Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 6511-6523

http://www.ripublication.com

Characterizations of semigroups in terms of $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideals

Pongpun Julatha[1] and Manoj Siripitukdet[2]

[1]*Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand.*

[2]*Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand.*
Research center for Academic Excellence in Mathematics, Naresuan University, Phitsanulok 65000, Thailand.
Corresponding author

Abstract

In this paper, we introduce the concepts of (α, β)-fuzzy generalized bi-ideals of semigroups, where $\alpha, \beta \in \{\in_\Delta, q_\Delta, \in_\Delta \wedge q_\Delta, \in_\Delta \vee q_\Delta\}$ and $\alpha \neq \in_\Delta \wedge q_\Delta$. Special attention is paid to $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideals, which are the more general form of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideals, of semigroups and their related properties are provided. Completely regular semigroups and groups are characterized in terms of their $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideals.

Keywords: $(\in_\Delta, \in_\Delta \vee q_\Delta)$- fuzzy generalized bi-ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideals, semigroups, completely regular semigroups, groups.

2010 MS Subject Classification: 20N25, 20M12, 03E72, 08A72.

1. INTRODUCTION

The fundamental concept of a fuzzy set, introduced by Zadeh [15], provides a natural framework for generalizing many basic concepts of algebraic structures as set theory, semigroup theory, group theory, ring theory, ect. Kuroki applied the concept of a fuzzy set to semigroup theory and studied the theory of fuzzy semigroups in his papers [2, 3, 4]. Fuzzy semigroup theory plays a vital role in mathematics with wide applications in many other branches. For example, the monograph by Mordeson and Malik [7] deals with the application of fuzzy sets to the concepts of automata and formal languages. In [8], Mordeson et al. systematically explain theoretical results on fuzzy semigroups and their application in fuzzy finite machines, fuzzy coding and fuzzy languages.

Kuroki [4] gave the concept of a fuzzy generalized bi-ideal of a semigroup and characterized regular semigroups and both intra-regular and left quasi-regular semigroups via their fuzzy generalized bi-ideals. Completely regular semigroups and groups were also characterized in terms of fuzzy generalized bi-ideals [8]. A new type of fuzzy generalized bi-ideal, that means the (α,β)-fuzzy generalized bi-ideal, was introduced in the paper of Shabir et al. [12] by using the notions of a "quasi-coincident with" relation (q) and a "belong to" relation $(\in)$ of a fuzzy set and a fuzzy point. In particular, the notion of an $(\in, \in\vee q)$-fuzzy generalized bi-ideal of a semigroup is a useful generalization of Kuroki's fuzzy generalized bi-ideal [4]. They characterized semigroups by using $(\in, \in\vee q)$-fuzzy generalized bi-ideals [12]. Generalizing the notion of the "quasi-coincident with" relation, Shadir [13] gave definitions of $(\in, \in\vee q_k)$-fuzzy generalized bi-ideals of semigroups and characterized semigroups by using these fuzzy subsets. Recently, Shadir and Ali [11] introduced the concepts of (α,β)-fuzzy generalized bi-ideals (bi-ideals, left ideals, right ideals, ideals) of semigroups, where α and β are any two elements of $\{\in_\gamma, q_\delta, \in_\gamma\wedge q_\delta, \in_\gamma\vee q_\delta\}$ with $\alpha \neq \in_\gamma\wedge q_\delta$, and characterizations of semigroups are investigated by using these fuzzy subsets. It is now natural to study similar types of $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy subsystems of other algebraic structures. Rehman and Shabir [9, 10] introduced notions of $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy generalized bi-ideals of ternary semigroups and gave characterizations of different classes of ternary semigroups in terms of these fuzzy subsystems. Huang et al. [1] initiated the notions of $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy hyperideals (bi-hyperideals, quasi-hyperideals) in semihyperrings and they use the notions to characterize hyper regular semihyperrings and left duo semihyperring. Shabir and Tariq [14] defined $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy generalized bi-hyperideals of semihypergroups and characterized semihypergroups by using these fuzzy sets.

In this paper, we introduce concepts of (α,β)-fuzzy generalized bi-ideals of semigroups, where $\alpha,\beta \in \{\in_\Delta, q_\Delta, \in_\Delta\wedge q_\Delta, \in_\Delta\vee q_\Delta\}$ and $\alpha \neq \in_\Delta\wedge q_\Delta$. Special attention is paid to $(\in_\Delta, \in_\Delta\vee q_\Delta)$-fuzzy generalized bi-ideals, which are the more general form of $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy generalized bi-ideals, of semigroups and their related properties are investigated. Completely regular semigroups and groups are characterized in terms of their ($\in_\Delta$, $\in_\Delta \vee q_\Delta$)-fuzzy generalized bi-ideals. In particular, it is shown that if we take $\Delta = (\gamma, 1]$, then we immediately obtain characterizations of such semigroups by their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideals.

2. PRELIMINARIES

Let S be a semigroup. A nonempty subset G of S is called a **generalized bi-ideal** of S if $GSG \subseteq G$. A subset P of S is called **semiprime** if for all $a \in S$, $a^2 \in P$ implies $a \in P$. A semigroup S is called **completely regular** if every element a of S there exists $x \in S$ such that $a = axa$ and $xa = ax$. The following two theorems are well known in semigroup theory.

Theorem 2.1. [8] Let S be a semigroup. Then S is completely regular if and only if every generalized bi-ideal of S is semiprime.

Theorem 2.2. [5, 6] A semigroup S is a group if and only if it contains no proper generalized bi-ideal.

A fuzzy subset f [15] of a nonempty set S is a function from S into $[0,1]$. Let f be a fuzzy subset of S and denote $\mathrm{Im}(f) = \{f(a) \mid a \in S\}$. For a subset A of S, the characteristic function of A is the function χ_A of S into $\{0,1\}$ defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. Let $t \in (0,1]$ and $x \in S$. A fuzzy subset x_t of S, defined by for all $y \in S$

$$x_t(y) = \begin{cases} t & \text{if } x = y; \\ 0 & \text{otherwise,} \end{cases}$$

is called a **fuzzy point** [8] with support x and value t. Let x_t be a fuzzy point, f be a fuzzy subset of a nonempty set S and $0 < \gamma < \delta \leq 1$. Define relations $\in_\gamma, q_\delta, \in_\gamma \wedge q_\delta$ and $\in_\gamma \vee q_\delta$ [11] between x_t and f as follow:

(1) $x_t \in_\gamma f$ if $f(x) \geq t > \gamma$,
(2) $x_t q_\delta f$ if $f(x) + t > 2\delta$,
(3) $x_t \in_\gamma \wedge q_\delta f$ if $x_t \in_\gamma f$ and $x_t q_\delta f$,
(4) $x_t \in_\gamma \vee q_\delta f$ if $x_t \in_\gamma f$ or $x_t q_\delta f$.

For $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \wedge q_\delta, \in_\gamma \vee q_\delta\}$, we say that $x_t \bar{\alpha} f$ [11] if $x_t \alpha f$ does not hold. These relations are a useful role to study algebraic structures seen in [1, 9, 10, 11, 14].

Now, we give relations between a fuzzy point and a fuzzy subset of a nonempty set as follows:

Definition 2.3. Let x_t be a fuzzy point, f a fuzzy subset of a nonempty set S and $\delta \in \Delta \subseteq (0,1]$ such that $\inf(\Delta) \neq \delta$. Define that

(1) $x_t \in_{\Delta} f$ if the following conditions are satisfied:
  (i) $t \in \Delta$;
  (ii) $f(x) \geq t > \inf(\Delta)$;
  (iii) $f(x) \in \Delta \cup [\sup(\Delta), 1]$,

(2) $x_t q_{\Delta} f$ if the following conditions are satisfied:
  (i) $t \in \Delta$;
  (ii) $f(x) + t > 2\delta$;
  (iii) if $\inf(\Delta) < f(x) < \sup(\Delta)$, then $f(x) \in \Delta$,

(3) $x_t \in_{\Delta} \vee\, q_{\Delta} f$ if $x_t \in_{\Delta} f$ or $x_t q_{\Delta} f$,

(4) $x_t \in_{\Delta} \wedge\, q_{\Delta} f$ if $x_t \in_{\Delta} f$ and $x_t q_{\Delta} f$.

For $\alpha \in \{\in_{\Delta}, q_{\Delta}, \in_{\Delta} \wedge q_{\Delta}, \in_{\Delta} \vee q_{\Delta}\}$, we denote $x_t \bar{\alpha} f$ if $x_t \alpha f$ does not hold. From Definition 2.3, the following statements hold.

(1) If $x_t \in_{\gamma} f$, then $x_t \in_{\Delta} f$ where $\Delta = (\gamma, 1]$.

(2) If $t \in \Delta$ and $x_t q_{\delta} f$, then $x_t q_{\Delta} f$ where $\Delta = (\gamma, 1]$.

3. (α, β)-FUZZY GENERALIZED BI-IDEALS

Throughout this paper, let $0 < \gamma < \delta \leq 1$, $\Delta \subseteq (0,1]$, $\inf(\Delta) < \delta \in \Delta$, $\alpha \in \{\in_{\Delta}, q_{\Delta}, \in_{\Delta} \vee q_{\Delta}\}$ and $\beta \in \{\in_{\Delta}, q_{\Delta}, \in_{\Delta} \wedge q_{\Delta}, \in_{\Delta} \vee q_{\Delta}\}$ unless otherwise specified. In this section, we introduce the concept of (α, β)-fuzzy generalized bi-ideals of semigroups and also investigate related properties.

Definition 3.1. Any fuzzy subset f of a semigroup S is said to be an (α, β)- **fuzzy generalized bi-ideal** of S if it satisfies

$$a_t \alpha f \text{ and } b_r \alpha f \Rightarrow (axb)_{\min\{t,r\}} \beta f$$

for all $a, b, x \in S$ and $t, r \in \Delta$.

If we take $\Delta = (\gamma, 1]$ in Definition 3.1, then it is reduced to the following definition.

Definition 3.2. [11] Let $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$ and $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \wedge q_\delta, \in_\gamma \vee q_\delta\}$. Any fuzzy subset f of a semigroup S is said to be an (α, β)-**fuzzy generalized bi-ideal** of S if it satisfies

$$a_t \alpha f \text{ and } b_r \alpha f \Rightarrow (axb)_{\min\{t,r\}} \beta f$$

for all $a, b, x \in S$ and $t, r \in (0,1]$.

Example 3.3. The set of all positive integers $\mathbb{N}$ is a semigroup under the usual multiplication.

(1) Let f be a fuzzy subset of $\mathbb{N}$ defined by

$$f(n) = \begin{cases} 0.6 & \text{if } n \in 2\mathbb{N}; \\ 0.7 & \text{if } n \notin 2\mathbb{N} \end{cases}$$

for all $n \in \mathbb{N}$.
If $\gamma = 0.1, \delta = 0.2$ and $\Delta = \{0.1, 0.2, 0.6\}$, then by the routine calculations, we have that f is an (α, β)-fuzzy generalized bi-ideal of $\mathbb{N}$ for all $\alpha \in \{\in_\Delta, q_\Delta, \in_\Delta \vee q_\Delta\}$ and $\beta \in \{\in_\Delta, q_\Delta, \in_\Delta \wedge q_\Delta, \in_\Delta \vee q_\Delta\}$.
However, f is not an $(\in_\gamma, \in_\gamma)$-fuzzy generalized bi-ideal of $\mathbb{N}$, because

$$3_{0.7} \in_\gamma f, 5_{0.7} \in_\gamma f \text{ but } \big((3)(2)(5)\big)_{0.7} = 30_{0.7} \overline{\in_\gamma} f.$$

(2) Let g be a fuzzy subset of $\mathbb{N}$ defined by

$$g(n) = \begin{cases} 0.8 & \text{if } n \in 4\mathbb{N}; \\ 0.5 & \text{if } n \in 2\mathbb{N}\backslash 4\mathbb{N}; \\ 0.6 & \text{if } n \in \mathbb{N}\backslash 4\mathbb{N} \end{cases}$$

for all $n \in \mathbb{N}$.
If $\gamma = 0.4$, $\delta = 0.8$ and $\Delta = (0.4, 0.8] \setminus \{0.6\}$, then by the routine calculations, we have that g is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of $\mathbb{N}$. However, g is not an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of $\mathbb{N}$, because

$$3_{0.6} \in_\gamma g \text{ but } \big((3)(2)(3)\big)_{0.6} = 18_{0.6} \overline{\in_\gamma \vee q_\delta} g.$$

Let f be a fuzzy subset of a nonempty set S. Define the subset S_f of S as follows:

$$S_f := \{a \in S \mid f(a) \in \Delta \cup [\sup(\Delta), 1] \text{ and } f(a) > \inf(\Delta)\}.$$

Remark 3.4. If we take $\Delta = (\gamma, 1]$, then $S_f = \{a \in S \mid f(a) > \gamma\}$.

Theorem 3.5. Let $\max(\Delta)$ exist and $2\delta = \max(\Delta) + \inf(\Delta)$. If f is an (α, β)-fuzzy generalized bi-ideal of a semigroup S, then $S_f (\neq \emptyset)$ is a generalized bi-ideal of S.

Proof. Suppose on the contrary that $axb \notin S_f$ for some $a, b \in S_f$, $x \in S$. Then

$$f(axb) \leq \inf(\Delta) \text{ or } f(axb) \notin \Delta \cup [\max(\Delta), 1].$$

We consider two cases as follows:

Case (1): $\alpha \in \{\in_\Delta, \in_\Delta \vee q_\Delta\}$. Since $a, b \in S_f$, we get $f(a) > \inf(\Delta), f(b) > \inf(\Delta)$ and $f(a), f(b) \in \Delta \cup [\max(\Delta), 1]$. Choose $t := \min\{f(a), f(b), \delta\}$, we get $\inf(\Delta) < t \in \Delta$. Then $a_t \in_\Delta f$ and $b_t \in_\Delta f$ which imply that $a_t \alpha f$ and $b_t \alpha f$. If $f(axb) \leq \inf(\Delta)$, then $f(axb) < t$ and so $(axb)_t \overline{\in_\Delta} f$. Since $\inf(\Delta) + \max(\Delta) = 2\delta$, we get $f(axb) + t \leq 2\delta$ which implies that $(axb)_t \overline{q_\Delta} f$. If $f(axb) > \inf(\Delta)$, then $f(axb) \notin \Delta \cup [\max(\Delta), 1]$. Thus $(axb)_t \overline{\in_\Delta} f$ and $(axb)_t \overline{q_\Delta} f$. This shows that $(axb)_t \bar{\beta} f$ for all $\beta \in \{\in_\Delta, q_\Delta, \in_\Delta \wedge q_\Delta, \in_\Delta \vee q_\Delta\}$ which is contradicted by f is an (α, β)-fuzzy generalized bi-ideal of S.

Case (2): $\alpha = q_\Delta$. Since $\inf(\Delta) + \max(\Delta) = 2\delta$ and $a, b \in S_f$, we have $a_{\max(\Delta)} q_\Delta f$ and $b_{\max(\Delta)} q_\Delta f$. Then $a_{\max(\Delta)} \alpha f$ and $b_{\max(\Delta)} \alpha f$. If $f(axb) \leq \inf(\Delta)$, then $f(axb) < \max(\Delta)$ and $f(axb) + \max(\Delta) \leq 2\delta$ which imply that $(axb)_{\max(\Delta)} \overline{\in_\Delta} f$ and $(axb)_{\max(\Delta)} \overline{q_\Delta} f$. If $f(axb) > \inf(\Delta)$, then $f(axb) \notin \Delta \cup [\max(\Delta), 1]$. Thus $(axb)_{\max(\Delta)} \overline{\in_\Delta} f$, $(axb)_{\max(\Delta)} \overline{q_\Delta} f$. This shows that $(axb)_{\max(\Delta)} \bar{\beta} f$ for all $\beta \in \{\in_\Delta, q_\Delta, \in_\Delta \wedge q_\Delta, \in_\Delta \vee q_\Delta\}$ which is contradicted by f is an (α, β)-fuzzy generalized bi-ideal of S. Hence $axb \in S_f$ for all $a, b \in S_f$ and $x \in S$. Therefore S_f is a generalized bi-ideal of S.

∎

By taking $\Delta = (\gamma, 1]$ in Theorem 3.5, we get the following corollary.

Corollary 3.6. [11] Let $2\delta = 1 + \gamma, \alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$ and $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \wedge q_\delta, \in_\gamma \vee q_\delta\}$. If f is an (α, β)-fuzzy generalized bi-ideal of a semigroup S, then $\{a \in S \mid f(a) > \gamma\} (\neq \emptyset)$ is a generalized bi-ideal of S.

Theorem 3.7. Let $\max(\Delta)$ exist and $2\delta = \max(\Delta) + \inf(\Delta)$. Then any nonempty subset G of a semigroup S is a generalized bi-ideal of S if and only if the fuzzy subset f of S, defined as follow:

$$f(a) = \begin{cases} t_1 \in [\delta, 1] \cap (\Delta \cup [\max(\Delta), 1]) & \text{if } a \in G; \\ t_2 \in [0, \inf(\Delta)] & \text{otherwise} \end{cases}$$

for all $a \in S$, is an $(\alpha, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S.

Proof. $(\Rightarrow)$ (i) Assume that $a_t \in_\Delta f$ and $b_r \in_\Delta f$. Thus $\min\{t, r\} \in \Delta$, $f(a) = t_1$ and $f(b) = t_1$, so $a, b \in G$. Since G is a generalized bi-ideal of S, we have $axb \in G$. Hence $f(axb) = t_1 \geq \delta$ and $f(axb) \in \Delta \cup [\max(\Delta), 1]$. If $\min\{t, r\} \leq \delta$, then $(axb)_{\min\{t,r\}} \in_\Delta f$. On the other hand, if $\min\{t, r\} > \delta$, then we have $(axb)_{\min\{t,r\}} q_\Delta f$. Thus f is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S.

(ii) Assume that $a_t q_\Delta f$ and $b_r q_\Delta f$. Then $\min\{t,r\} \in \Delta$, $f(a) + t > 2\delta$ and $f(b) + r > 2\delta$. Since $2\delta = \max(\Delta) + \inf(\Delta)$, we have

$$f(a) > 2\delta - t \geq 2\delta - \max(\Delta) = \inf(\Delta).$$

Similarly, we get $f(b) > \inf(\Delta)$. Thus $a, b \in G$ and so $axb \in G$. Hence $f(axb) = t_1 \geq \delta$ and $f(axb) \in \Delta \cup [\max(\Delta), 1]$. Then $(axb)_{\min\{t,r\}} \in_\Delta \vee q_\Delta f$. Therefore f is an $(q_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S.

(iii) Assume that $a_t \in_\Delta f$ and $b_r q_\Delta f$. Then $\min\{t, r\} \in \Delta, f(a) \geq t > \inf(\Delta)$, and

$$f(b) > 2\delta - r \geq 2\delta - \max(\Delta) = \inf(\Delta).$$

Thus $a, b \in G$ and so $axb \in G$. Analogous to (i) and (ii) we get $(axb)_{\min\{t,r\}} \in_\Delta \vee q_\Delta f$. Therefore f is an $(\in_\Delta \vee q_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S.

($\Leftarrow$) Let f be an $(\alpha, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S. Then $G = S_f$. Hence it follows from Theorem 3.5 that G is a generalized bi-ideal of S.

∎

Corollary 3.8. Let $\max(\Delta)$ exist, $2\delta = \max(\Delta) + \inf(\Delta)$ and let G be a nonempty subset of a semigroup S. Then G is a generalized bi-ideal of S if and only if χ_G is an $(\alpha, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S.

Proof. It follows from Theorem 3.7.

∎

It is easy to prove that every (α, β)-fuzzy generalized bi-ideal of a semigroup S is an $(\alpha, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S. However, the converse is not true and we can see in the following example.

Example 3.9. Let $S = \{a, b, c, d, e\}$ and the binary operation "$\cdot$" be defined on S as follows:

$\cdot$	a	b	c	d	e
a	a	a	a	a	e
b	a	a	a	a	e
c	a	a	b	a	e
d	a	a	b	b	e
e	e	e	e	e	a

Then S is a semigroup under the binary operation "$\cdot$". Define a fuzzy subset f of S by

$$f(a) = 0.9, \quad f(b) = 0.4, \quad f(c) = 0.8, \quad f(d) = 0.3, \quad f(e) = 0.5.$$

Choose $\Delta = \{0.2, 0.3, 0.5, 0.6\}$ and $\delta = 0.5$. The following statements are true.

(1) f is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S.
(2) f is not an $(\in_\Delta, \in_\Delta)$-fuzzy generalized bi-ideal of S since $a_{0.6} \in_\Delta f$ and $(aea)_{0.6}\ \overline{\in_\Delta}\ f$.
(3) f is not an $(\in_\Delta, q_\Delta)$-fuzzy generalized bi-ideal of S since $a_{0.3} \in_\Delta f$ and $(aea)_{0.3}\ \overline{q_\Delta} f$.

Proposition 3.10. Any $(\in_\Delta \vee q_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of a semigroup S is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S.

Proof. By using the fact that $a_t \in_\Delta f$ implies $a_t \in_\Delta \vee q_\Delta f$, this proposition holds. ∎

4. $(\in_\Delta, \in_\Delta \vee q_\Delta)$-FUZZY GENERALIZED BI-IDEALS

In this section, we investigate characterizations of $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideals of semigroups. Also characterizations of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideals of semigroups are given by take $\Delta = (\gamma, 1]$.

Definition 4.1. A fuzzy subset f of a semigroup S is called an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-**fuzzy generalized bi-ideal** of S if

$$a_t \in_\Delta f \text{ and } b_r \in_\Delta f \Rightarrow (axb)_{\min\{t,r\}} \in_\Delta \vee q_\Delta f$$

for all $a, b, x \in S$ and $t, r \in \Delta$.

If we take $\Delta = (\gamma, 1]$ in Definition 4.1, then it is reduced to the following definition.

Definition 4.2. A fuzzy subset f of a semigroup S is said to be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-**fuzzy generalized bi-ideal** of S if

$$a_t \in_\gamma f \text{ and } b_r \in_\gamma f \Rightarrow (axb)_{\min\{t,r\}} \in_\gamma \vee q_\delta f$$

for all $a, b, x \in S$ and $t, r \in \Delta$.

Remark 4.3. If f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S, then f is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S where $\Delta = (\gamma, 1]$.

Note that if $S_f \neq \emptyset$, then f is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S. In the following theorem, an equivalent condition for an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal f of S is given by using the set S_f.

Theorem 4.4. A fuzzy subset f of a semigroup S is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S if and only if S_f $(\neq \emptyset)$ is a generalized bi-ideal of S and

$$f(axb) \geq \min\{f(a), f(b), \delta\} \text{ for all } a, b \in S_f, x \in S.$$

Proof. ($\Rightarrow$) Let $a, b \in S_f$ and $x \in S$. Suppose on the contrary that

$$f(axb) < \min\{f(a), f(b), \delta\}.$$

Choose $t := \min\{f(a), f(b), \delta\}$, we have $\inf(\Delta) < t \in \Delta$. Thus $f(a) \geq t, f(b) \geq t$, $f(axb) < t$ and $f(axb) + t < 2\delta$. Hence $a_t, b_t \in_\Delta f$ but $(axb)_t \overline{\in_\Delta \vee q_\Delta} f$. It is contradicted by f is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S. Thus $f(axb) \geq \min\{f(a), f(b), \delta\}$. Next, we shall show that $axb \in S_f$. Since $a, b \in S_f$ and f is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S, we obtain $(axb)_t \in_\Delta \vee q_\Delta f$. Thus $axb \in S_f$.
($\Leftarrow$) Assume that $a_t, b_r \in_\Delta f$. Then $a, b \in S_f$ and $\min\{f(a), f(b)\} \geq \min\{t, r\}$. By the assumption, we get $axb \in S_f$ and $f(axb) \geq \min\{t, r, \delta\} > \inf(\Delta)$. Thus $(axb)_{\min\{t,r\}} \in_\Delta \vee q_\Delta f$. Therefore f is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S. ■

By taking $\Delta = (\gamma, 1]$ in Theorem 4.4, we obtain the following corollary.

Corollary 4.5. [11] Any fuzzy subset f of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S if and only if

$$\max\{f(axb), \gamma\} \geq \min\{f(a), f(b), \delta\}$$

for all $a, b, x \in S$.

Note that if $\Delta \neq (\gamma, 1]$ and f is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of a semigroup S, then f is not an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S in general.

Example 4.6. The set of all positive integers $\mathbb{N}$ is a semigroup under the usual multiplication. Let f be a fuzzy subset of $\mathbb{N}$ defined by

$$f(n) = \begin{cases} 0.6 & \text{if } n \in \{1, 2, \ldots, 5\}; \\ \dfrac{n-1}{n} - 0.4 & \text{if } n \in \{2(m+2) \mid m \in \mathbb{N}\}; \\ 1 & \text{if } n \in \{2m+5 \mid m \in \mathbb{N}\} \end{cases}$$

for all $n \in \mathbb{N}$. If $\Delta = (0.3, 0.8] \setminus \{0.6\}$ and $\delta = 0.7$, then by the routine calculations, we have that f is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of $\mathbb{N}$ but not an $(\in_{0.3}, \in_{0.3} \vee q_{0.7})$-fuzzy generalized bi-ideal of $\mathbb{N}$, because

$$\max\{0.3, f(10)\} = 0.5 < 0.6 = \min\{f(2), f(5), 0.7\}.$$

Definition 4.7. [11] Let f be a fuzzy subset of a nonempty set S. Define the fuzzy subset $\tilde{f}$ of S by $\tilde{f}(a) = \max\{\min\{f(a), \delta\}, \gamma\}$ for all $a \in S$.

Definition 4.8. Let f be a fuzzy subset of a nonempty set S. The fuzzy subset f^* of S is defined by

$$f^*(a) = \begin{cases} \min\{f(a), \delta\} & \text{if } a \in S_f; \\ \inf(\Delta) & \text{otherwise} \end{cases}$$

for all $a \in S$.

Remark 4.9. If we take $\Delta = (\gamma, 1]$ in Definition 4.8, then $\tilde{f} = f^*$.

Note that if f is a fuzzy subset of a nonempty set S, then $S_f = S_{f^*}$. In the following theorem, an characterization of any $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal f of semigroups is considered via the fuzzy subset f^*.

Theorem 4.10. A fuzzy subset f of a semigroup S is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S if and only if f^* is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S.

Proof. Let $a_t \in_\Delta f^*$ and $b_r \in_\Delta f^*$. Then $f^*(a) \geq t$, $f^*(b) \geq r$ and $a, b \in S_{f^*} = S_f$. Thus $f^*(a) = \min\{f(a), \delta\}$ and $f^*(b) = \min\{f(b), \delta\}$. By Theorem 4.4, we get $axb \in S_f = S_{f^*}$ and $f(axb) \geq \min\{f(a), f(b), \delta\}$. Hence $f^*(axb) \geq \min\{t, r, \delta\}$, so $(axb)_{\min\{t,r\}} \in_\Delta \vee q_\Delta f^*$. Therefore f^* is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S.

Conversely, let $a_t \in_\Delta f$ and $b_r \in_\Delta f$. Then $f(a) \geq t$, $f(b) \geq r$ and $a, b \in S_f = S_{f^*}$. Thus $f^*(a) \geq \min\{t, \delta\}$ and $f^*(b) \geq \min\{r, \delta\}$. By Theorem 4.4, we have $axb \in S_{f^*} = S_f$ and

$$f^*(axb) \geq \min\{f^*(a), f^*(b), \delta\} \geq \min\{t, r, \delta\}.$$

Thus $f(axb) \geq f^*(axb) \geq \min\{t, r, \delta\}$. Hence $(axb)_{\min\{t,r\}} \in_\Delta \vee q_\Delta f$. Therefore f is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S.

■

5. CHARACTERIZATIONS OF SEMIGROUPS

In this section, we characterize completely regular semigroups and groups in terms of $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideals. If we take $\Delta = (\gamma, 1]$ in Theorem 5.1 and Theorem 5.3, then we immediately have characterizations of such semigroups via their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideals.

Theorem 5.1. A semigroup S is completely regular if and only if $f^*(a) = f^*(a^2)$ for all $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal f of S and $x \in S$.

Proof. Let f be an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S and $a \in S$. If $a \notin S_f$ and $a^2 \notin S_f$, then $f^*(a) = \inf(\Delta) = f^*(a^2)$. Suppose that $a \in S_f$ or $a^2 \in S_f$. Since S is completely regular, we get $a = a^2ba^2$ for some $b \in S$. By Theorem 4.4 and Theorem 4.10, we have

$$\begin{aligned} f^*(a) &= f^*(a^2ba^2) & &\geq \min\{f^*(a^2), \delta\} \\ &= f^*(a^2) & &= f^*(a^3ba^2) \\ &\geq \min\{f^*(a), \delta\} & &= f^*(a). \end{aligned}$$

This show that $f^*(a) = f^*(a^2)$.

Conversely, let G be a generalized bi-ideal of S and $a \in$ S such that $a^2 \in G$. Then χ_G is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S. By the assumption, we have $\chi_G^*(a^2) = \chi_G^*(a)$ and so $a \in G$. Hence G is semiprime. By Theorem 2.1, we get S is completely regular.

■

If we take $\Delta = (\gamma, 1]$ in Theorem 5.1, then it is reduced to the following corollary.

Corollary 5.2. Let S be a semigroup. Then S is a completely regular semigroup if and only if $\tilde{f}(a) = \tilde{f}(a^2)$ for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f of S and $a \in S$.

In the following theorem, we characterize semigroups that are groups by using their $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideals.

Theorem 5.3. A semigroup S is a group if and only if $|\mathrm{Im}(f^*)| = 1$ for each $(\in_\Delta, \in_\Delta \vee q_\Delta)$- fuzzy generalized bi-ideal f of S.

Proof. Let e be identity of a group S, f be an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S and $x \in S$. By using Theorem 4.4 and Theorem 4.10, we get $f^*(exe) \geq \min\{f^*(e), \delta\}$ and so

$$\begin{aligned} f^*(x) &\geq f^*(e) && = f^*(e^2) \\ &= f^*((xx^{-1})(x^{-1}x)) && \geq \min\{f^*(x), \delta\} \\ &= f^*(x)\,. \end{aligned}$$

Hence $f^*(x) = f^*(e)$. This show that $|\mathrm{Im}(f^*)| = 1$.

Conversely, let G be a generalized bi-ideal of S. Thus χ_G^* is an $(\in_\Delta, \in_\Delta \vee q_\Delta)$-fuzzy generalized bi-ideal of S. Since the assumption, we obtain $\chi_G^*(x) = \delta$ for all $x \in S$. Thus $G = S$. Therefore it follows from Theorem 2.2 that S is a group. ∎

By taking $\Delta = (\gamma, 1]$ in Theorem 5.3, we get the following corollary.

Corollary 5.4. A semigroup S is a group if and only if $|\mathrm{Im}(\tilde{f})| = 1$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f of S.

ACKNOWLEDGEMENTS

This research is supported by by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand and by Pibulsongkram Rajabhat University, Phitsanulok, Thailand.

REFERENCES

[1] Huang, X., Yin, Y., and Zhan, J., 2013, "Characterizations of semihyperrings by their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideals," Journal of Applied Mathematics, 1, pp.1-13.

[2] Kuroki, N., 1979, "Fuzzy bi-ideals in semigroups," Comment Math Univ St Paul, 5, pp.128-132.

[3] Kuroki, N., 1981, "On fuzzy ideals and fuzzy bi-ideals in semigroups," Fuzzy Sets and Systems, 5(2), pp.203-215.

[4] Kuroki, N., 1992, "Fuzzy generalized bi-ideals in semigroups," Information Sciences, 66, pp.235-243.

[5] Lajos, S., 1975, "Generalized bi-ideals in semigroups," K Marx Univ Economics, Dept Math Budapest, 5, pp.1-16.

[6] Lajos, S., 1984, "Note on generalized bi-ideals in semigroups," Soochow Journal of Mathematics, 10, pp.55-59.

[7] Mordeson, J.N., and Malik, D.S., 2002, "Fuzzy automata and languages: theory and applications," Chapman and Hall/CRC, Boca Raton.

[8] Mordeson, J.N., Malik, D.S., and Kuroki, N., 2003, "Fuzzy semigroups, Studies in Fuzziness and Soft Computing," 131, Springer-Verlag, Berlin.

[9] Rehman, N., and Shabir, M., 2014, "Some characterizations of ternary semigroups by the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals," Journal of Intelligent and Fuzzy Systems, 26(5), pp.2107-2117.

[10] Rehman, N., and Shabir, M., 2013, "Some kinds of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of ternary semigroups," Iranian Journal of Science and Technology (Sciences), 37(3.1), pp.365-378.

[11] Shabir, M., and Ali, M., 2013, "Characterizations of semigroups by the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals," Iranian Journal of Science and Technology (Sciences), 37(2), pp.117-131.

[12] Shabir, M., Jun, Y.B., and Nawaz, Y., 2010, "Characterizations of regular semigroups by (α, β)-fuzzy ideals,"Computers and mathematics with applications, 59(1),pp.161-175.

[13] Shabir, M., Jun, Y.B., and Nawaz, Y., 2010, "Semigroups characterized by $(\in, \in \vee q_k)$-fuzzy ideals," Computers and Mathematics with Applications,60(5), pp.1473-1493.

[14] Shabir, M., and Mahmood, T., 2015, "Semihypergroups characterized by$(\in_\gamma, \in_\gamma \vee q_\delta)$- fuzzy hyperideals," Journal of Intelligent and Fuzzy Systems, 28, pp.2667-2678.

[15] Zadeh, L.A., 1965, "Fuzzy sets," Information and control, 8(3), pp.338-353.