A Study on Star Chromatic Number of Some Special Classes of Graphs

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Abstract

Let G= (V, E) be an undirected, simple, finite and connected graph. The star chromatic number of a graph G is the least number of colours needed to colour the path on four vertices using three distinct colours. The aim of this paper is to study the star colouring of some Circulant graphs and to determined the star chromatic number of some graph families formed from the Cartesian product of some simple graphs.

Keywords: Proper colouring, Chromatic number, Star colouring, Star chromatic number, Circulant graphs, Harary graphs, Andrásfai graph, Cocktail Party graph, Musical graph, Crown graph Cartesian product of simple graphs, Prisnm graph, Barbell graph, Fan graph Windmill graph and Lollipop graph.

1. INTRODUCTION

In this paper, we have taken the graphs to be finite, undirected, connected and simple. The path P of a graph is a walk in which no vertices are repeated. In 1973 the concept of star colouring was introduced by Grünbaum [10] and also he introduced the notion
of star chromatic number. His works were developed further by Bondy and Hell [4]. According to them the star colouring is the proper colouring on the paths with four vertices by giving 3- distinct colours on it. In graph theory, a Circulant graph [5] is an undirected graph that has a cyclic group of symmetries that takes any vertex to any other vertex. In this paper we have established the star chromatic number of some Circulant graphs. The Cartesian product of a graph $G=G_1\square G_2$ is the graph with vertex set $V=V_1 \times V_2$ with $u$ and $v$ are adjacent whenever $u_1=v_1$ and $u_2$ adjacent to $v_2$ or $u_2=v_2$ and $u_1$ adjacent to $v_1$. In this paper, we have obtained the star chromatic number of some graphs formed from the Cartesian product of two simple graphs.

2. DEFINITIONS

**Definition 2.1- Vertex Colouring**
Let $G$ be a graph and let $V(G)$ be the set of all vertices of $G$ and let $\{1,2,3..k\}$ denotes the set of all colours which are assigned to each vertex of $G$. A proper vertex colouring of a graph is a mapping $c: V(G) \rightarrow \{1, 2, 3 \ldots k\}$ such that $c(u) \neq c(v)$ for all arbitrary adjacent vertices $u, v \in V(G)$.

**Definition 2.2- Chromatic Number**
If $G$ has a proper vertex colouring then the chromatic number of $G$ is the minimum number of colours needed to colour $G$. The chromatic number of $G$ is denoted by $\chi(G)$.

**Definition 2.3- Star colouring**
A proper vertex colouring of a graph $G$ is called star colouring [10], if every path of $G$ on four vertices is not 2- coloured.

**Definition 2.4- Star Chromatic Number**
The star chromatic number is the minimum number of colours needed to star colour $G$ [1] and is denoted by $\chi_s(G)$.

Let us consider the following example:
Let $G$ be a path graph and $V(G) = \{v_1, v_2, v_3, v_4\}$

Here $c(v_1) = c(v_4) = 1$, $c(v_2) = 2$, $c(v_3) = 3$

Then the path $P_4$ of $G$ is $v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4$

$c(P_4) = c(v_1) \rightarrow c(v_2) \rightarrow c(v_3) \rightarrow c(v_4) = 1 - 2 - 3 - 1$

Hence, $\chi_s(P_4) = 3 \implies \chi_s(G) = 3$

**Definition 2.5- Circulant graphs**

A circulant graph [5] is a graph of $n$ vertices in which the $i^{th}$ vertex is adjacent to the $(i+j)^{th}$ and $(i-j)^{th}$ vertices for each $j$ by the cyclic group of symmetry.

**Example 2.6:** Schläfli graph

**Definition 2.7- Harary Graphs**

The structure of harary graphs [4] denoted by $H_{k,m}$ depends on the parities of $k$ and $m$ where $k$ denotes the vertex connectivity and $m$ denotes the number of vertices.

**Case 1:** when $k$ and $m$ are even:

Let $k = 2n$, then $H_{2n,m}$ is constructed as follows. It has vertices $0, 1, 2, \ldots, m-1$ and two vertices $i$ and $j$ are joined if $i-n \leq j \leq i+n$ (when addition is taken modulo $m$). $H_{4,8}$ is shown in fig(a).
**Case 2**: when $k$ is odd and $m$ is even:

Let $k = 2n+1$, then $H_{(2n+1,m)}$ is constructed by first drawing $H_{(2n,m)}$ and then adding edges joining vertex $i$ to vertex $i+(m/2)$ for $1 \leq i \leq m/2$. $H_{(5,8)}$ is shown in fig(b).

**Case 3**: when $k$ and $m$ are odd:

Let $k = 2n+1$, then $H_{(2n+1,m)}$ is constructed by first drawing $H_{(2n,m)}$ and then adding edges joining vertex 0 to the vertices $(m-1)/2$ and $(m+1)/2$ and vertex $i$ to vertex $i+(m+1)/2$ for $1 \leq i \leq (m-1)/2$. $H_{(5,9)}$ is shown in fig(c).

**Example: 2.8**

**Definition 2.9 - Andrásfai Graphs**

The $n$-Andrásfai graph [6] is a circulant graph on $3n-1$ vertices and whose indices are given by the integers $1, \ldots, 3n-1$ that are congruent to 1 (mod 3). The Andrásfai graphs have graph diameter 2 for $n \geq 1$ and is denoted by $A_n$. 
Example: 2.10

Graph $A_5$ with $n=14$

**Definition 2.11-Cocktail party Graph**

The cocktail party graph [3] is the graph consisting of two rows of paired vertices in which all vertices except the paired ones are connected with an edge and is denoted by $C_{p_n}$.

Example: 2.12

Graph $C_{P_4}$ with $n=4$
Definition 2.13 - Musical Graph

The musical graph [8] of order n consists of two parallel copies of cycle graphs $C_n$ in which all the paired vertices and its neighborhood vertices are connected with an edge and is denoted by $M_{2n} \forall n \geq 3$.

Example: 2.14

![Graph M_{2n} with n=4](image)

Definition 2.15 - Crown Graph

The crown graph $S_n^0$ [9] for an integer $n \geq 3$ is the graph with vertex set \{x_0, x_1, x_2, \ldots, x_{n-1}, y_0, y_1, y_2, \ldots, y_{n-1}\} and edge set \{(x_i, y_j) : 0 \leq i, j \leq n - 1, i \neq j\}$. $S_n^0$ is therefore equivalent to the complete bipartite graph $K_{n,n}$ with horizontal edges removed.

Example: 2.16

![Graph $S_n^0$ with n=4](image)
Definition 2.17- Cartesian product of graphs:
The Cartesian product of simple graphs G and H is the G□H [6] whose vertex set is V(G) × V(H) and whose edge set is the set of all pairs (u_i,v_i), (u_j,v_j) such that u_i,u_j∈V(G) and v_i,v_j∈V(H).

Definition 2.18 Empty graph:
An empty graph [10] on n vertices consists of n isolated n vertices with out edges, Such graphs are sometimes also called edgeless graph and is commonly denoted by \( \overline{K}_n \).

Definition : 2.19 Prism graph
A generalized prism graph \( Y_{n,m} \) [5] is a simple graph obtained by the Cartesian product of two graphs say cycle \( C_n \) and path \( P_m \).

Definition 2.20-Fan graph:
A fan graph \( F_{m,n} \) [5] is defined as the graph by joining \( \overline{K}_m \) and \( P_n \) graphs where \( \overline{K}_m \) is the empty graph on m vertices and \( P_n \) is the path graph on n vertices.

Definition 2.21- Barbell graph:
The n-barbell graph [6] is the simple graph obtained by connecting two copies of a complete graph \( K_n \) by a bridge and is denoted by \( B(K_n, K_n) \).

Definition 2.22.-Lollipop graph
The (m,n)-lollipop graph [6] is the graph obtained by joining a complete graph \( K_m \) to a path graph \( P_n \) with a bridge and is denoted by \( L_{m,n} \).

Definition 2.23- Windmill graph
The windmill graph \( W_n^{(m)} \) [7] is the graph obtained by taking m copies of the complete graph \( K_n \) with a vertex in common.

3. PRELIMINARIES:

Theorem 3.1:[10] (Star Colouring of Cycle)
Let \( C_n \) be the cycle with n ≥ 3 vertices. Then \( \chi_s(C_n) = \begin{cases} 4, & \text{when } n = 5 \\ 3, & \text{otherwise} \end{cases} \)

Theorem 3.2:[10] (Star Colouring of complete graph)
If \( K_n \) is the complete graph with n vertices. Then \( \chi_s(k_n) = n, \forall n \geq 3 \)

Theorem 3.3:[10] (Star Colouring of path graph)
Let $P_n$ be the path graph with $n$ vertices. Then $\chi_s(P_n) = 3$, $\forall n \geq 4$

**Theorem 3.4** [10] (Star Colouring of Cycle)

Let $C_n$ be the cycle with $n \geq 3$ vertices. Then $\chi_s(C_n) = \begin{cases} 4, & \text{when } n = 5 \\ 3, & \text{otherwise} \end{cases}$

**Theorem 3.5** [10] (Star Colouring of complete graph)

If $K_n$ is the complete graph with $n$ vertices. Then $\chi_s(K_n) = n$, $\forall n \geq 3$

**Theorem 3.6** [10] (Star Colouring of path graph)

Let $P_n$ be the path graph with $n$ vertices. Then $\chi_s(P_n) = 3$, $\forall n \geq 4$.

4. STAR COLOURING OF SOME GRAPHS FORMED FROM THE CARTESIAN PRODUCT OF SIMPLE GRAPHS:

In this section we have obtained the star chromatic number of various graphs formed from the Cartesian product of simple graphs

4.1. Star chromatic number of Harary graphs:

**Theorem 4.1**:
The star chromatic number of Harary graphs when both $k$ and $m$ are even.

Let $H((k,m))$ be a Harary graph with $k=2n$ and $m=4n$, then star chromatic number

$$\chi_s(H((k,m))) = \begin{cases} 3, & \text{when } n = 1 \\ (k + 2), & \text{otherwise} \end{cases}, \forall n \geq 1$$

**Proof**:

Let $H((k,m))$ be a harary graph [3] with $m$ vertices, where $k = 2n$ and $m = 4n$.

Therefore the vertex set of $H((k,m))$ is given by $V(H((k,m))) = \{v_0, v_1, v_2, \ldots, v_{m-1}\}$ we know that the graph $H((k,m))$ contains $m$ vertices and the degree of each vertex is $k$. By the definition of harary graph we know that any two vertices $v_i$ and $v_j$ are joined if $i-n \leq j \leq i+n$ (when addition is taken modulo $m$)

Let us now star colour the graph $H((k,m))$

**Case 1**: for $n = 1$,

we have $k = 2$ and $m=4$ then $H((2,4))$ has four vertices and each vertex is of degree 2

We know that a graph with four vertices and each vertex is of degree 2 is cycle $C_4$. Therefore the graph $H((2,4))$ is the graph $C_4$ and we know that $\chi_s(C_4) = 3$ [4] therefore $\chi_s(H((2,4))) = 3$.

Hence $\chi_s(H((k,m))) = 3$ for $n=1$
**Case 2:** for \( n \geq 2 \) and \( k, m \) are even and \( H_{(k,m)} \) has 4n vertices

Let us now see the procedure to star colour the graph \( H_{(k,m)} \) as follows

- Assign colours from 1, 2, 3, …, \( k+2 \) to vertices \( v_1, v_2, \ldots, v_{k+2} \) respectively.
- Assign even colours from the colour set \{1, 2, …, \( k+2 \}\} to the vertices with odd indices and assign odd colours from the colour set \{1, 2, …, \( k+2 \}\} to the vertices with even indices.

Here one can observe that there is no possibility for any path on four vertices to be bicoloured.

Thus there is a proper star colouring and Therefore \( \chi_s(H_{(k,m)}) = k+2 \forall n \geq 2 \)

When \( k=2n \) and \( m=4n \)

Therefore \( \chi_s(H_{(k,m)})=\begin{cases} 3, & \text{when } n = 1 \\ (k + 2), & \text{otherwise} \end{cases} \)

Hence proved

**Example: 4.2**

When \( n=4 \), we have \( k=8 \) and \( m=16 \)

Here \( c(v_0)=c(v_7)=7, \ c(v_1)=1 \ c(v_2)=2 \ c(v_3)=3, \ldots, \ c(v_{10})=10, \ldots, \ c(v_{15})=4 \)

Clearly this graph accepts proper star colouring.

Therefore \( \chi_s(H_{(8,16)})=10. \ (8+2) \)
**Theorem 4.3**

The star chromatic number of Harary graphs when \( k \) is odd and \( m \) is even

Let \( H_{(k,m)} \) be the harary graph with \( k=2n+1 \) and \( m=2n+2 \), then the star chromatic number \( \chi_s(H_{(k,m)})= m \) where \( k=2n+1 \) and \( m=2n+2 \), \( \forall \ n \geq 1 \)

**Proof:**

Let \( H_{(k,m)} \) be the harary graph with the parities \( k \) and \( m \) and take that \( k \) is odd and \( m \) is even. Let us take \( k=2n+1 \) and \( m=2n+2 \) \( \forall \ n \geq 1 \). By the definition of harary graphs [3] for the case \( k \)- odd and \( m \)- even is constructed as follows :

- First \( H_{(2n,2n+2)} \) is constructed.
- Then edges are joined from vertex \( i \) to vertex \( i+\left(\frac{2n+2}{2}\right) \) for \( 1 \leq i \leq \left(\frac{2n+2}{2}\right) \).

Let us now assign colours to star colour the graph.

We can observe that the harary graph \( H_{(2n+1,2n+2)} \) to be a complete graph with \( 2n+2 \) vertices. Since every pair of vertices is adjacent to each other. In a complete graph, since all the vertices are adjacent, each vertex receives different colours. Thus for any vertex \( v_i \) its neighborhood vertices are assigned with distinct colours. Therefore for any path on four vertices is not bicoloured thus the star chromatic number is equal to the chromatic number.

And we know that the star chromatic number of the complete graph is equal to the number of vertices in it.[4]

(i.e.) \( \chi_s(K_n)=n \).

Since the harary graph \( H_{(2n+1,2n+2)} \) is a complete graph, the above discussion also holds for the harary graph \( H_{(2n+1,2n+2)} \). Thus the star chromatic number of the harary graph \( H_{(2n+1,2n+2)} \) is equal to the number of vertices. Here \( m \) denotes the number of vertices.

Therefore \( \chi_s(H_{(k,m)})= m \)

(i.e.) \( \chi_s(H_{(2n+1,2n+2)})= 2n+2 \ \forall \ n \geq 1 \).

Hence proved.

**Example: 4.4**

When \( n=11 \) then \( k=2\times11+1 =23 \) and \( m= 2\times11+2 =24 \)
Here $c(v_0)=1$, $c(v_1)=2$, $c(v_2)=3$, …, $c(v_{23})=24$

Clearly the graph accepts proper star colouring

Therefore $\chi_s(H_{(23,24)})=24$.

**Theorem: 4.5**

(Star chromatic number of Andrásfai graphs)

Let $A_n$ be the Andrásfai graph with $k$ vertices where $k=3n-1$. Then the star chromatic number of $A_n$ is given by $\chi_s(A_n) = \begin{cases} 4, & \text{when } n = 2 \\ k - 2, & \text{otherwise} \end{cases}$

**Proof:**

We know that Andrásfai graph is a circulant graph on $3n-1$ vertices, the degree of each vertex is $n$ and any two vertices $v_i$ and $v_j$ are joined by taking $1 \mod (3)$ [7]. Let us now star colour the graph $A_n$.

**Case 1:** for $n = 2$

we have $k=3(2)-1 = 5$ (i.e.) $k=5$

Clearly it is the cycle with five vertices and we know that $\chi_s(C_5)=4$
Thus $\chi_s(A_2) = \chi_s(C_5) = 4$

Therefore $\chi_s(A_n) = 4$ for $n=2$.

**Case 2:** for $n > 2$

Since $A_n$ is a circulant graph [5] all the vertices are adjacent to each other in a symmetrical manner. So it is possible to colour the vertices with $k-2$ colours. Suppose if we colour the vertices with less than $k-2$ colours, we can observe that either certain paths on four vertices are bicoloured or certain vertices which are adjacent to each other are assigned with same colour. Therefore assigning colours less than $k-2$ colours is neither a proper colouring nor a proper star colouring.

So assign colours $1,2,3\ldots, k-2$ to the vertices consecutively to all the $k-2$ vertices. There are 2 vertices that are left uncoloured. Assign any two colours from the colour set $1,2,3\ldots,k-2$ randomly by checking the possible paths that are not to be bicoloured from those two vertices. This is a proper star colouring.

Hence $\chi_s(A_n) = k-2$

From case 1 and case 2,

$$\chi_s(A_n) = \begin{cases} 
4, & \text{when } n = 2 \\
 k - 2, & \text{otherwise} 
\end{cases}$$

hence proved.

**Example: 4.6**

for $n=7$ we have $k=3(7)-1 = 20$

Here $c(v_1)=1$, $c(v_2)=2$, $c(v_3)=3\ldots$, $c(v_{18})=18$, $c(v_{19})=8$, $c(v_{20})=14$.

Clearly the graph accepts proper star colouring

Therefore $\chi_s(A_n) = 18$ (20-2).
Theorem: 4.7
(Star chromatic number of cocktail party graphs)

Let $C_{p_n}$ be the cocktail party graph with $m$ vertices where $m=2n$. Then the star chromatic number of $C_{p_n}$ is given by $\chi_s(C_{p_n}) = 2n-1 \forall n \geq 2$.

Proof:

The cocktail party graph is formed from two rows of paired vertices in which all the vertices are connected except the paired ones [8]. Here $v(C_{p_n}) = 2n$. Let $A= \{v_1, v_2, v_3, \ldots, v_n\}$ be one set of vertices and let $B= \{w_1, w_2, w_3, \ldots, w_n\}$ be another set of vertices. Here for any $i$, $v_i$ is connected to all the vertices of the vertex set $A$ and to all the vertices of the vertex set $B$ except $w_i$.

In this graph $v_1$ is non adjacent to $w_1$, $v_2$ is non adjacent to $w_2$, etc.

Let us now see the procedure to star colour the graph $C_{p_n}$:

- Assign colours from 1 to $n$ successively to all the vertices of the vertex set $A$.

(i.e.) assign colours 1,2,...,$n$ to the vertices $v_1, v_2, v_3, \ldots, v_n$.

- Assign colour 1 to vertex $w_1$. 
• Then assign colours from \(n+1\) to \(2n-1\) to the remaining vertices of the vertex set of \(B\) (i.e.) \(n+1, n+2, \ldots, 2n-1\) to the vertices \(w_2, w_3, \ldots, w_n\).

Clearly no path on four vertices is bicoloured, which is a valid star colouring. Hence \(\chi_s(C_{pn}) = 2n-1\) \(\forall n \geq 2\).

Hence proved.

**Example: 4.8**

When \(n = 15\) we have \(m = 30\)

Here \(c(v_1) = 1, c(v_2) = 2, c(v_3) = 3, \ldots, c(v_{15}) = 15, c(w_1) = 1, c(w_2) = 16, c(w_3) = 17, \ldots, c(w_{15}) = 29\).

Clearly the graph accepts proper star colouring

Therefore \(\chi_s(C_{15}) = 29 (2 \times 15 - 1)\).

**Theorem 4.9**

(Star chromatic number of musical graph)

Let \(M_n\) be the musical graph with \(k\) vertices where \(k = 2n\). Then the star chromatic number of \(M_n\) is given by \(\chi_s(M_n) = 6 \forall n \geq 3\).
Proof:

From the definition of musical graph [10], it consists of two parallel cycles $C_n$. Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and be the vertex set of the exterior cycle and let $W = \{w_1, w_2, w_3, \ldots, w_n\}$ be the vertex set of the interior cycle. For any $i$, $v_i$ is adjacent to $w_i$ and $N(w_i)$.

Let us star colour the graph $M_n$ by the following procedure:

Colour the vertices $w_1, w_2, w_3, \ldots, w_n$ with colour 1, 2, 3 accordingly to satisfy the star colouring condition except for the cycle $C_5$ (since $\chi_s(C_5) = 4$)

Colour the vertices $v_1, v_2, v_3, \ldots, v_n$ with colours 4, 5, 6 consecutively by satisfying the star colouring condition

Hence we observe that no path on four vertices is bi-coloured. This is a proper star colouring.

Hence $\chi_s(M_n) = 6 \forall n \geq 3$.

Hence proved.

Example: 4.10

Here $c(v_1)=4$, $c(v_2)=5$, $c(v_3)=6$, $c(v_{10})=6$, $c(w_1)=1$, $c(w_2)=2$, $c(w_3)=3$, $c(w_{10})=3$.

Clearly the graph accepts proper star colouring

Therefore $\chi_s(M_{10})= 6$
**Theorem: 4.11**

(Star chromatic number of crown graph)

Let $S_n^0$ be the crown graph with $n \geq 3$ vertices. Then the star chromatic number of $S_n^0$ is given by $\chi_s(S_n^0) = n \forall n \geq 3$.

**Proof:**

The crown graph $S_n^0$ [9] has $2n$ vertices. Let $V(S_n^0)$ be the vertex set of $S_n^0$. Let $V(S_n^0)$ be bipartitioned into two vertex subsets $X$ and $Y$ such that $X = \{x_0, x_1, \ldots, x_{n-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{n-1}\}$. Here $S_n^0$ is equivalent to a complete bipartite graph in which the paired edges $(x_i, y_i)$ are removed.

The procedure is to star colour the graph $S_n^0$ as given as follows:

- Assign colours $1, 2, 3, \ldots, n$ to the vertices $x_0, x_1, \ldots, x_{n-1}$ [4].
- Assign colours $1, 2, 3, \ldots, n$ to the vertices $y_0, y_1, \ldots, y_{n-1}$ [4].

Since $x_i$ and $y_i$ are non-adjacent there is no possibility for any path on four vertices to be bicoloured. This is proper star colouring. Hence $\chi_s(S_n^0) = n \forall n \geq 3$

Hence proved.

**Example: 4.12**

Here $c(x_0) = 1, c(x_1) = 2, c(x_2) = 3, \ldots, c(x_9) = 10, c(y_0) = 1, c(y_1) = 2, c(y_2) = 3, \ldots, c(y_9) = 10$.

Clearly the graph accepts proper star colouring

Therefore $\chi_s(S_{10}^0) = 10$
5. STAR COLOURING OF SOME GRAPHS FORMED FROM THE CARTESIAN PRODUCT OF SIMPLE GRAPHS:

In this section we have obtained the star chromatic number of various graphs formed from the Cartesian product of simple graphs

**Theorem:** 5.1

(Star colouring of prism graph)

Let $Y_{n,m}$ be the prism graph then the star chromatic number of $Y_{n,m}$ is given by

$$
\chi_s(Y_{n,m}) = \begin{cases} 
4, & \text{when } n = 5 \\
3, & \text{otherwise, when } m = 1, \forall \ n \geq 3 \text{ and } \\
6, & \text{when } n = 5 \\
5, & \text{otherwise, when } m \geq 2, \forall \ n \geq 3 .
\end{cases}
$$

**Proof:**

Let $Y_{n,m} = C_n \square P_m$ be the prism graph is formed by the Cartesian product of $C_n$ and $P_m$

The star colouring of $Y_{n,m}$ are discussed in two cases

**Case 1:** For $m = 1$, the prism graph $Y_{n,1}$ is a cycle graph $C_n$ with $n \geq 3$ also we know that

$$
\chi_s(C_n) = \begin{cases} 
4, & \text{when } n = 5 \\
3, & \text{otherwise}.
\end{cases} \quad \text{[4]}
$$

Therefore

$$
\chi_s(Y_{n,1}) = \begin{cases} 
4, & \text{when } n = 5 \\
3, & \text{otherwise} \quad \to \quad (1)
\end{cases}
$$

**Case 2:** for $m \geq 2$, the prism graph $Y_{n,m}$ is a $m$-concentric copies of cycle graph $C_n$ in which all the corresponding vertices are adjoined.

When $n = 5$, assign colours 1, 2, 3, 4 consecutively to the vertices of the innermost cycle of the graph $Y_{5,m}$. Now assign colours 1, 2, 3, 4, 5, 6 to all the vertices of the remaining cycle of $Y_{5,m}$ in such a way that the star colouring condition is satisfied.

When $n \neq 5$, assign colours 1, 2, 3 consecutively to the vertices of the innermost cycle of the graph $Y_{n,m}$ now all the vertices of the remaining cycle of $Y_{n,m}$ are assigned with colours 1, 2, 3, 4, 5 such that there is no possibility for all the paths on four vertices to be bicoloured.

Therefore this is a proper star colouring

Hence

$$
\chi_s(Y_{n,m}) = \begin{cases} 
6, & \text{when } n = 5 \\
5, & \text{otherwise} \quad \to \quad (2)
\end{cases}
$$

From (1) and (2) we get,

$$
\chi_s(Y_{n,m}) = \begin{cases} 
4, & \text{when } n = 5 \\
3, & \text{otherwise, when } m = 1, \forall \ n \geq 3 \text{ and } \\
6, & \text{when } n = 5 \\
5, & \text{otherwise, when } m \geq 2, \forall \ n \geq 3 .
\end{cases}
$$
Hence proved.

Example: 5.2

\[
\begin{align*}
c(v_1) &= 1, \ c(v_2) = 2, \ldots, \ c(v_6) = 3 \\
\text{Here no path on four vertices is bicoloured.} \\
\text{Therefore } \chi_s(Y_{6,3}) &= 5.
\end{align*}
\]

\[
\begin{align*}
c(v_1) &= c(v_4) = c(v_{13}) = c(v_{15}) = 1, \ c(v_2) = c(v_3) = c(v_{14}) = c(v_6) = c(v_{10}) = 2, \\
c(v_5) &= c(v_7) = c(v_9) = c(v_{11}) = c(v_{12}) = c(v_{16}) = 3, \ c(v_8) = c(v_{17}) = 4 \\
\text{Here no path on four vertices is bicoloured.} \\
\text{Therefore } \chi_s(Y_{6,3}) &= 5.
\end{align*}
\]

Theorem: 5.3

(Star colouring of lollipop graph)

Let \( L_{m,n} \) be the lollipop graph, then the star chromatic number of \( L_{m,n} \) is given by

\[ \chi_s(L_{m,n}) = m, \ \forall \ m, n \geq 3, \]
Proof:

By the definition of lollipop graph, it is obtained by joining a complete graph $K_m$ to a path graph $P_n$ [6]. Let $A$ denote the vertex set of the complete graph $K_m$ (i.e.) $A = \{v_1, v_2, v_3 \ldots v_m\}$. Let $B$ denote the vertex set of the path graph $P_n$ (i.e.) $B = \{u_1, u_2, u_3 \ldots u_n\}$.

Let us now star colour the graph $L_{m,n}$ by the following procedure:

- Assign colours 1, 2, 3, ..., $m$ to all the vertices of the vertex set $A$ [4] (since all the vertices $v_i$ are adjacent to each other).
- Assign colours 1, 2, 3 consecutively to all the vertices of the vertex set $B$ [4].

Thus by combining the above steps we get no paths on four vertices is bicoloured. This is a proper star colouring. Hence $\chi_s(L_{m,n}) = m$, $\forall m, n \geq 3$.

Example: 5.4

![Diagram of L_{12,5} = K_{12} \circ P_5]

$c(v_1) = c(u_3) = 1$, $c(v_2) = c(u_4) = 2$, $c(v_3) = c(u_2) = c(u_5) = 3$, $c(v_4) = 4$, ..., $c(v_{12}) = 12$

Here no path on four vertices is bicoloured. Therefore $\chi_s(L_{12,5}) = 12$.

Theorem: 5.5

(Star colouring of barbell graph)

Let $B(K_n, K_n)$ be the barbell graph, then the star chromatic number of $B(K_n, K_n)$ is given by $\chi_s(B(K_n, K_n)) = n+1$, $\forall n \geq 3$. 

Proof:
We know that a barbell graph is obtained by connecting two copies of complete graph $K_n$ by a bridge [6]. Let $A$ be the first copy of the complete graph $K_n$ and let $B$ be the second copy of the complete graph $K_n$. Let $V = \{v_1, v_2, v_3 \ldots v_n\}$ be the vertices of $A$ and let $U = \{u_1, u_2, u_3 \ldots u_n\}$ be the vertices of $B$.

The procedure to star colour the graph $B(K_n, K_n)$ is as follows:
- Assign colours 1, 2, 3, ..., $n$ to $v_1, v_2, v_3 \ldots v_n$ consecutively [4]
- Assign colour $n+1$ to the particular vertex $u_j$ of $B$ which is adjacent to the particular vertex $v_i$ of $A$.
- Assign colours 1, 2, 3, ... $n-1$ to the remaining vertices of $B$

Thus there is no possibility for all the paths to be bicoloured hence this is a proper star colouring.

Thus $\chi_s(B(K_n, K_n)) = n+1$. $\forall \ n \geq 3$

Hence proved.

Example: 5.6

![Barbell Graph Diagram]

$c(v_1)=1$, $c(v_2)=2$, $\ldots$, $c(v_{13})=13$, $c(u_1)=1$, $c(u_2)=2$, $\ldots$, $c(u_{12})=12$, $c(u_{13})=14$

Here no path on four vertices is bicoloured.

Therefore $\chi_s(B(K_{13}, K_{13})) = 14$. $(13+1)$

Theorem: 5.7
(Star colouring of windmill graph)

Let $W_n^{(m)}$ be the windmill graph then the star chromatic number of $W_n^{(m)}$ is given by $\chi_s(W_n^{(m)}) = n \forall \ n \geq 3$ and $\forall \ m \geq 1$. 
Proof:

From the definition of windmill graph $W_n^{(m)}$ and we know that it is formed by taking $m$- copies of the complete graph $K_n$ with a vertex in common [7].

Since all the vertices of $K_n$ are adjacent to each other, each vertex receives distinct colours (i.e.) for a complete graph $K_i$, receives $i$ colours for $i$.

Let us star colour the graph $W_n^{(m)}$

- First assign colour 1 to the vertex in common of $W_n^{(m)}$
- Assign colours from 2 to $n$ to the remaining vertices of one copy $K_n$

Here the vertex with colour 1 is the common vertex to all the copies of $K_n$.
- We shall assign colours from 2 to $n$ to the vertices of the remaining copies of $K_n$.

Here for any vertex $v_i$ with colour $i$ its neighboring vertices are assigned with distinct colours so there is no possibility for the paths on four vertices to be bicoloured. Thus it is a proper star colouring.

Hence $\chi_s(W_n^{(m)}) = n$, $\forall$ $n \geq 3$ and $\forall$ $m \geq 1$.

Hence proved.

Example:5.8

$c(v_1)=1$, $c(v_2)=2,\ldots$, $c(v_{25})=7$
Here no path on four vertices is bicoloured.
\[ \chi_s(W_7^{(4)}) = 7. \]

**Theorem: 5.9**
(Star colouring of fan graph)

Let \( F_{m,n} \) be the fan graph, then the star chromatic number of \( F_{m,n} \) are given as follows:

When \( m=1, \forall n \geq 2 \)
\[ \chi_s(F_{m,n}) = \begin{cases} 3, & \text{when } n \leq 3 \\ 4, & \text{otherwise} \end{cases} \]
when \( m \geq 2, \forall n \geq 2 \)
\[ \chi_s(F_{m,n}) = n+1. \]

**Proof:**

From the definition of fan graph we know that \( F_{m,n} = \overline{K}_m \square P_n \) [5] where \( \overline{K}_m \) is the empty graph [10] on \( m \) vertices and \( P_n \) is the path graph on \( n \) vertices.

Let us now star colour the graph \( F_{m,n} \)

**Case 1:** when \( m=1 \)

If \( m=1 \) then \( \overline{K}_1 \) is the empty graph with one vertex, thus \( F_{1,n} = \overline{K}_1 \square P_n \)

On star colouring \( F_{1,n} \) we have two sub cases

**Sub case 1:** \( \chi_s(F_{1,n})=3, n \leq 3 \)
If \( n=2 \) then \( F_{1,2} \) is clearly the cycle graph \( C_3 \) (i.e.) \( F_{1,2} = C_3 \)

w.r.t \( \chi_s(C_3) = 3 \) [4] therefore \( \chi_s(F_{1,2}) = 3 \) \( \rightarrow (1) \)

if \( n=3 \) we get \( F_{1,3} \) we have to assign colour 1 the vertex of the empty graph then the remaining three vertices of the path of \( F_{1,3} \) is assigned with colours 2, 3 consecutively thus is a proper star colouring hence \( \chi_s(F_{1,3}) = 3 \) \( \rightarrow (2) \)

From (1) and (2) \( \chi_s(F_{1,n}) = 3, n \leq 3 \) \( \rightarrow (3) \)

**Sub case 2:** \( \chi_s(F_{1,n})=4, n > 3 \)

Assign colour 1 to the vertex of the empty graph then the remaining vertices are the vertices of the path graph and w.r.t \( \chi_s(P_n) = 3 \) [4]. \( \forall n \geq 4 \) now assign colours 2, 3, 4 consecutively to all the vertices of the path. Clearly we say that this colouring is a proper star colouring therefore \( \chi_s(F_{1,n}) = 4, n > 3 \) \( \rightarrow (4) \)

From (3), (4) we have \( \chi_s(F_{m,n}) = \begin{cases} 3, & \text{when } n \leq 3 \\ 4, & \text{otherwise} \end{cases} \)

Hence case 1
Case 2: when \( m \geq 2 \)

Since all the vertices of \( \overline{K}_m \) are non adjacent we can assign colour 1 to all the vertices of \( \overline{K}_m \). now we are left with the vertices of the path

Suppose we star colour the vertices of the path with repetition of colours 2, 3, 4 then there is a possibility of having certain paths to be bicoloured.

Therefore without repetition of colours we star colour the \( n \) vertices of the path with \( n+1 \) colours so any vertex \( v_i \) on the path graph its neighborhood vertices are assigned with colour 1 along with two distinct colours. Therefore no path on four vertices is bicoloured.

Hence this is a proper star colouring.

\[
\chi_s(F_{m,n}) = n+1 \quad \text{when} \quad m \geq 2, \quad \forall \quad n \geq 2 \quad \rightarrow (6)
\]

hence case(2)

from (5), (6) we have

When \( m=1, \quad \forall \quad n \geq 2 \)

\[
\chi_s(F_{m,n}) = \begin{cases} 
3, & \text{when } n \leq 3 \\
4, & \text{otherwise}
\end{cases}
\]

and when \( m \geq 2, \quad \forall \quad n \geq 2 \)

\[
\chi_s(F_{m,n}) = n+1.
\]

Hence proved

---

Example: 5.10

\[
c(v_1) = 1, \quad c(u_1) = 2, \quad c(u_2) = 3, \quad c(u_3) = 4, \quad c(u_4) = 2, \quad c(u_5) = 3,
\]

Here no path on four vertices is bicoloured.

\[
\chi_s(F_{1,5}) = 4
\]
6. CONCLUSION

In this paper, we attained bounds for star chromatic number of some Circulant graphs. We also obtained the star chromatic number of various graphs formed from the Cartesian product of simple graphs. This work can be further extended to various Circulant graphs and several graphs formed from the Cartesian product of simple graphs.

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