(t=0^+\right)$,

which is singular at $t=0^+$. In this theory presented in the previous section our main concern is the regularity of the transformed solution. For the present problem, by employing nonlinear transformation $t=z^2$, the smooth solution

$$\bar{u}(z)=y\left(z^2\right)=\frac{\sin z^2}{z}$$

is obtained. The main purpose is to check the convergence behavior of numerical solutions with respect to the polynomial degrees *N* for several α and β.

In Fig. 1, we plot the $L^2_{w^{\alpha,\beta}}$-errors and L^∞-errors in semi-log scale. To confirm the theoretical prediction, we plot the errors as functions of the polynomial degrees *N* for $\alpha=-1/2,\ \beta=-1/2$. As expected, the errors show an exponential decay, since in this semi-log representation one observes that the error variations are essentially linear versus the degrees of polynomial. This indicates that the convergence of the spectral collocation method is exponential.

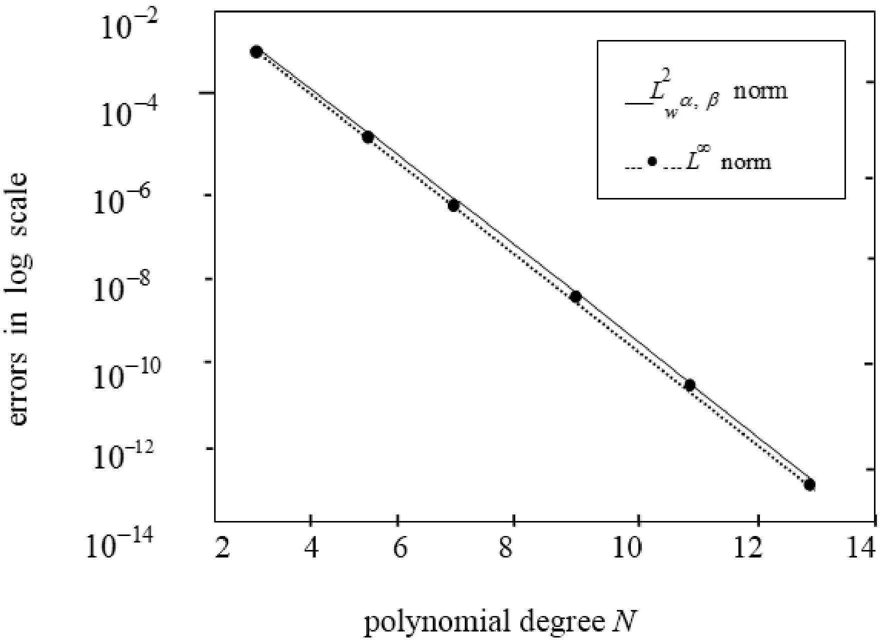

Fig.1, L^∞ and $L^2_{w^{\alpha,\beta}}$ errors versus the polynomial degree $N(\alpha=-1/2, \beta=-1/2)$

CONCLUSION

In this paper we apply Jacobi Collocation method to solve Abel-Volterra integral equations with singular kernel $K(t,s)=(t-s)^{-1/2}$. We make convergence analysis for this method. We also provide error estimate under the assumption that the exact solution is regular.

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