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On Common Fixed Points of three Maps in Generalized Metric Space

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Abstract

The aim of this paper is to prove common fixed point theorem of three self maps in a complete G-metric space without condition of continuity.

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1. Introduction

In 1922 [1], proved a fixed point theorem in complete metric space called Banach contraction principle.Many authors generalized and modified Banach fixed point theorem in several ways.Inspired by the fact that fixed point theory has many applications in science and engineering field.Many authors proved various fixed point theorems for different mappings,contraction satisfying certain conditions.Some Authors generalized the notion of metric space to obtained more and more generalized structure of metric space.And starting from Banach contraction principle proved all results in new generalized structure.

In 1992 B.C. Dhage [2] introduced the concept of D-metric space.In the geometric sense a D-metric D(x,y,z) is the perimeter of the triangle with vertices $x, y, z, \in R^2$.But later in 2003, Mustafa and Sims [3] show that some topological properties are not true.Also the D-metric is not continuous of its variables.

In 2006 Z. Mustafa and Sims [4] introduced a new structure of generalized metric space which overcomes the drawbacks of D-metric space. many authors proved fixed point theorems in this context.

The study of existence and uniqueness of common fixed points of functions which satisfies different contractive condition has been an interesting field of Mathematics from 1922.

In 1986 Sesa [5] introduced a concept of weakly commuting mappings and obtained some common fixed point theorems in complete metric space. Then Jungck [6] defined compatible mappings and proved some common fixed point results.In 2009 Aage C.T. and Salunke J.N. [7] proved common fixed point results of three maps which are compatible of type A. The purpose of this paper is to prove common foxed point theorem for three maps in the context of G-metric space.

2. Preliminaries

Definition 2.1. G-metric Space [4] Let X be a non empty set and $G : X^3 \to R^+$ which satisfies the following conditions

1. $G(a, b, c) = 0 \; if a = b = c \; i.e. \; every \; a, b, c \; in \; X \; coincides.$
2. $G(a, a, b) > 0 \; for \; every \; a, b, c \in X \; s.t. \; a \neq b$
3. $G(a, a, b) \leq G(a, b, c), \; \forall a, b, c \in \; X \; s.t. \; c \neq b$
4. $G(a, b, c) = G(b, a, c) = G(c, b, a) = ...$
   (symmetrical in all three variables)
5. $G(a, b, c) \leq G(a, x, x) + G(x, b, c)$, for all a,b,c,x in X
   (rectangle inequality)

Then the function G is said to be generalized metric or simply G-metric on X and the pair (X,G) is said to be G-metric space.

Example 2.2. [4]Let $G : X^3 \to R^+$ s.t. $G(a, b, c)$ = perimeter of the triangle with vertices at a,b,c in R^2, also by taking p in the interior of the triangle then rectangle inequality is satisfied and the function G is a G-metric on X.

Remark 2.3. G-metric space is the generalization of the ordinary metric space that is every G-metric space is (X,G) defines ordinary metric space (X,d_G) by d_G(a,b)=$G(a, b, b)+ G(a, a, b)$.

Example 2.4. Let (X,d) be the usual metric space. Then the function $G : X^3 \to R^+$ defined by

$$G(a, b, c) = max.\{d(a, b), d(b, c), d(c, a)\}$$

for all a,b,c $\in$ X is a G-metric space.

Definition 2.5. A G-metric space (X,G) is said to be symmetric if $G(a, b, b) = G(a, a, b)$ for all $a, b \epsilon X$ and if $G(a, b, b) \neq G(a, a, b)$ then G is said to be non symmetric G-metric space.

Example 2.6. Let $X = \{x, y\}$ and $G : X^3 \to R^+$ defined by $G(x, x, x) = G(y, y, y) = 0$, $G(x, x, y) = 1$, $G(x, y, y) = 2$ and extend G to all of X^3 by symmetry in the variables. Then X is a G-metric space but It is non symmetric. Since $G(x, x, y) \neq G(x, y, y)$.

Definition 2.7. Let (X,G) be a G-metric space, Let $\{a_n\}$ be a sequence of elements in X. The sequence $\{a_n\}$ is said to be G-convergent to a
if

$$lim_{m,n\to\infty} G(a, a_n, a_m) = 0$$

i.e for every $\epsilon > 0$ there is N s.t. $G(a, a_n, a_m) < \epsilon$ for all $m, n \geq N$. It is denoted as $a_n \to a$ or $lim_{n\to\infty} a_n = a$.

Proposition 2.8. [4] If (X,G) be a G-metric space.Then the following are equivalent

1. $\{a_n\}$ is G-convergent to a.
2. $G(a_n, a_n, a) \to 0 \; as \; n \to \infty$
3. $G(a_n, a, a) \to 0 \; as \; n \to \infty$
4. $G(a_m, a_n, a) \to 0 \; as \; m, n \to \infty$.

Definition 2.9. Let (X,G) be a G-metric space a sequence $\{a_n\}$ is called G-Cauchy if, for each $\epsilon > 0$ there is an N ϵI^+ (set of positive integers) s.t.

$$G(a_n, a_m, a_l) < \epsilon \; for \; all \; n, m, l \geq N$$

Proposition 2.10. Let (X,G) be a G-metric space then the function G(a,b,c) is said to be jointly continuous in all three of its variables.

Proposition 2.11. [4] Let (X,G) be a G-metric space. Then, for any a,b,c,x in X it gives that

1. $if \; G(a, b, c) = 0 \; then \; a = b = c$
2. $G(a, b, c) \leq G(a, a, b) + G(a, a, c)$
3. $G(a, b, b) \leq 2G(b, a, a)$
4. $G(a, b, c) \leq G(a, x, c) + G(x, b, c)$
5. $G(a, b, c) \leq \frac{2}{3}(G(a, x, x) + G(b, x, x) + G(c, x, x))$

Now we begin with the contraction mapping in G-metric space.

Definition 2.12. Let (X,G) be a G-metric space and $T : X \to X$ be a self mapping on X.Now T is said to be a contraction if

$$G(Ta, Tb, Tc) \leq \alpha G(a, b, c) \ for\ all\ a, b, c \epsilon X\ where\ 0 \leq \alpha < 1$$

Aage C.T. And J.N. Salunke proved common fixed point theorem for three self maps of compatible of type A in Complete metric space.

Thoerem 2.13. [7] Suppose S,T and I are three self mappings of a complete metric space (X,d) satisfying the following conditions.

1. $S(X) \subset T(X) \subset I(X)$.
2. $d(Sx, Ty) \leq \alpha d(Ix, Iy) + \beta[d(Sx, Ix) + d(Ty, Iy)] + \gamma[d(Ix, Ty) + d(Iy, Sx)]$, for all $x, y \epsilon X$ and α, β and γ are non negative reals such that $\alpha + 2\beta + 2\gamma < 1$
3. One of S,T and I is continuous.
4. (S,I) and (T,I) are compatible of type A.Then S,T and I have a unique common fixed point.

3. Main Result

In this part we prove common fixed point result for three maps in the context of G-metric space.

Theorem 3.1. Let A,S and T be self mappings on a complete G-metric space X which satisfies

$$\begin{aligned} G(Aa, Sb, Tc) \leq & \alpha G(a, b, c) + \beta[G(Aa, a, a) + G(b, Sb, b) + G(c, c, Tc)] + \\ & \gamma[G(Aa, b, c) + G(a, Sb, c) + G(a, b, Tc)] + \delta G(a, b, c) \end{aligned} \tag{3.1}$$

for all a,b,c in X , where $\alpha, \beta, \gamma, \delta > 0$ and $1 < \alpha + 3\beta + 4\gamma + \delta$. Then any fixed point of A is a fixed point of S and T and Conversely.Furthermore A,S and T have a unique common fixed point in X.

Proof. We will prove the theorem in two steps,
Step I
In first step we prove that any fixed point of A is a fixed point of S and T.Assume that $p \epsilon X$ be a fixed point of A i.e. Ap=p. Now we prove that Sp=p and Tp=p.We prove this by method of contradiction.If possible suppose that $p \neq Sp$ and $p \neq Tp$.

$$G(p, Sp, Tp) = G(Ap, Sp, Tp)$$

$$\begin{aligned}G(p,Sp,Tp) \leq& \alpha G(p,p,p)+\beta[G(Ap,p,p)+G(p,Sp,p)+G(p,p,Tp)]\\&+\gamma[G(Ap,p,p)+G(p,Sp,p)+G(p,p,Tp)]+\delta G(p,p,p)\\&=(\beta+\gamma)[G(p,Sp,p)+G(p,p,Tp)\\&\leq(\beta+\gamma)[G(p,Sp,Tp)+G(p,Sp,Tp)\\&=(2\beta+2\gamma)G(p,Sp,Tp)\end{aligned}$$

$\therefore$ this is a contradiction. $\therefore$ p=Sp and p=Tp.Similarly we prove that p=Sp and p=Tp.

Step II

In this step we prove that A,S,T have a unique common fixed point.Suppose a_0 be an arbitrary point in X.We construct a sequence $\{a_n\}$ by $a_{3n+1} = Aa_{3n}, a_{3n+2} = Sa_{3n+1}, a_{3n+3} = Ta_{3n+2}, n = 0, 1, 2, ...$

Case I

If $a_n = a_{n+1}$ for some n,with n=3m,then $p = a_{3n}$ is a fixed point of A and,then by the first step, p is a common fixed point of A,S and T.The same is true if n=3m+1,n=3n+2.

Case II

Now if $a_n \neq a_{n+1}$ for all $n\epsilon N$.

Consider

$$G(a_{3n+1},a_{3n+2},a_{3n+3}) = G(Aa_{3n},Sa_{3n+1},Ta_{3n+2})$$

$$\begin{aligned}G(a_{3n+1},a_{3n+2},a_{3n+3}) \leq& \alpha G(a_{3n},a_{3n+1},a_{3n+2})+\beta[G(Aa_{3n},a_{3n},a_{3n})\\&+G(a_{3n+1},Sa_{3n+1},a_{3n+1})\\&+G(a_{3n+2},a_{3n+2},Ta_{3n+2})]+\gamma[G(Aa_{3n},a_{3n+1},a_{3n+2})\\&+G(a_{3n},Sa_{3n+1},a_{3n+2})+G(a_{3n},a_{3n+1},Ta_{3n+2})]\\&+\delta G(a_{3n},a_{3n+1},a_{3n+2})\\&=\alpha G(a_{3n},a_{3n+1},a_{3n+2})+\beta[G(a_{3n+1},a_{3n},a_{3n})\\&+G(a_{3n+1},a_{3n+2},a_{3n+1})+G(a_{3n+2},a_{3n+2},a_{3n+3})]\\&+\gamma[G(a_{3n+1},a_{3n+1},a_{3n+2})+G(a_{3n},a_{3n+2},a_{3n+2})\\&+G(a_{3n},a_{3n+1},a_{3n+3})]\\&+\delta G(a_{3n},a_{3n+1},a_{3n+2})\\&\leq\alpha G(a_{3n},a_{3n+1},a_{3n+2})+\beta[G(a_{3n},a_{3n+1},a_{3n+2})\\&+G(a_{3n},a_{3n+1},a_{3n+2})\\&+G(a_{3n+1},a_{3n+2},a_{3n+3})]+\gamma[G(a_{3n},a_{3n+1},a_{3n+2})\\&+G(a_{3n},a_{3n+1},a_{3n+2})\\&+\{G(a_{3n},a_{3n+1},a_{3n+2})+G(a_{3n+1},a_{3n+2},a_{3n+3})\}]\\&+\delta G(a_{3n},a_{3n+1},a_{3n+2})\end{aligned}$$

i.e.

$$(1-\beta-\gamma)G(a_{3n+1},a_{3n+2},a_{3n+3}) \leq (\alpha+2\beta+3\gamma+\delta.)G(a_{3n},a_{3n+1},a_{3n+2})$$

Hence,

$$G(a_{3n+1}, a_{3n+2}, a_{3n+3}) \leq \lambda G(a_{3n}, a_{3n+1}, a_{3n+2})$$

where

$$\lambda = \frac{(\alpha + 2\beta + 3\gamma + \delta.)}{(1 - \beta - \gamma)}$$

as $0 < \lambda < 1$. In the same way it can be shown that

$$G(a_{3n+2}, a_{3n+3}, a_{3n+4}) \leq \lambda G(a_{3n+1}, a_{3n+2}, a_{3n+3})$$

also

$$G(a_{3n+3}, a_{3n+4}, a_{3n+5}) \leq \lambda G(a_{3n+2}, a_{3n+3}, a_{3n+4})$$

Therefore for all n,

$$\begin{aligned} G(a_{n+1}, a_{n+2}, a_{n+3}) \leq& \lambda G(a_n, a_{n+1}, a_{n+2}) \\ &\leq .... \leq \lambda^{n+1} G(a_0, a_1, a_1) \end{aligned}$$

Therefore for all l, m, n with $l > m > n$,

$$\begin{aligned} G(a_n, a_m, a_l) \leq& G(a_n, a_{n+1}, a_{n+1}) + G(a_{n+1}, a_{n+2}, a_{n+3}) \\ &+ .... + G(a_{l-1}, a_{l-1}, a_l) \\ &\leq G(a_n, a_{n+1}, a_{n+2} + G(a_{n+1}, a_{n+2}, a_{n+3}) \\ &+ .... + G(a_{l-2}, a_{l-1}, a_l) \\ &\leq [\lambda^n + \lambda^{n+1} + ... + \lambda^{l-2}] G(a_0, a_1, a_1) \\ &\leq \frac{\lambda^n}{1 - \lambda} G(a_0, a_1, a_1) \end{aligned}$$

The same is true if $l = m > n$ and if $l > m = n$ then we have

$$G(a_n, a_m, a_l) \leq \frac{\lambda^{n-1}}{1 - \lambda} G(a_0, a_1, a_1)$$

letting limit as $n, m, l \to \infty$, this gives $G(a_n, a_m, a_l) \to 0 \therefore \{a_n\}$ is a G-Cauchy sequence.Since X is a complete $\therefore$ there exists $u \epsilon X$ s.t. $\{a_n\}$ converges to u, as $n \to \infty$. Now we claim that Au=u.We prove this by the method of contradiction, suppose $Au \neq u$.

Now consider

$$
\begin{aligned}
G(Au, a_{3n+2}, a_{3n+3}) =& G(Au, Sa_{3n+1}, Ta_{3n+2}) \\
& \leq \alpha G(u, a_{3n+1}, a_{3n+2}) + \beta[G(Au, u, u) \\
& + G(a_{3n+1}, Sa_{3n+1}, a_{3n+1}) \\
& + G(a_{3n+2}, a_{3n+2}, Ta_{3n+2})] + \gamma[G(Au, a_{3n+1}, a_{3n+2}) \\
& + G(u, Sa_{3n+1}, a_{3n+2}) \\
& + G(u, a_{3n+1}, Ta_{3n+2})] + \delta G(u, a_{3n+1}, a_{3n+2}) \\
& = \alpha G(u, a_{3n+1}, a_{3n+2}) + \beta[G(Au, u, u) \\
& + G(a_{3n+1}, a_{3n+2}, a_{3n+1}) \\
& + G(a_{3n+2}, a_{3n+2}, a_{3n+3})] + \gamma[G(Au, a_{3n+1}, a_{3n+2}) \\
& + G(u, a_{3n+2}, a_{3n+2}) \\
& + G(u, a_{3n+1}, a_{3n+3})] + \delta G(u, a_{3n+1}, a_{3n+2})
\end{aligned}
$$

letting limit as $n \to \infty$, we have

$$G(Au, u, u) \leq (\beta + \gamma)G(Au, u, u)$$

This is a contradiction and hence $Au \neq u$ is wrong ∴ Au=u.Similarly we can prove that Su=u and Tu=u.
To prove uniqueness, If possible suppose that v is another common fixed point of A,S and T, Then

$$
\begin{aligned}
G(u, v, v) =& G(Au, Sv, Tv) \\
& \leq \alpha G(u, v, v) + \beta[G(Au, u, u) + G(v, Sv, v) + G(v, v, Tv)] \\
& + \gamma[G(Au, v, v) + G(u, Sv, v) + G(u, v, Tv)] + \delta G(u, v, v) \\
& = \alpha G(u, v, v) + \beta[G(u, u, u) + G(v, v, v) + G(v, v, v)] \\
& + \gamma[G(u, v, v) + G(u, v, v) + G(u, v, v)] + \delta G(u, v, v) \\
& = (\alpha + 3\gamma + \delta)G(u, v, v)
\end{aligned}
$$

This is not possible ∴ G(u,v,v)=0 , ∴ $u = v$. Hence u is a Unique Common fixed point of A,S and T. ■

Corollary 3.2. Let A,S and T be self mappings on a complete G-metric space X which satisfies the following conditions.

$$
\begin{aligned}
G(A^m a, S^m b, T^m c) \leq& \alpha G(a, b, c) + \beta[G(A^m a, a, a) + G(b, S^m b, b) + G(c, c, T^m c)] \\
& + \gamma[G(A^m a, b, c) + G(a, S^m b, c) + G(a, b, T^m c)] + \delta G(a, b, c)
\end{aligned} \tag{3.2}
$$

for all a,b,c in X , where $\alpha, \beta, \gamma, \delta > 0$ and $\alpha + 3\beta + 4\gamma + \delta < 1$. Then A,S and T have a unique common fixed point in X.Furthermore, any fixed point of A is a fixed point of S and T and conversely.

Proof. Using Theorem (1) A^m, S^m and T^m have a unique fixed point p. as $A(p) = A(A^m(p)) = A^{m+1}(p) = A^m(A(p))$,
$S(p) = S(S^m(p)) = S^{m+1}(p) = S^m(S(p))$, and
$T(p) = T(T^m(p)) = T^{m+1}(p) = T^m(T(p))$,
This gives Ap,Sp and Tp are also fixed points A^m, S^m and T^m is unique, we deduce that p=Ap=Sp=Tp.We know that every fixed point of A is fixed point of S and T and conversely. ■

4. Conclusion

Thus we have proved common fixed point theorem for three maps without condition of continuity, compatibility,weakly compatibility.

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