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Effectiveness of the Extended Kalman Filter Through Difference Equations

Parveen Banu M
Department of Mathematics,
Mazharul Uloom College, India.

Abstract

This paper develops a new approach about the Extended Kalman Filter (EKF) for the output $y(\tau)$ of a process over the interval $[t_0, t_f]$ of the form

$$\dot{x} = A(t)x + B(t)u,$$

$$y = C(t)x$$

with the steady state Extended Kalman Filter of the quadratic control $\dot{x} = eAx + Bu$ with (A, B) as stabilizable. Further analysis is carried out related to its deterministicness and autonomy.

AMS subject classification:
Keywords:

1. Introduction

The Kalman Filter [3] is a process whose inputs are noisy and incomplete measurements, whose outputs are optimal estimates of system variables [11]. The KF presents a recursive approach to discrete linear filtering and prediction problems [4]. The KF is so popular because it produces good results, easy to formulate and implement, convenient form for online real time processing [9]. Theoretically, a KF is an estimator, also called as the linear quadratic Gaussian (LQG) estimator, is used to estimate the instantaneous state of a linear dynamical system perturbed by Gaussian noise [8].

2. Preliminaries

Definition 2.1. State observer. A state observer is a system that provides an estimate of the internal state of a given real system, from measurements of the input and output of the real system [1].

Definition 2.2. Controllability. A system is said to be controllable if every state vector $x(k)$ can be transformed to a desired state in finite time by the application of unconstrained control inputs $u(k)$. An uncontrollable system is one where some elements of the state vector $x(k)$ cannot be affected by the control input [5].

Definition 2.3. Observability. A system is said to be observable at a time step k_0 if for a state $x(k_0)$ at that time, there is a finite $k_1 > k_0$ such that knowledge of the outputs z from k_0 to k_1 are sufficient to determine state k_0 [10].

Definition 2.4. Optimal control. The Optimal control theory is an extension of the calculus of variations, which is a mathematical optimization method for deriving control policies. An optimal control is a set of difference equations describing the paths of the control variables that minimize the cost functional [10].

Definition 2.5. Noise measurement. It relates to the sensitivity of communications systems, the purity of signals, or the quality of audio systems. The concept is to define the noise level below which signals cannot reliably be detected. It can be thought of as uncertainty of the information being carried over a communications channel [7].

Now we discuss about the Kalman Filter. The State Estimator design problem is to choose L in

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y(t) - C\hat{x}),$$

$$\dot{\tilde{x}} = (A - LC)\tilde{x}.$$

So that the Observer Error dynamics is stable, if A-LC is stable.

The related State feedback problem is to choose K in

$$\dot{x} = A^T x + C^T u; \quad u = -Kx.$$

$$\Rightarrow \dot{x} = (A^T - C^T K)x.$$

$A^T - C^T K$ is stable. Choose $L = K^T$ for the observer, which is ensured to be stable [2] [3].

3. Steady State Kalman Filter

Consider the Infinite Horizon LQ(Linear Quadratic) control $\dot{x} = Ax + Bu$. We can easily solve the positive definite P_∞ in the Algebraic Riccati Equation (ARE),

$$A^T P_\infty + P_\infty A - P_\infty B R^{-1} B^T P_\infty + Q = 0.$$

We can get the optimal feedback gain $u = -Kx$, where $K = R^{-1}B^T P_\infty$
Since (A, B) and $(A, Q^{\frac{1}{2}})$ are stabilizable and detectable, $A - BK$ has eigenvalues on the open left half plane.

The stable observer problem is to find L in

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

so that $A - LC$ is stable.
Let us solve LQ control problem for the dual problem

$$\dot{x} = A^T x + C^T u \tag{3.1}$$

Transform the ARE as $C^T \to B$, $A^T \to A$, then we have

$$AP_\infty + P_\infty A^T - P_\infty C^T R^{-1} C P_\infty + Q = 0.$$

Stabilizing feedback gain K for (3.1) is

$$K = L^T = R^{-1} C P_\infty \Rightarrow L = P_\infty C^T R^{-1}.$$

This generates the observer $\dot{\hat{x}} = A\hat{x} + Bu + P_\infty C^T R^{-1}(y - C\hat{x})$, with observer error dynamics $\dot{\tilde{x}} = (A - P_\infty C^T R^{-1} C)\tilde{x}$ which will be stable, as long as (A^T, C^T) is stabilizable, which is the same as (A, C) being detectable and $(A^T, Q^{\frac{1}{2}})$ is detectable, which is the same as $(A, Q^{\frac{1}{2}})$ being stabilizable [10].

4. Deterministic Kalman Filter

The Problem statement of the Kalman Filter is given by [11],

Suppose we are given the output $y(\tau)$ of a process over the interval $[t_0, t_f]$. It is anticipated that $y(\tau)$ is generated from a process of the form,

$$\dot{x} = A(t)x + B(t)u,$$

$$y = C(t)x.$$

To account for model uncertainty and measurement noise, the process model used is modified to,

$$\begin{aligned} \dot{\hat{x}} &= A(t)\hat{x} + B(t)u + B_w(t)w \\ y &= C(t)\hat{x} + v. \end{aligned} \tag{4.1}$$

The state $\hat{x}(t|t_f)$ is the state estimate of the actual process at time t given all the measurements up to time t_f . $\dot{\hat{x}}(t|t_f)$ denotes derivative with respect to t. The final time t_f is assumed to be fixed. In (4.1), w and v are the process and measurement noises that are assumed to be unknown. The initial state $\hat{x}(t_0|t_f)$ is also assumed to be unknown.

Choose the initial state $\hat{x}(t_0|t_f)$ and the measurement and process noises$w(\tau|t_f)$, $v(\tau|t_f)$ for $\tau?[t_0, t_f]$, such that
1. The measured output is

$$y(\tau) = C(\tau)\hat{x}(\tau|t_f) + v(\tau|t_f) \tag{4.2}$$

for all $\tau?[t_0, t_f]$.
2. Minimize the following objective function by the LQ control

$$J(y(.), \hat{x}(t_0|t_f), w(.|t_f)) = \frac{1}{2}\hat{x}^T(t_0|t_f)S^{-1}\hat{x}(t_0|t_f)$$

$$+\frac{1}{2}\int_{t_0}^{t_f} w^T(\tau|t_f)Q^{-1}(\tau)w(\tau|t_f) + v^T(\tau|t_f)R^{-1}(\tau)v(\tau|t_f)d\tau$$

where $v(\tau|t_f) = y(\tau) - C(\tau)x(\tau|t_f)$.
Thus, the goal is to use as little noise or initial condition as possible so as to explain the measured output $y(\tau)$.
Hence we have to set

- R^{-1} large when measurement is accurate, i.e. assume little measurement noise v.
- Q^{-1} large when process model is accurate, i.e. assume little process noise w.
- S^{-1} large when estimate of $\hat{x}(t_0|t_f) = 0$ is a confident guess of the initial state.

If a non-zero estimate (say x_0) for the initial state is desired, it is possible to modify the cost function so that,

$$\frac{1}{2}\hat{x}^T(t_0|t_f)S^{-1}\hat{x}(t_0|t_f) \Rightarrow \frac{1}{2}(\hat{x}(t_0|t_f) - x_0)^T S^{-1}(\hat{x}(t_0|t_f) - x_0).$$

It is clear that the optimal estimate at t_0 before any measurement is made is $\hat{x}(t_0|t_0) = x_0$.

Lemma 4.1. Let f be a random vector with values in $\mathbb{C}^k$. Let M be a subspace of random variables, and Y be the subspace generated by the random vector y in $\mathbb{C}^m$. Set $H = M \vee Y$.Then $H = M \oplus E$ where E is the subspace generated by the vector $\varphi = y - P_M y$. Moreover, if R_φ is invertible, then

$$P_H f = P_M f + R_{f\varphi}R_\varphi^{-1}\varphi. \tag{4.3}$$

This decomposition is orthogonal, that is, $P_M f$ is orthogonal to $R_{f\varphi}R_\varphi^{-1}\varphi = P_\epsilon f$. Finally, the error covariance for $f - P_H f$ is given by [4]

$$E(f - P_H f)(f - P_H f)^* = E(f - P_M f)(f - P_M f)^*?R_{f\varphi}R_\varphi^{-1}R_{f\varphi}^*. \tag{4.4}$$

Theorem 4.2. Consider the state space system [4]

$$x(n+1) = Ax(n) + Bu(n), y(n) = Cx(n) + Dv(n) \tag{4.5}$$

where $u(n)$ and $v(n)$ are independent noise random processes, which are independent of the initial condition $x(0)$. Then the optimal estimate $\hat{x}(k) = P_{M_{k-1}}x(k)$ of the state $x(k)$ given the past $\{y(j)\}_0^{k-1}$ is recursively computed by

$$\hat{x}(n+1) = A\hat{x}(n) + \Lambda_n(y(n) - C\hat{x}(n)), \tag{4.6}$$

$$\Lambda_n = AQ_nC^*(CQ_nC^* + DD^*)^{-1} \tag{4.7}$$

The state covariance error $Q_k = E(x(k) - \hat{x}(k))(x(k) - \hat{x}(k))^*$ is recursively computed by solving the Riccati difference equation

$$Q_{n+1} = AQ_nA^* + BB^* - AQ_nC^*(CQ_nC^* + DD^*)^{-1}CQ_nA^*, \tag{4.8}$$

subject to the initial condition $Q_0 = Ex(0)x(0)^*$.

Proof. Let us give a proof of the Kalman filter by implementing the above lemma with $H = M_n = M_{n-1} \vee Y$. In our setting Y is the span of $y(n)$, the subspace $M = M_{n-1}$ and $\varphi = \varphi(n) = y(n) - P_{M_{n-1}}y(n)$. Recall that the solution to the difference equation in (4.5) is given by

$$x(n) = \Psi(n-1, -1)x(0) + \sum_{j=0}^{n-1} \Psi(n-1, j)B(j)u(j), \tag{4.9}$$

$$y(n) = C(n)\Psi(n-1, -1)x(0) + \sum_{j=0}^{n-1} C(n)\Psi(n-1, j)B(j)u(j) + D(n)v(n).$$

Here $\Psi(n, v) = A(n)A(n-1)......A(v+1)$ and $\Psi(k, k) = I$. This shows that

$$M_n = \vee_{k=0}^{n} y(k) \subset \vee\{x(0), u(0), u(1), ..., u(n-1), v(0), v(1), ..., v(n)\}. \tag{4.10}$$

Because $u(k)$ and $v(k)$ are independent noise processes and orthogonal to $x(0)$, the random vector $v(n)$ is orthogonal to M_{n-1}. In particular, $P_{M_{n-1}}v(n) = 0$. Recall that the optimal state estimate is given by $\hat{x}(n) = P_{M_{n-1}}x(n)$. Using this along with the fact that C and D are not random, we have

$$\begin{aligned}\varphi(n) &= y(n) - P_{M_{n-1}}y(n)\\ &= y(n) - P_{M_{n-1}}(Cx(n) + Dv(n))\\ &= y(n) - CP_{M_{n-1}}x(n)\\ &= y(n) - C\hat{x}(n).\end{aligned}$$

Hence $\varphi(n) = y(n) - C\hat{x}(n)$. By definition the state estimation error $\tilde{x}(n) = x(n) - \hat{x}(n)$. Since $y(n) = Cx(n) + Dv(n)$, we have $\varphi(n) = C\tilde{x}(n) + Dv(n)$. This yields the following two useful formulas,

$$\varphi(n) = y(n) - C\hat{x}(n) = C\tilde{x}(n) + Dv(n). \tag{4.11}$$

By consulting (4.9) and (4.10), we see that $v(n)$ is orthogonal to both $x(n)$ and M_{n-1}. Hence $v(n)$ is orthogonal to $\tilde{x}(n) = x(n) - \hat{x}(n)$ and $\varphi(n) = C\tilde{x}(n) + Dv(n)$, implies that

$$\begin{aligned} E\varphi(n)\varphi(n)^* &= E(C\tilde{x}(n) + Dv(n))(C\tilde{x}(n) + Dv(n))^* \\ &= CE\tilde{x}(n)\tilde{x}(n)^*C^* + DD^*. \end{aligned}$$

By definition $Q_n = E\tilde{x}(n)\tilde{x}(n)^*$ is the error covariance. Therefore

$$R_{\varphi(n)} = E\varphi(n)\varphi(n)^* = CQ_nC^* + DD^*. \tag{4.12}$$

Equation (4.10) shows that $u(n)$ is orthogonal to M_{n-1}. In other words,$P_{M_{n-1}}u(n) = 0$. By employing (4.3) from the above Lemma with $\varphi = \varphi(n)$ and $y = y(n)$ and $M_n = M_{n-1} \vee y(n)$, we obtain

$$\begin{aligned} \hat{x}(n+1) = P_{M_n}x(n+1) &= P_{M_{n-1}}x(n+1) + R_{x(n+1)\varphi(n)}R_\varphi^{-1}(n)\varphi(n) \\ &= P_{M_{n-1}}(Ax(n) + Bu(n)) + R_{x(n+1)\varphi(n)}R_\varphi^{-1}(n)\varphi(n) \\ &= A\hat{x}(n) + R_{x(n+1)\varphi(n)}R_\varphi^{-1}(n)\varphi(n) \end{aligned} \tag{4.13}$$

We need an expression for $R_{x(n+1)\varphi(n)}$. Since $\hat{x}(n)$ is contained in M_{n-1}, the random vector $\varphi(n) = y(n) - C\hat{x}(n)$ is contained in M_n. Hence $\varphi(n)$ is orthogonal to $u(n)$ (by 4.10). Moreover, $v(n)$ is orthogonal to $x(n)$ (by 4.9). Using $\varphi(n) = C\tilde{x}(n) + Dv(n)$, we have

$$\begin{aligned} Ex(n+1)\varphi(n)^* &= E(Ax(n) + Bu(n))\varphi(n)^* = AEx(n)\varphi(n)^* \\ &= AEx(n)(C\tilde{x}(n) + Dv(n))^* = AEx(n)\tilde{x}(n)^*C^* \\ &= AE(\hat{x}(n) + \tilde{x}(n))\tilde{x}(n)^*C^* = AE\tilde{x}(n)\tilde{x}(n)^*C^* \\ &= AQ_nC^*. \end{aligned}$$

The second from the last equality follows from the fact that $\hat{x}(n)$ is orthogonal to $\tilde{x}(n)$. The previous calculation yields the result

$$R_{x(n+1)\varphi(n)} = Ex(n+1)\varphi(n)^* = AQ_nC^*. \tag{4.14}$$

Substituting $Rx(n+1)\varphi(n) = AQ_nC^*$ and the expression for $R_{\varphi(n)}$ in (4.11) into (4.12) yields

$$\hat{x}(n+1) = A\hat{x}(n) + AQ_nC^*(CQ_n^*C^* + DD^*)^{-1}\varphi(n). \tag{4.15}$$

Finally, using $\varphi(n) = y(n) - C\hat{x}(n)$ gives the state space formula for $\hat{x}(n)$ in (4.6).

Now let us use (4.4) from the above Lemma to derive the discrete time Riccati equation in (4.8). Recall that $u(n)$ is orthogonal to M_{n-1}. Using $P_{M_{n-1}}u(n) = 0$ along with the optimal state estimate $\hat{x}(n) = P_{M_{n-1}}x(n)$, we obtain

$$\begin{aligned} x(n+1) - P_{M_{n-1}}x(n+1) &= Ax(n) + Bu(n) - P_{M_{n-1}}(Ax(n) + Bu(n)) \\ &= Ax(n) + Bu(n) - A\hat{x}(n) = A\tilde{x}(n) + Bu(n). \end{aligned}$$

This readily implies that

$$x(n+1) - P_{M_{n-1}}x(n+1) = A\tilde{x}(n) + Bu(n). \tag{4.16}$$

By (4.9) we see that $u(n)$ is orthogonal to $x(n)$. Since the optimal estimate $\hat{x}(n) = P_{M_{n-1}}x(n)$ is a vector in M_{n-1}, the random vector $u(n)$ is also orthogonal to $\hat{x}(n)$ (by 4.10). Hence $u(n)$ is orthogonal to the error $\tilde{x}(n) = x(n) - \hat{x}(n)$. Using this fact, in (4.15) along with $E\tilde{x}(n)\tilde{x}(n)^* = Q_n$, we arrive at

$$E(x(n+1) - P_{M_{n-1}}x(n+1))(x(n+1) - P_{M_{n-1}}x(n+1))^* = AQ_nA^* + BB^*.$$

Recall that $Ex(n+1)\varphi(n)^* = AQ_nC^*$ (by 4.13). Finally, by employing equation (4.4) in above Lemma with the expression for $R_{\varphi(n)}$ in (4.11) , we have

$$\begin{aligned} Q_{n+1} &= E(x(n+1) - P_{M_n}x(n+1))(x(n+1) - P_{M_n}x(n+1))^* \\ &= E(x(n+1) - P_{M_{n-1}}x(n+1))(x(n+1) - P_{M_{n-1}}x(n+1))^* \\ &- Ex(n+1)\varphi(n)^*R_{\varphi}^{-1}(n)(Ex(n+1)\varphi(n)^*)^* \\ &= AQ_nA^* + BB^* - AQ_nC^*(CQ_nC^* + DD^*)^{-1}CQ_nA^*. \end{aligned}$$

This is the Riccati difference equation in (4.8). To obtain the initial condition,recall that $M_{-1} = 0$, that is, $y(-1) = 0$.
Hence $\tilde{x}(0) = x(0) - P_{M-1}x(0) = x(0)$.
Thus $Q_0 = E\tilde{x}(0)\tilde{x}(0)^* = Ex(0)x(0)^*$.
This completes the proof. ■

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