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# Cayley Bipolar Fuzzy Graphs Induced By Loops

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**Abstract**

In this paper, we introduce a class of Cayley bipolar fuzzy graphs induced by Loops and then study its various graph theoretic properties in terms of algebraic properties. Moreover, we study the concepts of various connectedness of this graph in terms of algebraic properties. This did not attract much attention in the literature.

**AMS subject classification:**
**Keywords:** fuzzy graph, cayley graph, vertex transitive graph, loop.

## 1. Introduction

Let $(V, *)$ be a group and $A$ be any subset of $V$. Then the Cayley graph induced by $(V, *, A)$ is the graph $G = (V, R)$, where $R = \{(x, y) : x^{-1}y \in A\}$. A fuzzy subset $\nu$ of any set $X$ is a function $\nu : X \to [0, 1]$. A *fuzzy binary relation* on $V$ is a fuzzy subset $\mu$ on $V \times V$. By a *fuzzy relation* we mean a fuzzy binary relation given by $\mu : V \times V \to [0, 1]$. Let $(V, *)$ be a group and $\nu$ be a fuzzy subset of $V$. Then the fuzzy relation $R$ defined on $V$ by $R(x, y) = \nu(x^{-1} * y)$ for all $x, y \in V$ induces a fuzzy graph $G = (V, R)$ called the *Cayley fuzzy graph* [3] induced by the triplet $(V, *, \nu)$. Let $V$ be a nonempty set. A *bipolar fuzzy set* [7] $B$ in $V$ is an object of the form $B = \{(x, \mu_B^P(x), \mu_B^N(x)) : x \in V\}$, where $\mu_B^P$ and $\mu_B^N$ are respectively the functions, $\mu_B^P : V \to [0, 1]$ and $\mu_B^N : V \to [-1, 0]$. A *bipolar fuzzy relation* [7] $R = (\mu_R^P(x, y), \mu_R^N(x, y))$ in a universe $X \times Y$ is a bipolar fuzzy set of the form $R = \{((x, y), \mu_R^P(x, y), \mu_R^N(x, y)) : (x, y) \in X \times Y\}$, where $\mu_R^P : X \times Y \to [0, 1]$ and $\mu_R^N : X \times Y \to [-1, 0]$. Let $V$ be a non empty set. A *bipolar fuzzy digraph* [4] of a digraph $(V, E)$ is a pair $(A, B)$ where $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set in $V$ and $B = (\mu_B^P, \mu_B^N)$ is a bipolar fuzzy relation on $E \subseteq V \times V$ such that $\mu_B^P(x, y) \leq min(\mu_A^P(x), \mu_A^P(y))$, $\mu_B^N(x, y) \geq max(\mu_A^N(x), \mu_A^N(y))$ for all $x, y \in V$. Let $(V, *)$ be a group and let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset of $V$. Then the bipolar fuzzy relation $R$ on $V$ defined by $R(x, y) = \{(\mu_A^P(x^{-1}y), \mu_A^N(x^{-1}y))\ \forall\, x, y \in V\}$ induces a bipolar fuzzy digraph $G = (V, R)$, called the *Cayley bipolar fuzzy graph* [6] induced by the triplet $(V, *, A)$. In [3] Madhavan Namboothiri N.M. et al. introduced a class of Cayley fuzzy graphs induced by groups and studied the properties of Cayley fuzzy graph in terms of algebraic properties. Also, N. O. Alshehri and M. Akram in [6] introduced Cayley bipolar fuzzy graphs and discussed its properties in terms of algebraic properties. In this paper, we generalise the results of N. O. Alshehri and M. Akram and prove that Cayley bipolar fuzzy graphs could be induced by loops.

## 2. Cayley Bipolar Fuzzy Graphs Induced by Loops

In this section we introduce a class of Cayley Bipolar Fuzzy graphs induced by Loops and discuss some of its basic properties.

**Definition 2.1.** Let $(V, *)$ be a loop and let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset of $V$ satisfying the condition, $\mu_A^P(y/x) = \mu_A^P(ay/ax)$ and $\mu_A^N(y/x) = \mu_A^N(ay/ax)$, for all $a, x, y \in V$, where $y/x$ denote the solution of $y = xt$ in $V$. Then this bipolar fuzzy subset of $V$ is called *scaled bipolar fuzzy subset* of the loop $(V, *)$.

**Definition 2.2.** Let $(V, *)$ be a loop and let $A = (\mu_A^P, \mu_A^N)$ be a scaled bipolar fuzzy subset of $V$. Define a bipolar fuzzy relation $R = (\mu_R^P, \mu_R^N)$ by $R(x, y) = \{(\mu_A^P(y/x), \mu_A^N(y/x)), \forall\, x, y \in V\}$. That is, $\mu_R^P(x, y) = \mu_A^P(y/x)$ and $\mu_R^N(x, y) = \mu_A^N(y/x)$. Then $G = (V, R)$ is called the *Cayley bipolar fuzzy graph* induced by the loop $(V, *)$ and is denoted by $CayF_B(V, A)$.

**Definition 2.3.** Let $(V, *)$ be a loop and let $\alpha \in [0, 1]$. For any bipolar fuzzy subset $A = (\mu_A^P, \mu_A^N)$ of $V$, $\{x : \mu_A^P(x) \geq \alpha$, and $\mu_A^N(x) \leq \alpha\}$ is called $\alpha$-cut of $A$ and $\{x : \mu_A^P(x) > \alpha$, and $\mu_A^N(x) < \alpha\}$ is called *strong* $\alpha$-cut of $A$ and are denoted respectively by $A_\alpha$ and $A_\alpha^+$.

**Theorem 2.4.** $CayF_B(V, A)$ is vertex transitive.

*Proof.* Let $a, b \in V$ and $b = z_\circ a,\ z_\circ \in V$. Define $\Psi : V \to V$ by $\Psi(x) = z_\circ x$. Clearly, $\Psi$ is a bijective map. For each $x, y \in V$,

$$\begin{aligned}
R(\Psi(x), \Psi(y)) &= \left(\mu_R^P(\Psi(x), \Psi(y)), \mu_R^N(\Psi(x), \Psi(y))\right) \\
&= \left(\mu_R^P(z_\circ x, z_\circ y), \mu_R^N(z_\circ x, z_\circ y)\right) \\
&= \left(\mu_A^P(z_\circ y/z_\circ x), \mu_A^N(z_\circ y/z_\circ x)\right) \\
&= \left(\mu_A^P(y/x), \mu_A^N(y/x)\right) \\
&= R(x, y).
\end{aligned}$$

Therefore, $R(\Psi(x), \Psi(y)) = R(x, y)$. Hence $\Psi$ is an automorphism on $CayF_B(V, A)$. Also $\Psi(a) = b$. Hence $CayF_B(V, A)$ is vertex transitive. ■

**Theorem 2.5.** $CayF_B(V, A)$ is regular.

*Proof.* Every vertex transitive bipolar fuzzy graphs are regular and $CayF_B(V, A)$ is vertex transitive. Hence $CayF_B(V, A)$ is regular. ■

### 2.1. Basic Results

**Theorem 2.6.** $CayF_B(V, A)$ is reflexive if and only if $\mu_A^P(1) = 1$ and $\mu_A^N(1) = -1$.

*Proof.* $R$ is reflexive if and only if $R(x, x) = (1, -1)$ for all $x \in V$. Now, $R(x, x) = \big(\mu_A^P(x/x), \mu_A^N(x/x)\big) = \big(\mu_A^P(1), \mu_A^N(1)\big)$. Therefore, $R$ is reflexive implies $\mu_A^P(1) = 1$ and $\mu_A^N(1) = -1$. Hence $R$ is reflexive if and only if $\mu_A^P(1) = 1$ and $\mu_A^N(1) = -1$. ■

**Theorem 2.7.** $CayF_B(V, A)$ is symmetric if and only if

$$(\mu_A^P(x), \mu_A^N(x)\big) = \big(\mu_A^p(1/x), \mu_A^N(1/x)\big), \ \ for\ all\ x \in V.$$

*Proof.* Suppose that $CayF_B(V, A)$ is symmetric. Then for any $x \in V$,

$$\begin{aligned}
\big(\mu_A^P(x), \mu_A^N(x)\big) &= \big(\mu_A^P(x^2/x), \mu_A^N(x^2/x)\big) \\
&= R(x, x^2) \\
&= R(x^2, x) \\
&= \big(\mu_A^P(x/x^2), \mu_A^N(x/x^2)\big) \\
&= (\mu_A^P(1/x), \mu_A^N(1/x)).
\end{aligned}$$

Conversely, suppose that $\big(\mu_A^P(x), \mu_A^N(x)\big) = \big(\mu_A^P(1/x), \mu_A^N(1/x)\big)$ for all $x \in V$. Then,

$$\begin{aligned}
R(x, y) &= \big(\mu_A^P(y/x), \mu_A^N(y/x)\big) \\
&= \big(\mu_A^P(xt/x), \mu_A^N(xt/x)\big) \\
&= \big(\mu_A^P(t), \mu_A^N(t)\big) \\
&= \big(\mu_A^P(1/t), \mu_A^N(1/t)\big) \\
&= \big(\mu_A^P(x/xt), \mu_A^N(x/xt)\big) \\
&= \big(\mu_A^P(x/y), \mu_A^N(x/y)\big) \\
&= R(y, x).
\end{aligned}$$

Hence $CayF_B(V, A)$ is symmetric. ■

**Theorem 2.8.** $CayF_B(V, A)$ is antisymmetric if and only if

$$\big\{x : \big(\mu_A^P(x), \mu_A^N(x)\big) = \big(\mu_A^P(1/x), \mu_A^N(1/x)\big)\big\} = \{1\}.$$

*Proof.* First let $R$ is antisymmetric.
Let $x \in \{x : (\mu_A^P(x), \mu_A^N(x)) = (\mu_A^P(1/x), \mu_A^N(1/x))\}$. Then $\mu_A^P(x) = \mu_A^P(1/x)$ and $\mu_A^N(x) = \mu_A^N(1/x)$. Therefore, $R(1, x) = R(x, 1)$ which implies $x = 1$. Hence, $\{x : (\mu_A^P(x), \mu_A^N(x)) = (\mu_A^P(1/x), \mu_A^N(1/x))\}$
$= \{1\}$.

Conversely, let $\{x : (\mu_A^P(x), \mu_A^N(x)) = (\mu_A^P(1/x), \mu_A^N(1/x))\} = \{1\}$. Then,

$$\begin{aligned} R(x, y) = R(y, x) &\Leftrightarrow (\mu_A^P(y/x), \mu_A^N(y/x)) = (\mu_A^P(x/y), \mu_A^N(x/y)) \\ &\Leftrightarrow (\mu_A^P(xt/x), \mu_A^N(xt/x)) = (\mu_A^P(x/xt), \mu_A^N(x/xt)), \\ &\quad\ t = y/x \\ &\Leftrightarrow (\mu_A^P(t), \mu_A^N(t)) = (\mu_A^P(1/t), \mu_A^N(1/t)) \\ &\Leftrightarrow t = 1 \\ &\Leftrightarrow y = x. \end{aligned}$$

Therefore, $R$ is antisymmetric. Hence the proof. ■

**Definition 2.9.** Let $(V, *)$ be a loop. Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset of $V$. Then $A$ is said to be a bipolar fuzzy sub quasigroup of $V$ if for all $x, y \in V$, $\mu_A^P(xy) \geq \mu_A^P(x) \wedge \mu_A^P(y)$ and $\mu_A^N(xy) \leq \mu_A^N(x) \vee \mu_A^N(y)$.

**Theorem 2.10.** *$CayF_B(V, A)$ is transitive if and only if $A$ is a bipolar fuzzy sub quasigroup of $V$.*

*Proof.* Suppose $R$ is transitive and let $x, y \in V$. Then $R^2 \leq R$. That is $\mu_{R^2}^P \leq \mu_R^P$ and $\mu_{R^2}^N \geq \mu_R^N$.
Now,

$$\begin{aligned} \mu_A^P(x) \wedge \mu_A^P(y) &\leq \vee\{\mu_A^P(z) \wedge \mu_A^P(xy/z) : z \in V\} \\ &= \vee\{\mu_R^P(1, z) \wedge \mu_R^P(z, xy) : z \in V\} \\ &= \mu_{R^2}^P(1, xy) \\ &\leq \mu_R^P(1, xy) \\ &= \mu_A^P(xy) \end{aligned}$$

and

$$\begin{aligned} \mu_A^N(x) \vee \mu_A^N(y) &\geq \wedge\{\mu_A^N(z) \vee \mu_A^N(xy/z) : z \in V\} \\ &= \wedge\{\mu_R^N(1, z) \wedge \mu_R^N(z, xy) : z \in V\} \\ &= \mu_{R^2}^N(1, xy) \\ &\geq \mu_R^N(1, xy) \\ &= \mu_A^N(xy). \end{aligned}$$

Therefore, $\mu_A^P(xy) \geq \mu_A^P(x) \wedge \mu_A^P(y)$ and $\mu_A^N(xy) \leq \mu_A^N(x) \vee \mu_A^N(y)$. Hence $A$ is a bipolar fuzzy sub quasigroup of $(V, *)$.

Conversely, suppose that $A$ is a bipolar fuzzy sub quasigroup of $(V, *)$. For $x, y \in V$, choose an arbitrary $z \in V$. Then there exist some $t, t_\circ, t_1 \in V$ such that $y = xt, z = xt_\circ, y = zt_1$. Then,

$$\begin{aligned}
\mu_R^P(x, y) &= \mu_A^P(y/x) \\
&= \mu_A^P((xt_\circ)t_1/x) \\
&= \mu_A^P(t_\circ t_1') \\
&\geq \mu_A^P(t_\circ) \wedge \mu_A^P(t_1') \\
&= \mu_A^P(z/x) \wedge \mu_A^P(t_\circ t_1'/t_\circ) \\
&= \mu_A^P(z/x) \wedge \mu_A^P((xt_\circ)t_1/xt_\circ) \\
&= \mu_A^P(z/x) \wedge \mu_A^P(y/z).
\end{aligned}$$

Therefore, $\mu_R^P(x, y) \geq \mu_A^P(z/x) \wedge \mu_A^P(y/z)$, for any $z \in V$. That is,

$$\begin{aligned}
\mu_R^P(x, y) &\geq \vee\{\mu_A^P(z/x) \wedge \mu_A^P(y/z) : z \in V\} \\
&= \vee\{\mu_R^P(x.z) \wedge \mu_R^P(z, y) : z \in V\} \\
&= \mu_{R^2}^P(x, y).
\end{aligned}$$

Therefore, $\mu_{R^2}^P(x, y) \leq \mu_R^P(x, y)$. Similarly, $\mu_{R^2}^N(x, y) \geq \mu_R^N(x, y)$. Hence $R = (\mu_R^P, \mu_R^N)$ is transitive. ■

**Theorem 2.11.** *$R$ is a partial order if and only if $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy sub quasigroup of $(V, *)$ satisfying:*

$$(i)\ \mu_A^P(1) = 1 \text{ and } \mu_A^N(1) = -1.$$
$$(ii)\big\{x : \big(\mu_A^P(x), \mu_A^N(x)\big) = \big(\mu_A^P(1/x), \mu_A^N(1/x)\big)\big\} = \{1\}.$$

**Theorem 2.12.** *$R$ is a linear order if and only if $(\mu_A^P, \mu_A^N)$ is a bipolar fuzzy sub quasigroup of $(V, *)$ satisfying:*

$$(i)\ \mu_A^P(1) = 1 \text{ and } \mu_A^N(1) = -1,$$
$$(ii)\ \big\{x : \big(\mu_A^P(x), \mu_A^N(x)\big) = \big(\mu_A^P(1/x), \mu_A^N(1/x)\big)\big\} = \{1\} \text{ and}$$
$$(iii)\ \{x : \mu_A^P(x) \vee \mu_A^P(1/x) > 0,\ \mu_A^N(x) \wedge \mu_A^N(1/x) < 0\} = V.$$

*Proof.* First, suppose that $R$ is a linear order. Then the conditions $(i)$ and $(ii)$ are satisfied and $(\mu_A^N, \mu_A^N)$ is a bipolar fuzzy sub quasigroup of $(V, *)$. For $x \in V$, $(\mu_R^P \vee \mu_{R^{-1}}^P)(1, x) > 0$ and $(\mu_R^N \wedge \mu_{R^{-1}}^N)(1, x) < 0$, which implies, $\mu_R^P(1, x) \vee \mu_R^P(x, 1) > 0$, and $\mu_R^N(1, x) \wedge \mu_R^N(x, 1) < 0$. Then $\mu_A^P(x) \vee \mu_A^P(1/x) > 0$ and $\mu_A^N(x) \wedge \mu_A^N(1/x) < 0$.

That is $x \in \{x : \mu_A^P(x) \vee \mu_A^P(1/x) > 0,\ \mu_A^N(x) \wedge \mu_A^N(1/x) < 0\}$. Hence, condition $(iii)$ is satisfied.

Conversely, suppose that the conditions $(i)$, $(ii)$, $(iii)$ hold and $(\mu_A^P, \mu_A^N)$ is a bipolar fuzzy sub quasigroup of $(V, *)$. $(\mu_A^P, \mu_A^N)$ is a bipolar fuzzy sub quasigroup and the conditions $(i)$ and $(ii)$ together implies that $R$ is a partial order. For $x, y \in V,\ t = y/x \in V$. Then by assumption $(iii)$, $\mu_A^P(t) \vee \mu_A^P(1/t) > 0$ and $\mu_A^N(t) \wedge \mu_A^N(1/t) < 0$. But we have, $\mu_A^P(1/t) = \mu_A^P(x/xt) = \mu_A^P(x/y)$. Therefore, $\mu_A^P(y/x) = \mu_A^P(x/y) > 0$ and $\mu_A^N(y/x) \wedge \mu_A^N(x/y) < 0$, which implies, $\mu_R^P(x, y) \vee \mu_R^P(y, x) > 0$ and $\mu_R^N(x, y) \wedge \mu_R^N(y, x) < 0$. That is, $(\mu_R \vee \mu_{R-1}^p)(x, y) > 0$ and $(\mu_R^N \wedge \mu_{R-1}^N)(x, y) < 0$. Thus, $R$ is a linear order.
Hence the proof. ∎

**Theorem 2.13.** *$R$ is an equivalence relation if and only if $(\mu_A^P, \mu_A^N)$ is a bipolar fuzzy sub quasigroup of $(V, *)$ satisfying:*

$$(i)\ \mu_A^P(1) = 1 \ and \ \mu_A^N(1) = -1 \ and$$
$$(ii)\ \big(\mu_A^P(x), \mu_A^N(x)\big) = \big(\mu_A^P(1/x), \mu_A^N(1/x)\big) \ for\ all\ x \in V.$$

**Theorem 2.14.** *$CayF_B(V, A)$ is a Hasse diagram if and only if it is connected and for any collection $x_1, x_2, ..., x_n$ of vertices in $V$ with $n \geq 2$ and $\mu_A^P(x_i) > 0,\ \mu_A^N(x_i) < 0$, for $i = 1, 2, ..., n$ we have, $\mu_A^P\big((...(x_1x_2)...)x_n\big) = 0$ and $\mu_A^N\big((...(x_1x_2)...)x_n\big) = 0$.*

*Proof.* Suppose $CayF_B(V, A)$ is a Hasse diagram and let $x_1, x_2, ..., x_n$ be vertices in $V$ with $n \geq 2$ and $\mu_A^P(x_i) > 0,\ \mu_A^N(x_i) < 0$, for $i = 1, 2, ..., n$. Then $R\big((...(x_1x_2)...)x_{i-1}\big) = \big(\mu_A^P(x_i), \mu_A^N(x_i)\big)$, implies $1, x_1, x_1x_2, ..., (...$
$(x_1x_2)...)x_n$ is a path from 1 to $(...(x_1x_2)...)x_n$. Since $CayF_B(V, A)$ is a Hasse diagram, we have, $R\big(1, (...(x_1x_2)...)x_n\big) = 0$. Therefore, $\mu_A^P\big((...(x_1x_2)...\ )x_n\big) = 0$ and $\mu_A^N\big((...(x_1x_2)...)x_n\big) = 0$.

Conversely, suppose that $CayF_B(V, A)$ is connected and for any collection $x_1, x_2, ..., x_n$ of vertices in $V$ with $n \geq 2$ and $\mu_A^P(x_i) > 0,\ \mu_A^N(x_i) < 0$, for $i = 1, 2, ..., n$ we have, $\mu_A^P\big((...(x_1x_2)...)x_n\big) = 0$ and $\mu_A^N\big((...(x_1x_2)...)x_n\big) = 0$. Let $(x_\circ, x_1, ..., x_n)$ be a path in $CayF_B(V, A)$ from $x_\circ$ to $x_n$ with $n \geq 2$. Then, $\mu_A^P(x_i) > 0$ and $\mu_A^N(x_i) < 0$ for $i = 1, 2, ..., n$. Let $x_1 = x_\circ t_1,\ x_2 = x_1t_2, ..., x_n = x_{n-1}t_n$. Then, $\mu_A^P(t_i) = \mu_A^P(x_i/x_{i-1}) > 0$ for $i = 1, 2, ..., n$. We have, $x_n = x_{n-1}t_n = ... = (...((x_\circ t_1)t_2)...)t_n = (...(x_\circ(t_1t_2')\ )...)t_n = ... = x_\circ(...((t_1t_2')t_3')...)t_n')$. Therefore, $\mu_R^P(x_\circ, x_n) = \mu_A^P(x_n/x_\circ) = \mu_A^P((...((t_1t_2')t_3')...)t_n')$. We have, $\mu_A^P(t_1) > 0$ and $\mu_A^P(t_2') = \mu_A^P(t_1t_2'/t_1) = \mu_A^P(x_\circ(t_1t_2')/x_\circ t_1) = \mu_A^P((x_\circ t_1)t_2\ /x_\circ t_1) = \mu_A^P(t_2) > 0$. In general, $\mu_A^P(t_i') = \mu_A^P(t_i) > 0$ for $i = 2, 3, ..., n$. Therefore, since $t_1, t_i' \in V, i = 2, 3, ..., n, n \geq 2$ and $\mu_A^P(t_1) > 0, \mu_A^P(t_i') > 0$, for $i = 2, 3, ..., n$, we have,$\mu_R^P(x_\circ, x_n) = \mu_A^P((...((t_1t_2')t_3')...)t_n') = 0$. Similarly, we can prove that $\mu_R^N(x_\circ, x_n) = \mu_A^N((...((t_1t_2')t_3')...)t_n') = 0$, both together gives $R(x_\circ, x_n) = 0$.
Hence $CayF_B(V, A)$ is a Hasse diagram. ∎

**Definition 2.15.** Let $(V,*)$ be a loop and let $A=(\mu_A^P,\mu_A^N)$ be a bipolar fuzzy subset induced by $V$. Then the sub-loop generated by $A$ is the meeting of all bipolar fuzzy sub-loops of $V$ which contains $A$. It is denoted by $\langle A\rangle$.

**Theorem 2.16.** Let $(V,*)$ be a loop and $A=(\mu_A^P,\mu_A^N)$ be a bipolar fuzzy subset of $V$. Then the fuzzy subset $\langle A\rangle$ is precisely given by $\langle\mu_A^P\rangle(x)=\vee\{\mu_A^P(x_1)\wedge\mu_A^P(x_2)\wedge ...\wedge\mu_A^P(x_n) : x=(...((x_1x_2)x_3)...)x_n$ with a finite positive integer $n$, $x_i\in V$ and $\mu_A^P(x_i)>0$ for $i=1,2,...,n\}$, $\langle\mu_A^N\rangle(x)=\wedge\{\mu_A^N(x_1)\vee\mu_A^N(x_2)\vee ...\vee\mu_A^N(x_n) : x=(...((x_1x_2)x_3)...)x_n$ with a finite positive integer $n$, $x_i\in V$ and $\mu_A^N(x_i)>0$ for $i=1,2,...,n\}$ for any $x\in V$.

*Proof.* Let $A'=(\mu_A^{P'},\mu_A^{N'})$ be the bipolar fuzzy subset of $V$ defined by $\mu_A^{P'}(x)=\vee\{\mu_A^P(x_1\wedge\mu_A^P(x_2)\wedge...\wedge\mu_A^P(x_n) : x=(...((x_1x_2)x_3)...)x_n$ with a finite positive integer $n$, $x_i\in V$ and $\mu_A^P(x_i)>0$ for $i=1,2,...,n\}$, $\mu_A^{N'}(x)=\wedge\{\mu_A^N(x_1)\vee\mu_A^N(x_2)\vee ...\vee\mu_A^N(x_n) : x=(...((x_1x_2)x_3)...)x_n$ with a finite positive integer $n$, $x_i\in V$ and $\mu_A^N(x_i)>0$ for $i=1,2,...,n\}$ for any $x\in V$. If $y\in V$, by definition of $\mu_A^{P'}$ and $\mu_A^{N'}$, it is clear that $\mu_A^{P'}(y)\geq\mu_A^P(y)$ and $\mu_A^{N'}(y)\leq\mu_A^N(y)$. Thus, we have $\mu_A^P\leq\mu_A^{P'}$ and $\mu_A^N\geq\mu_A^{N'}$. Let $x,y\in V$. If $\mu_A^P(x)=0$ or $\mu_A^P(y)=0$, $\mu_A^P(x)\wedge\mu_A^P(y)=0$ and if $\mu_A^N(x)=0$ or $\mu_A^N(y)=0$, $\mu_A^N(x)\vee\mu_A^N(y)=0$. Then, $\mu_A^{P'}(xy)\geq\mu_A^P(x)\wedge\mu_A^P(y)$ and $\mu_A^{N'}(xy)\leq\mu_A^N(x)\wedge\mu_A^N(y)$. Again, if $\mu_A^P(x)\neq 0$ and $\mu_A^P(y)\neq 0$, then by definition of $\mu_A^{P'}$, we have $\mu_A^{P'}(xy)\geq\mu_A^P(x)\wedge\mu_A^P(y)$ and if $\mu_A^N(x)\neq 0$ and $\mu_A^N(y)\neq 0$, by definition of $\mu_A^{N'}$, we have $\mu_A^{N'}(xy)\leq\mu_A^N(x)\wedge\mu_A^N(y)$. Hence $A'$ is a bipolar fuzzy sub-loop of $V$ containing $A$.

Now let $L$ be any fuzzy sub-loop of $V$ containing $A$. Then, for any $x\in V$ with $x=(...((x_1x_2)x_3)...)x_n$ with a finite positive integer $n$, $x_i\in V$ and $\mu_A^P(x_i)>0$, $\mu_A^N(x_i)<0$ for $i=1,2,...,n$, we have $\mu_L^P(x)\geq\mu_L^P(x_1)\wedge\mu_L^P(x_2)\wedge ...\wedge\mu_L^P(x_n)\geq\mu_A^P(x_1)\wedge\mu_A^P(x_2)\wedge ...\wedge\mu_A^P(x_n)$ and $\mu_L^N(x)\leq\mu_L^N(x_1)\vee\mu_L^N(x_2)\vee ...\vee\mu_L^N(x_n)\leq\mu_A^N(x_1)\vee\mu_A^N(x_2)\vee...\vee\mu_A^N(x_n)$ , which implies that $\mu_L^P(x)\geq\vee\{\mu_A^P(x_1)\wedge\mu_A^P(x_2)\wedge...\wedge\mu_A^P(x_n) : x=(...((x_1x_2)x_3)...)x_n$with a finite positive integer $n$, $x_i\in V$and $\mu_A^P(x_i)>0$ for $i=1,2,...,n\}$ and $\mu_A^N(x)\leq\wedge\{\mu_A^N(x_1)\vee\mu_A^N(x_2)\vee...\vee\mu_A^N(x_n) : x=(...((x_1x_2)x_3)...)x_n$ with a finite positive integer $n$, $x_i\in V$ and $\mu_A^N(x_i)>0$ for $i=1,2,...,n\}$ for any $x\in V$. Therefore, $\mu_L^P(x)\geq\mu_A^{P'}(x)$ and $\mu_L^N(x)\leq\mu_A^{N'}$ for all $x\in V$. That is, $\langle\mu_A^P\rangle(x)=\vee\{\mu_A^P(x_1)\wedge\mu_A^P(x_2)\wedge...\wedge\mu_A^P(x_n) : x=(...((x_1x_2)x_3)...)x_n$ with a finite positive integer $n$, $x_i\in V$ and $\mu_A^P(x_i)>0$ for $i=1,2,...,n\}$ and $\langle\mu_A^N\rangle(x)=\wedge\{\mu_A^N(x_1)\vee\mu_A^N(x_2)\vee ...\vee\mu_A^N(x_n) : x=(...((x_1x_2)x_3)...)x_n$ with a finite positive integer $n$, $x_i\in V$ and $\mu_A^N(x_i)>0$ for $i=1,2,...,n\}$ for any $x\in V$. Thus, $A'=(\mu_A^{P'},\mu_A^{N'})=\langle(\mu_A^P,\mu_A^N)\rangle=\langle A\rangle$. Hence the proof. ■

## 3. Connectedness in Cayley Bipolar Fuzzy Graphs Induced by Loops

**Theorem 3.1.** Let $(V, *)$ be a loop and $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset of $V$. Then, for any $\alpha \in [-1, 1]$,
$(\langle\mu^P_{A\alpha}\rangle, \langle\mu^N_{A\alpha}\rangle) = (\langle\mu_A^P\rangle_\alpha, \langle\mu_A^N\rangle_\alpha)$ and $(\langle\mu^{P+}_{A\alpha}\rangle, \langle\mu^{N+}_{A\alpha}\rangle) = (\langle\mu_A^P\rangle_\alpha^+, \langle\mu_A^N\rangle_\alpha^+)$, where $(\langle\mu^P_{A\alpha}\rangle, \langle\mu^N_{A\alpha}\rangle)$ denotes the bipolar fuzzy sub-loop generated by $(\mu^P_{A\alpha}, \mu^N_{A\alpha})$ and $(\langle\mu_A^P\rangle, \langle\mu_A^N\rangle)$ denotes the bipolar fuzzy sub-loop generated by $(\mu_A^P, \mu_A^N)$.

*Proof.* Observe that

$$\begin{aligned}
x \in (\langle\mu^P_{A\alpha}\rangle, \langle\mu^N_{A\alpha}\rangle) &\Leftrightarrow \exists\, x_1, x_2, ..., x_n \text{ in } A_\alpha \ni x = (...(x_1x_2)...)x_n \\
&\Leftrightarrow \exists\, 1, x_2, ..., x_n \text{ in } V \ni \mu_A^P(x_i) \geq \alpha, \mu_A^N(x_i) \\
&\quad\ \leq \alpha\ \forall\, i = 1, 2, ..., n \text{ and } x = (...(x_1x_2)...)x_n \\
&\Leftrightarrow \langle\mu_A^P(x)\rangle \geq \alpha, \langle\mu_A^N(x)\rangle \leq \alpha \\
&\Leftrightarrow x \in \langle\mu_A^P\rangle_\alpha, x \in \langle\mu_A^N\rangle_\alpha.
\end{aligned}$$

Therefore, $(\langle\mu^P_{A\alpha}\rangle, \langle\mu^N_{A\alpha}\rangle) = (\langle\mu_A^P\rangle_\alpha, \langle\mu_A^N\rangle_\alpha)$.
Similarly, we can prove that $(\langle\mu^{P+}_{A\alpha}\rangle, \langle\mu^{N+}_{A\alpha}\rangle) = (\langle\mu_A^P\rangle_\alpha^+, \langle\mu_A^N\rangle_\alpha^+)$. ■

**Remark 3.2.** Let $(V, *)$ be a loop and $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset of $V$. Then by Theorem 3.1, we have $\langle supp(A)\rangle = supp(\langle A\rangle)$.

**Theorem 3.3.** Let $A'$ be a right associative subset of a loop $(V', *)$ and $G' = (V', R')$ be the bipolar fuzzy graph with $R' = \{(x, y) : y/x \in A\}$. Then $G'$ is connected if and only if $\langle A'\rangle \supseteq V' - \{1\}$.

**Theorem 3.4.** $CayF_B(V, A)$ is connected if and only if $supp\langle A\rangle \supseteq V - \{1\}$.

*Proof.* Suppose $CayF_B(V, A)$ is connected. Let $x \in V - \{1\}$. Since $CayF_B(V, A)$ is connected, there exist a path from 1 to $x$, say, $1, x_1, x_2, ..$
$., x_n = x$. This implies that, there exist $t_1, t_2, ..., t_n \in supp(A)$ such that $x = x_n = x_{n-1}t_n,\ x_{n-1} = x_{n-2}t_{n-1}, ..., x_2 = x_1t_2,\ x_1 = 1.t_1$. Therefore, $x = x_n = x_{n-1}t_n = (x_{n-2}t_{n-1})t_n = ... = (...((t_1t_2)t_3)...)t_n,\ t_i \in supp(A),\ i = 1, 2, ..., n$, which implies that $x \in \langle supp(A)\rangle$. Therefore, $V - \{1\} \subseteq \langle supp(A)\rangle = supp\langle A\rangle$.

Conversely, let $V - \{1\} \subseteq supp\langle A\rangle$. Let $x, y$ be two distinct elements in $V$. Then, there exist $z \neq 1 \in V$ such that $y = xz$. Since $z \neq 1,\ z \in V - \{1\} \subseteq \langle supp(A)\rangle$. Then, there exist $z_1, z_2, ..., z_m \in supp(A)$ such that $z = z_1z_2...z_m$.
Clearly, $1, z_1, z_1z_2, ..., (...((z_1z_2)z_3)...)z_m = z$ is a path from 1 to $z$.
Then $x, xz_1, x(z_1z_2), x((z_1z_2)z_3), ..., x((...((z_1z_2)z_3)...)\ z_m) = xz = y$ is a path from $x$ to $y$. Therefore, $CayF_B(V, A)$ is connected. ■

**Theorem 3.5.** Let $A'$ be a right associative subset of a loop $(V', *)$ and $G' = (V', R')$ be the bipolar fuzzy graph with $R' = \{(x, y) : y/x \in A'\}$. Then, $G'$ is weakly connected if and only if $\langle A' \cup A'_\ell\rangle \supseteq V' - \{1\}$, where $A'_\ell = \{a_\ell : 1 = a_\ell a,\ a \in A'\}$.

**Theorem 3.6.** $CayF_B(V, A)$ is weakly connected if and only if $supp(\langle A \vee A_\ell \rangle) \supseteq V - \{1\}$ where $A_\ell(x_\ell) = A(x),\ 1 = x_\ell x$.

*Proof.* Suppose that $CayF_B(V, A)$ is weakly connected.
Then $CayF_B(V, A \vee A_\ell)$ is connected. Thus by Theorem 3.4, we have $V - \{1\} \subseteq supp(\langle A \vee A_\ell \rangle)$. This completes the proof. ∎

**Theorem 3.7.** Let $A'$ be a right associative subset of a loop $(V', *)$ and $G' = (V', R')$ be the bipolar fuzzy graph with $R' = \{(x, y) : y/x \in A'\}$. $G'$ is semi-connected if and only if $\langle A' \rangle \cup \langle A' \rangle_\ell \supseteq V' - \{1\}$.

**Theorem 3.8.** $CayF_B(V, A)$ is semi-connected if and only if

$$\langle supp(A) \rangle \cup \langle supp(A) \rangle_\ell \supseteq V - \{1\}.$$

*Proof.* First assume that $CayF_B(V, A)$ is semi-connected. Let $x \in V - \{1\}$. Since $CayF_B(V, A)$ is semi-connected, there exist a path from $x$ to 1 or a path from 1 to $x$. Suppose there exist a path $1, x_1, x_2, ..., x_n, x$ from 1 to $x$. Then, there exist $t_1, t_2, ..., t_{n+1} \in supp(A)$ such that $x_1 = 1t_1,\ x_2 = x_1 t_2, ..., x = x_n t_{n+1}$. Then, $x = x_n t_{n+1} = (x_{n-1} t_n) t_{n+1} = ... = (...((t_1 t_2) t_3)...) t_{n+1},\ t_i \in supp(A),\ i = 1, 2, ..., n+1$, which implies that $x \in \langle supp(A) \rangle$. Or suppose there exist a path $x, y_1, y_2, ..., y_m, 1$ from $x$ to 1. Then, there exist $k_i \in supp(A)$ for $i = 0, 1, ..., m$ such that $y_1 = xk_\circ,\ y_2 = y_1 k_1, ...,\ 1 = y_m k_m$. Then, $1 = (...((xk_\circ)k_1)...)k_m = (...(x(k_\circ k_1'))...)k_m = ... = x((...(k_\circ k_1')...)k_m')$. Here, $k_i' \in supp(A)$, for $i = 0, 1, ..., m$, since $supp(A)$ is right associative. This implies $1 = xk$, where $k = (...((k_\circ k_1')k_2')...)k_m' \in \langle supp(A) \rangle$. Hence $x \in \langle supp(A) \rangle_\ell$. Therefore, $CayF_B(V, A)$ is semi-connected implies $\langle supp(A) \rangle \cup \langle supp(A) \rangle_\ell \supseteq V - \{1\}$.

Conversely, assume that $\langle supp(A) \rangle \cup \langle supp(A) \rangle_\ell \supseteq V - \{1\}$. Let $x, y \in CayF_B(V, A)$ be two distinct elements. Then $y = xz$ for some $z \in CayF_B(V, A)$. Then, $z \in \langle supp(A) \rangle \cup \langle supp(A) \rangle_\ell$.

If $z \in \langle supp(A) \rangle$, there exist $t_1, t_2, ..., t_n \in supp(A)$ such that $z = (...((t_1 t_2) t_3)...) t_n$. Clearly $1, t_1, t_1 t_2, ..., (...((t_1 t_2) t_3)...) t_n = z$ is a path from 1 to $z$. Then, $x, xt_1, ..., x((...((t_1 t_2) t_3)...) t_n)$ is a path from $x$ to $y$. Or else, if $z \in \langle supp(A) \rangle_\ell,\ 1 = zt$ for some $t \in \langle supp(A) \rangle$, which implies there exist $p_1, p_2, ..., p_m \in supp(A)$ such that $t = p_1 p_2 ... p_m$. Then, $1 = zt = z((...((p_1 p_2) p_3)...) p_m) = (z...((p_1 p_2) p_3)...) p_m' = ... = (...((zp_1) p_2')...) p_m',\ p_i \in supp(A)$, since $supp(A)$ is right associative and $p_i \in supp(A)$. Let $k_1 = zp_1$. $k_2 = k_1 p_2',\ k_3 = k_2 p_3', ..., k_m = k_{m-1} p_m'$. Then, $k_m = k_{m-1} p_m' = ... = (...((zp_1) p_2')...) p_m' = 1$. Clearly, $z, k_1, k_2, ..., k_m = 1$ is a path from $z$ to 1. Then, $xz, xk_1, xk_2, ..., xk_m = x$ is a path from $y$ to $x$. Thus, for any $x, y \in V$ there exist a path from $x$ to $y$ or a path from $y$ to $x$, which implies that $CayF_B(V, A)$ is semi-connected.
This completes the proof. ∎

**Theorem 3.9.** Let $A'$ be a right associative subset of a loop $(V', *)$ and $G' = (V', R')$ be the bipolar fuzzy graph with $R' = \{(x, y) : y/x \in A'\}$. Then, $G'$ is locally connected if and only if $\langle A' \rangle = \langle A' \rangle_\ell$.

**Theorem 3.10.** $CayF_B(V, A)$ is locally connected if and only if

$$supp\langle A\rangle = supp\langle A\rangle_\ell.$$

*Proof.* First suppose that $CayF_B(V, A)$ is locally connected. Let $x \in \langle supp(A)\rangle$. Then, there exist $x_1, x_2, ..., x_n \in supp(A)$ such that $x = (...((x_1x_2)x_3)...)x_n$. Therefore $1, x_1, x_1x_2, ..., (...((x_1x_2)x_3)...)x_n$ is a path from 1 to $x$. Then, since $G$ is locally connected, there exist a path from $x$ to 1. Let $x, z_1, z_2, ..., z_{n-1}$ be a path from $x$ to 1 which implies there exist $a_1, a_2, ..., a_n \in supp(A)$ such that $z_1 = xa_1,\ z_2 = z_1a_2, ..., z_{n-1} = z_{n-2}a_{n-1},\ 1 = z_{n-1}a_n$. Then, $1 = z_{n-1}a_n = (z_{n-2}a_{n-1})a_n = ... = (...((xa_1)a_2)...)a_n$. Then, we have, $1 = (...(x(a_1a_2'))...)a_n = ... = x(...((a_1$
$a_2')a_3')...)a_n',\ a_i' \in supp(A)$. That is, $1 = xt,\ t \in \langle supp(A)\rangle$, implies $x \in \langle supp(A)\rangle_\ell$. Thus,

$$\langle supp(A)\rangle \subseteq \langle supp(A)\rangle_\ell. \tag{1}$$

Let $x_\ell \in \langle supp(A)\rangle_\ell$, which implies there exist an $x \in \langle supp(A)\rangle$ such that $1 = x_\ell x$. Then, $1, x_1, x_1x_2, ..., (...((x_1x_2)x_3)...)x_m$ is a path from 1 to $x$. Thus, since $CayF_B(V, A)$ is locally connected, there exist a path from $x$ to 1. We have, $1 = x_\ell x = x_\ell((...((x_1x_2)x_3)...)x_m) = x_\ell((...(x_1(x_2x_3'))...)x_m) = ... = (...((x_\ell x_1)x_2')...)x - m'\ x_1 \in supp(A)$, and here $x_i' \in supp(A),\ i = 1, 2, ..., m$, since $supp(A)$ is right associative. Now, let $t_1 = x_\ell x_1,\ t_2 = t_1x_2', ..., 1 = t_m = t_{m-1}x_m'$. Then, $x_\ell, t_1, t_2, ..., t_{m-1}, t_m = 1$ is a path from $x_\ell$ to 1. This implies that there exist a path from 1 to $x_\ell$, since $CayF_B(V, A)$ is locally connected. Let $1, k_1, k_2, ..., k_r = x_\ell$ be a path from 1 to $x_\ell$. Then, there exist $p_i \in supp(A),\ i = 1, 2, ..., r$ such that $k_1 = 1.p_1,\ k_2 = k_1p_2, ..., k_r = k_{r-1}p_r$. Thus, $x_\ell = k_r = k_{r-1}p_r = (k_{r-2}p_{r-1})p_r = ... = (...((p_1p_2)p_3)...)p_r$, which implies that $x_\ell \in \langle supp(A)\rangle$. Hence,

$$\langle supp(A)\rangle_\ell \subseteq \langle supp(A)\rangle. \tag{2}$$

Therefore, from equations (1) and (2) we get $\langle supp(A)\rangle = \langle supp(A)\rangle_\ell$.

Conversely, suppose $\langle supp(A)\rangle = \langle supp(A)\rangle_\ell$. Let $x, y \in V$ and there exist a path from $x$ to $y$ say $x, x_1, x_2, ..., x_{n-1}, y$. Then, there exist $a_i \in supp(A)$ for $i = 1, 2, ..., n$ such that $x_1 = xa_1,\ x_2 = x_1a_2, ..., x_{n-1} = x_{n-2}a_{n-1},\ y = x_{n-1}a_n$. Thus, $y = x_{n-1}a_n = (x_{n-2}a_{n-1}a_n = ... = (...((xa_1)a_2)...)a_n$. Therefore, $y = x((...((a_1a_2')a_3')...)a_n')$, where $a_i' \in supp(A)$, since $supp(A)$ is right associative, which implies $y/x \in \langle supp(A)\rangle$. Then, $x/y \in \langle supp(A)\rangle_\ell$. Now, since $k = x/y \in \langle supp(A)\rangle_\ell$ and $\langle supp(A)\rangle = \langle supp(A)\rangle_\ell$, $k = x/y \in \langle supp(A)\rangle$. Then, there exist $k_1, k_2, ..., k_p \in supp(A)$ such that $k = (...((k_1k_2)k_3)...)k_p$.
Clearly, $1, k_1, k_1k_2, ..., (...((k_1k_2)k_3)...)k_p = k$ is a path from 1 to $k$.
Then, $y, yk_1, y(k_1k_2), ..., y((...((k_1k_2)k_3)...)k_p) = yk = x$ is a path from $y$ to $x$. Hence $G'$ is locally connected. ■

**Theorem 3.11. [2]** A finite digraph has a source if and only if it is quasi-connected.

**Theorem 3.12.** Let $A'$ be a right associative subset of a finite loop $(V', *)$. $G' = (V', R')$ be a bipolar fuzzy digraph with $R' = \{(x, y) : y/x \in A'\}$. Then $G'$ is quasi-connected if and only if it is connected.

**Theorem 3.13.** A finite Cayley bipolar fuzzy graph $CayF_B(V, A)$, where $(V, *)$ is a finite loop, is quasi-connected if and only if it is connected.

*Proof.* Every connected graphs are quasi-connected.
Therefore, if $CayF_B(V, A)$ is connected then $CayF_B(V, A)$ is quasi-connected. Now, suppose that $CayF_B(V, A)$ is quasi-connected. Note that $CayF_B(V, A)$ is finite. Thus by Theorem 3.11, $CayF_B(V, A)$ has a source, say $z$. Then for any $x \in V$ with $x \neq z$, there is a directed path from $z$ to $x$. Let $z, z_1, z_2, ..., z_n, x$ be a path from $z$ to $x$. Then, there exist $k_i \in supp(A)$ for $i = 1, 2, ..., n+1$ such that $z_1 = zk_1,\ z_2 = z_1k_2, ...,\ z_n = z_{n-1}k_n,\ x = z_nk_{n+1}$. Then, $x = (...((zk_1)k_2)...)k_{n+1} = (...(z(k_1k_2'))...)k_{n+1} = ... = z((...(k_1k_2')...)k_{n+1}')$. Here, $k_i' \in supp(A)$, for $i = 1, 2, ..., n+1$, since $supp(A)$ is right associative. Therefore, $x/z = (...(k_1k_2')...)k_{n+1}') \in \langle supp(A)\rangle$. Thus it is clear that $x/z \in \langle supp(A)\rangle$ for every $x \in V$ with $x \neq z$. Hence $\langle supp(A)\rangle \supseteq V - \{1\}$. Hence, by Theorem 3.4, $CayF_B(V, A)$ is connected. ■

## 4. Strength of Connectedness in Cayley Bipolar Fuzzy Graphs Induced by Loops

In this subsection, we prove the following theorems based on different types of $\alpha$-connectedness. Let $(V, *)$ be a loop, $A$ be a scaled fuzzy subset of $V$ and $G = (V, R)$ be the Bipolar Cayley fuzzy graph induced by $(V, *, A)$. Also, for any $\alpha \in [-1, 1]$, let $A_\alpha$ be the $\alpha$-cut of $A$. Then

$$R_\alpha = (\mu^P_{R_\alpha}, \mu^N_{R_\alpha}) = (\mu^P_{A_\alpha}, \mu^N_{A_\alpha}) = \{(x, y) \in V \times V : y/x \in A_\alpha\}.$$

**Definition 4.1.** Let $G = (V, R)$ be a Cayley bipolar fuzzy graph induced by the loop $(V, *)$. The $\mu^P_R$-strength of a path $x_\circ, x_1, x_2, ..., x_n$ is defined as $min(\mu^P_R(x_{i-1}, x_i))$ for $i = 1, 2, ..., n$ and is denoted as $S(\mu^P_R)$. The $\mu^N_R$-strength of this path is defined as $max(\mu^N_R(x_{i-1}, x_i))$ for $i = 1, 2, ..., n$ and is denoted as $S(\mu^N_R)$.

**Definition 4.2.** Strength of a path $P'$ in $G = (V, R)$, denoted by $strength(P')$ is defined to be $strength(P') = \big(S(\mu^P_R), S(\mu^N_R)\big)$ and is said to be greater than or equal to $\alpha$ if $S(\mu^P_R) \geq \alpha$ and $S(\mu^N_R) \leq \alpha$.

Let $\alpha \in [-1, 1]$. Then we define the following:

**Definition 4.3.** Let $G = (V, \rho)$ be a bipolar fuzzy graph. Then $G$ is said to be: (i) $\alpha$-*connected* if for every pair of vertices $x, y \in G$, there is a path $P$ from $x$ to $y$ such that $strength(P) \geq \alpha$, (ii) *weakly $\alpha$-connected* if the fuzzy graph $(V, R \vee R^{-1})$ is $\alpha$-connected, (iii) *semi $\alpha$-connected* if for every $x, y \in V$, there is a path of strength greater than or equal to $\alpha$ from $x$ to $y$ or from $y$ to $x$ in $G$ and (iv) *locally $\alpha$-connected* if for every pair of vertices $x$ and $y$, there is a path $P$ of strength greater than or equal to $\alpha$ from $x$ to $y$ whenever there is a path $P'$ of strength greater than or equal to $\alpha$ from $y$ to $x$. (v) *quasi $\alpha$-connected* if for every pair $x, y \in V$, there is some $z \in V$ such that there is a directed path from $z$ to $x$ of strength greater than or equal to $\alpha$ and there is a directed path from $z$ to $y$ of strength greater than or equal to $\alpha$. (vi) $\alpha$-*complete* if $R(x, y) \geq \alpha$ for all $x, y \in V$.

**Theorem 4.4.** $G = CayF_B(V, A)$ is $\alpha$-connected if and only if $\langle A\rangle_\alpha \supseteq V - \{1\}$, where $\langle A\rangle_\alpha = (\langle \mu_A^P\rangle_\alpha), \langle \mu_A^N\rangle_\alpha)$.

**Theorem 4.5.** $G$ is weakly $\alpha$-connected if and only if

$$\langle A \cup A_\ell\rangle_\alpha \supseteq V - \{1\}.$$

**Theorem 4.6.** $G$ is semi $\alpha$-connected if and only if

$$\langle A\rangle_\alpha \cup \langle A\rangle_{\alpha\ell} \supseteq V - \{1\}.$$

**Theorem 4.7.** $G$ is locally $\alpha$-connected if and only if

$$\langle A\rangle_\alpha = \langle A\rangle_{\alpha\ell}.$$

**Theorem 4.8.** The fuzzy graph $G$ is $\alpha$-complete if and only if

$$A_\alpha \supseteq V - \{1\}.$$

**Theorem 4.9.** If $G$ is finite then it is quasi $\alpha$-connected if and only if it is $\alpha$-connected.

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