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Products of Composition, Multiplication and Differentiation between Hardy Spaces and Weighted Growth Spaces of the Upper-Half Plane

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Abstract

Let ψ be a holomorphic function of the upper-half plane Λ^+ and φ a holomorphic self-map of Λ^+. Let C_φ , M_ψ and D denote, respectively, the composition, multiplication and differentiation operators. In this paper, we chacterize boundedness of the operators induced by products of these operators acting between Hardy and growth spaces of the upper-half plane.

Key words and phrases: Composition operator, Differentiation operator, Multiplication operator, Growth space, Hardy space, Upper-half plane.

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1. INTRODUCTION

Let G be a non-empty set, X a topological vector space, $F(G,X)$ the topological vector space of functions from G to X with point-wise vector space operations and $\varphi : G \longrightarrow G$ be a function such that $fo\varphi \in F(G,X)$ for all $f \in F(G,X)$. Then the linear transformation $C_\varphi : F(G,X) \longrightarrow F(G,X)$, defined as $C_\varphi(f) = fo\varphi$ for all $f \in F(G,X)$, is known as the composition transformation induced by φ on the space $F(G,X)$. If C_φ is continuous, then it is called the composition operator or substitution operator induced by φ on the space $F(G,X)$. Let $\Lambda^+ = \{x + iy: x, y \in \mathbb{R}, y > 0\}$ be the upper half-plane and $1 \le p < \infty$. Then the Hardy space $\mathcal{H}^p(\Lambda^+)$ is the collection of all analytic functions $f: \Lambda^+ \longrightarrow \mathbb{C}$ such that

$$\sup_{y>0} \int_{-\infty}^{+\infty} |f(x+iy)|^p dx < \infty.$$

It is well known that $\mathcal{H}^p(\Lambda^+)$ is a Banach space under the norm

$$\|f\|_{\mathcal{H}^p(\Lambda^+)} = \left[\sup_{y>0} \int_{-\infty}^{+\infty} |f(x+iy)|^p dx \right]^{1/p},$$

and $\mathcal{H}^2(\Lambda^+)$ is a Hilbert space under the inner product:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f_*(x)\, \overline{g_*(x)} dx, \quad f, g \in \mathcal{H}^2(\Lambda^+),$$

where

$$f_*(x) = \lim_{y \to 0} f(x+iy),$$

which exists almost everywhere on $\mathbb{R}$. These Hardy spaces fall under the category of the functional Banach spaces which consist of bonafide functions with continuous evaluation functionals. For any positive real number α, the growth space $\mathcal{A}^\alpha(\Lambda^+)$ consists of analytic functions $f : \Lambda^+ \longrightarrow \mathbb{C}$ such that

$$\|f\|_{\mathcal{A}^\alpha(\Lambda^+)} = \sup\{(Imz)^\alpha |f(z)| : z \in \Lambda^+\} < \infty.$$

With the norm $\|.\|_{\mathcal{A}^\alpha(\Lambda^+)}$, $\mathcal{A}^\alpha(\Lambda^+)$ is a Banach space. Note that $\mathcal{A}^1(\Lambda^+)$ is the usual growth space. For $\psi \in H(\Lambda^+)$ the multiplication operator M_ψ is defined by $M_\psi f = \psi f$. The product of composition and multiplication operators, denoted by $W_{\psi,\varphi}$ and defined as $W_{\psi,\varphi} = M_\psi o\, C_\varphi$, is known as weighted composition operator and has been studied intensively in recent times. The differentiation operator denoted by D is defined by $Df = f'$. As a consequence of the Little - wood Subordination principle, it is known that every analytic self-map φ of the open unit disk $\mathbb{D}$ induces a bounded composition operator on Hardy and weighted Bergman spaces of the open unit disk $\mathbb{D}$

(see [3] and [17]). However, if we move to Hardy and weighted Bergman spaces of the upper half-plane Λ^+, the situation is entirely different. In fact, there exist analytic self-maps of the upper half-plane which do not induce composition operators on the Hardy spaces and weighted Bergman spaces of the upper half-plane. Interesting work on composition operators on the spaces of upper Half plane have been done by many authors, to cite a few, Singh [10], Singh and Sharma [11, 12], Sharma [18] , Matache [7, 8], Sharma, Sharma and Shabir [19, 20], Stevic and Sharma [22, 23, 24, 26], Sharma, Sharma and Abbas [16]. Recently, some attention have been paid to the study concrete operators and their products between spaces of holomorphic functions, for example, Sharma and Abbas [14], Sharma, Sharma and Abbas [15], Sharma and Abbas [13], Bhat, Abbas and Sharma[2], Kumar and Abbas [6], Abbas and Kumar[1], Kumar and Abbas [5] and [4, 9, 21, 23, 25, 27] and the related references therein.

We can define the products of composition, multiplication and differentiation operators in the following six ways.

$(M_\psi C_\varphi Df)(z) = \psi(z) f'(\varphi(z)),$

$(M_\psi DC_\varphi f)(z) = \psi(z)\, \varphi'(z) f'(\varphi(z)),$

$(C_\varphi M_\psi Df)(z) = \psi(\varphi(z)) f'(\varphi(z)),$

$(DM_\psi C_\varphi f)(z) = \psi'(z) f(\varphi(z)) + \psi(z)\, \varphi'(z) f'(\varphi(z)),$

$(C_\varphi DM_\psi f)(z) = \psi'(\varphi(z)) f(\varphi(z)) + \psi(\varphi(z)) f'(\varphi(z)),$

$(DC_\varphi M_\psi f)(z) = \psi'(\varphi(z))\, \varphi'(z) f(\varphi(z)) + \psi(\varphi(z))\varphi'(z) f'(\varphi(z)),$

for $z \in \Lambda^+$ and $f \in H(\Lambda^+)$.

Note that the operator $M_\psi C_\varphi D$ induces many known operators. If $\psi = 1$, then $M_\psi C_\varphi D = C_\varphi D$, while when $\psi(z) = \varphi'(z)$, then we get the operator DC_φ. If we put $\varphi(z) = z$, then $M_\psi C_\varphi D = M_\psi D$, that is, the product of differentiation operator and multiplication operator. Also note that $M_\psi DC_\varphi = M_{\psi\varphi'} C_\varphi D$ and $C_\varphi M_\psi D = M_{\psi o \varphi} C_\varphi D$. Thus the corresponding characterizations of boundedness and compactness of $M_\psi DC_\varphi$ and $C_\varphi M_\psi D$ can be obtained by replacing ψ, respectively by $\psi\varphi'$ and $\psi\, o \varphi$ in the results stated for $M_\psi C_\varphi D$.

In order to treat these operators in a unified manner, we introduce the following operator

$$T_{g,h,\varphi}f(z) = g(z)f(\varphi(z)) + h(z)f'(\varphi(z))$$

where $g, h \in \mathcal{H}(\Lambda^+)$ and φ a holomorphic self-map of Λ^+. It is clear that composition, multiplication, differentiation operators and all the products of the composition, multiplication and differentiation operators defined above can be obtained from the operator $T_{g,h,\varphi}$ by fixing g and h. More specifically, we have $C_\varphi = T_{1,0,\varphi}$, $M_\psi = T_{\psi,0,z}$, $D = T_{0,1,z}$, $M_\psi C_\varphi = T_{\psi,0,\varphi}$, $C_\varphi M_\psi = T_{\psi o \varphi,0,\varphi}$, $C_\varphi D = T_{0,1,\varphi}$, $D C_\varphi = T_{0,\varphi',\varphi}$, $M_\psi D = T_{0,\psi,z}$, $D M_\psi = T_{\psi',\psi,z}$, $M_\psi C_\varphi D = T_{0,\psi,\varphi}$, $M_\psi D C_\varphi = T_{0,\psi\varphi',\varphi}$, $C_\varphi M_\psi D = T_{0,\psi o\varphi,\varphi}$, $DM_\psi C_\varphi = T_{\psi',\psi\varphi',\varphi}$, $C_\varphi DM_\psi = T_{\psi' o\varphi,\psi o\varphi,\varphi}$, $DC_\varphi M_\psi = T_{(\psi' o\varphi)\varphi',(\psi o\varphi)\varphi',\varphi}$.

In this paper we characterize the boundedness of the operator $T_{g,h,\varphi}$ acting between Hardy spaces and growth spaces of the upper-half plane. Throughout this paper, constants are denoted by C, they are positive and not necessarily the same at each occurrence.

2. BOUNDEDNESS OF $T_{g,h,\varphi} : \mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+)$

In this section, we characterize boundedness of $T_{g,h,\varphi} : \mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+)$.

Theorem 2.1. *Let* $1 \le p < \infty$ *and* φ *be a holomorphic self-map of* Λ^+. *Then* $T_{g,h,\varphi} : \mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+)$ *is bounded if and only if*

$$(i)\ M = \sup_{z\in\Lambda^+} \frac{(Im(z))^\alpha}{(Im(\varphi(z)))^{\frac{1}{p}}}|g(z)| < \infty,$$

$$(ii)\ N = \sup_{z\in\Lambda^+} \frac{(Im(z))^\alpha}{(Im(\varphi(z)))^{1+\frac{1}{p}}}|h(z)| < \infty.$$

Moreover if $T_{g,h,\varphi} : \mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+)$ *is bounded, then*

$$||T_{g,h,\varphi}||_{\mathcal{H}^p(\Lambda^+)\to\mathcal{A}^\alpha(\Lambda^+)} \sim M + N.$$

Proof: Firstly, suppose that (i) and (ii) hold, then

$$\|T_{g,h,\varphi}f\|_{\mathcal{A}^\alpha(\Lambda^+)} = sup\{Im(z)|(T_{g,h,\varphi}f)(z)| : z \in \Lambda^+\}.$$

Now

$Im(z)\left|\left(T_{g,h,\varphi}f\right)(z)\right|$

$= (Imz)^{\alpha}\left|f\left(\varphi(z)\right)g(z) + h(z)\, f'\left(\varphi(z)\right)\right|$

$\leq (Imz)^{\alpha}\left(\left|f\left(\varphi(z)\right)\right||g(z)| + |h(z)|\,\left|f'\left(\varphi(z)\right)\right|\right)$

$$\leq C\|f\|_{\mathcal{H}^p(\Lambda^+)}\left(\frac{\left(Im(z)\right)^{\alpha}}{\left(Im\left(\varphi(z)\right)\right)^{\frac{1}{p}}}|g(z)| + \frac{\left(Im(z)\right)^{\alpha}}{\left(Im\left(\varphi(z)\right)\right)^{1+\frac{1}{p}}}|h(z)|\right)$$

$\leq C(M+N)\|f\|_{\mathcal{H}^p(\Lambda^+)}.$

Thus,

$$\left\|\left(T_{g,h,\varphi}f\right)f\right\|_{\mathcal{A}^{\alpha}(\Lambda^+)} \leq C(M+N)\|f\|_{\mathcal{H}^p(\Lambda^+)}$$

and so $T_{g,h,\varphi} : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^{\alpha}(\Lambda^+)$ is bounded and

$$||T_{g,h,\varphi}||_{\mathcal{H}^p(\Lambda^+)\to\mathcal{A}^{\alpha}(\Lambda^+)} \leq C(M+N). \tag{2.1}$$

Conversely, suppose that $T_{g,h,\varphi} : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^{\alpha}(\Lambda^+)$ is bounded. Let $z_0 \in \Lambda^+$ be fixed and let $\omega = \varphi(z_0)$. Consider the function

$$f_{\omega}(z) = \frac{\left(Im(\omega)\right)^{2-\frac{1}{p}}}{\pi^{\frac{1}{p}}(z-\overline{\omega})^2} - 2i\frac{\left(Im(\omega)\right)^{3-\frac{1}{p}}}{\pi^{\frac{1}{p}}(z-\overline{\omega})^3} .$$

Writing $z = x + iy$ and $\omega = u + iv$ and using the elementary inequality $(x+y)^a \leq 2^a(x^a + y^a)$ which holds for all $x, y \geq 0$and $a > 0$, we have

$$\|f\|_{\mathcal{H}^p(\Lambda^+)} \leq 2^p\left[\sup_{y>0}\int_{-\infty}^{\infty}\frac{v^{2p-1}}{\pi|(x+iy)-(u-iv)|^{2p}}dx + \sup_{y>0}\int_{-\infty}^{\infty}\frac{2v^{3p-1}}{\pi|(x+iy)-(u-iv)|^{3p}}dx\right].$$

Again using the inequalities

$$|(x+iy)-(u+iv)|^{2p} \geq (v+y)^{2p-2}((x-u)^2 + (y+v)^2)$$

and

$$|(x+iy)-(u+iv)|^{3p} \geq (v+y)^{3p-2}((x-u)^2+(y+v)^2),$$

we get

$$\|f\|_{\mathcal{H}^p(\Lambda^+)} \leq 2^p \left[v^{2p-1} \sup_{y>0} \frac{1}{(y+v)^{2p-1}} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y+v}{(x-u)^2+(y+v)^2} dx \right.$$

$$\left. + v^{3p-1} \sup_{y>0} \frac{2}{(y+v)^{3p-1}} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y+v}{(x-u)^2+(y+v)^2} dx \right]$$

$$= 2^p \left[v^{2p-1} \sup_{y>0} \frac{1}{(y+v)^{2p-1}} + v^{3p-1} \sup_{y>0} \frac{2}{(y+v)^{3p-1}} \right] \leq 2^{p+2}.$$

Also,

$$f'_\omega(z) = \frac{-2\big(Im(\omega)\big)^{2-\frac{1}{p}}}{\pi^{\frac{1}{p}}(z-\overline{\omega})^3} + 6i \frac{\big(Im(\omega)\big)^{3-\frac{1}{p}}}{\pi^{\frac{1}{p}}(z-\overline{\omega})^4}$$

Moreover ,

$$f'_\omega\big(\varphi(z_0)\big) = \left(\frac{1}{4i} + \frac{3i}{8}\right) \frac{1}{\pi^{\frac{1}{p}}} \frac{1}{\big(Im(\omega)\big)^{1+\frac{1}{p}}}$$

and $f_\omega\big(\varphi(z_0)\big) = 0$.

Since $T_{g,h,\varphi} : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded, we have

$$\begin{aligned} \|T_{g,h,\varphi} f_\omega\|_{\mathcal{A}^\alpha(\Lambda^+)} &\leq ||T_{g,h,\varphi}||_{\mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+)} \|f_\omega\|_{\mathcal{H}^p(\Lambda^+)} \\ &\leq 2^{p+2} ||T_{g,h,\varphi}||_{\mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+)}. \end{aligned}$$

This implies for each $z \in \Lambda^+$, we have

$$\begin{aligned} 2^{p+2} ||T_{g,h,\varphi}||_{\mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+)} &\geq \big(Im(z)\big)^\alpha \left|\big(T_{g,h,\varphi} f\big)'(z)\right| \\ &= (Imz)^\alpha \left| f\big(\varphi(z)\big) g(z) + h(z) f'\big(\varphi(z)\big) \right|. \end{aligned}$$

In particular, take $z = z_0$, we get

$$2^{p+2}||T_{g,h,\varphi}||_{\mathcal{H}^p(\Lambda^+)\to\mathcal{A}^\alpha(\Lambda^+)} \geq \frac{\big(Im(z_0)\big)^\alpha |h(z)| \left|\frac{1}{4}+\frac{3i}{8}\right| \frac{1}{\pi^{\frac{1}{p}}}}{\big(Im\varphi(z_0)\big)^{1+\frac{1}{p}}}.$$

Since $z_0 \in \Lambda^+$ is arbitrary, we have

$$N = \sup_{z\in\Lambda^+} \frac{\big(Im(z)\big)^\alpha}{\Big(Im\big(\varphi(z)\big)\Big)^{1+\frac{1}{p}}} |\varphi''(z)| \leq \quad 2^{p+2}||T_{g,h,\varphi}||_{\mathcal{H}^p(\Lambda^+)\to\mathcal{A}^\alpha(\Lambda^+)}. \tag{2.2}$$

Again consider the function

$$f_\omega(z) = \frac{3i\big(Im(\omega)\big)^{2-\frac{1}{p}}}{\pi^{\frac{1}{p}}(z-\overline{\omega})^2} + 4\frac{\big(Im(\omega)\big)^{3-\frac{1}{p}}}{\pi^{\frac{1}{p}}(z-\overline{\omega})^3}\ , \quad \omega = \varphi(z_0).$$

Once again it is easy to prove that $\|f\|_{\mathcal{H}^p(\Lambda^+)} \leq 2^p \times 7$. Also,

$$f'_\omega(z) \ = \frac{-6i\big(Im(\omega)\big)^{2-\frac{1}{p}}}{\pi^{\frac{1}{p}}(z-\overline{\omega})^3} - 12\frac{\big(Im(\omega)\big)^{3-\frac{1}{p}}}{\pi^{\frac{1}{p}}(z-\overline{\omega})^4}$$

Thus $f'_\omega\big(\varphi(z_0)\big) = 0$ and

$$f_\omega\big(\varphi(z_0)\big) = \frac{1}{\pi^{\frac{1}{p}}}\left(\frac{i}{4}\right)\frac{1}{\big(Im(\omega)\big)^{\frac{1}{p}}}.$$

Since $T_{g,h,\varphi} : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded, there exists a positive constant C such that

$$\begin{aligned} 7\times 2^p||T_{g,h,\varphi}||_{\mathcal{H}^p(\Lambda^+)\to\mathcal{A}^\alpha(\Lambda^+)} &\geq \left\|T_{g,h,\varphi}f_\omega\right\|_{\mathcal{A}^\alpha(\Lambda^+)} \\ &\geq \big(Im(z_0)\big)^\alpha\left|f'\big(\varphi(z_0)\big)g(z_0) + h(z_0)f'\big(\varphi(z_0)\big)\right| \\ &\geq \frac{3}{8}\frac{\big(Im(z_0)\big)^\alpha}{\Big(Im\big(\varphi(z_0)\big)\Big)^{\frac{1}{p}}}|g(z_0)|. \end{aligned}$$

Since $z_0 \in \Lambda^+$ is arbitrary, we have

$$M = \sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^{\alpha}}{\left(Im(\varphi(z))\right)^{\frac{1}{p}}} |g(z)| \leq C\, ||T_{g,h,\varphi}||_{\mathcal{H}^p(\Lambda^+)\to\mathcal{A}^{\alpha}(\Lambda^+)}. \qquad (2.3)$$

From (2.2) and (2.3), we have

$$M + N \leq C\, ||T_{g,h,\varphi}||_{\mathcal{H}^p(\Lambda^+)\to\mathcal{A}^{\alpha}(\Lambda^+)}. \qquad (2.4)$$

From (2.1) and (2.4), we have

$$||T_{g,h,\varphi}||_{\mathcal{H}^p(\Lambda^+)\to\mathcal{A}^{\alpha}(\Lambda^+)} \sim M + N. \blacksquare$$

Corollary 2.2. Let $1 \leq p < \infty$ and φ be a holomorphic self-map of the upper half-plane Λ^+. Then $C_{\varphi} : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^{\alpha}(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^{\alpha}}{\left(Im(\varphi(z))\right)^{\frac{1}{p}}} < \infty.$$

Corollary 2.3. Let $1 \leq p < \infty$ and $\psi \in H(\Lambda^+)$ and $\alpha \geq \frac{1}{p}$. Then $M_{\psi} : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^{\alpha}(\Lambda^+)$ is bounded if and only if $\psi \in \mathcal{A}^{\alpha-\frac{1}{p}}(\Lambda^+)$ if $\alpha > \frac{1}{p}. \psi \in$ X, where

$$X = \begin{cases} \mathcal{A}^{\alpha-\frac{1}{p}} if \alpha > \frac{1}{p} \\ H^{\infty} if \alpha = \frac{1}{p} \end{cases}.$$

Corollary 2.4. Let $1 \leq p < \infty$ and φ be a holomorphic self-map of the upper half-plane Λ^+. Then $C_{\varphi}D : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^{\alpha}(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^{\alpha}}{\left(Im(\varphi(z))\right)^{1+\frac{1}{p}}} < \infty.$$

Corollary 2.5. Let $1 \leq p < \infty$ and φ be a holomorphic self-map of the upper half-plane Λ^+. Then $DC_{\varphi} : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^{\alpha}(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^{\alpha}}{\left(Im(\varphi(z))\right)^{1+\frac{1}{p}}} |\varphi'(z)| < \infty.$$

Corollary 2.6. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and φ be a holomorphic self-map of the upper half-plane Λ^+. Then $M_\psi C_\varphi : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^{\alpha}}{\left(Im(\varphi(z))\right)^{\frac{1}{p}}} |\psi(z)| < \infty.$$

Corollary 2.7. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and φ be a holomorphic self-map of the upper half-plane Λ^+. Then $C_\varphi M_\psi : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^{\alpha}}{\left(Im(\varphi(z))\right)^{\frac{1}{p}}} |\psi(\varphi(z))| < \infty.$$

Corollary 2.8. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and φ be a holomorphic self-map of the upper half-plane Λ^+. Then $M_\psi C_\varphi D : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^{\alpha}}{\left(Im(\varphi(z))\right)^{1+\frac{1}{p}}} |\psi(z)| < \infty.$$

Corollary 2.9. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and φ be a holomorphic self-map of the upper half-plane Λ^+. Then $M_\psi D C_\varphi : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^{\alpha}}{\left(Im(\varphi(z))\right)^{1+\frac{1}{p}}} |\psi(z)\varphi'(z)| < \infty.$$

Corollary 2.10. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and φ be a holomorphic self-map of the upper half-plane Λ^+. Then $C_\varphi M_\psi D : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^{\alpha}}{\left(Im(\varphi(z))\right)^{1+\frac{1}{p}}} |\psi(\varphi(z))| < \infty.$$

Corollary 2.11. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and φ be a holomorphic self-map of the upper half-plane Λ^+. Then $DM_\psi C_\varphi : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^\alpha}{\left(Im(\varphi(z))\right)^{\frac{1}{p}}} |\psi'(z)| < \infty$$

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^\alpha}{\left(Im(\varphi(z))\right)^{1+\frac{1}{p}}} |\psi(z)\varphi'(z)| < \infty.$$

Corollary 2.12. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and φ be a holomorphic self-map of the upper half-plane Λ^+. Then $C_\varphi DM_\psi : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^\alpha}{\left(Im(\varphi(z))\right)^{\frac{1}{p}}} \left|\psi'(\varphi(z))\right| < \infty$$

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^\alpha}{\left(Im(\varphi(z))\right)^{1+\frac{1}{p}}} |\psi(\varphi(z))| < \infty.$$

Corollary 2.13. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and φ be a holomorphic self-map of the upper half-plane Λ^+. Then $DC_\varphi M_\psi : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^\alpha}{\left(Im(\varphi(z))\right)^{\frac{1}{p}}} \left|\psi'(\varphi(z))\varphi'(z)\right| < \infty$$

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^\alpha}{\left(Im(\varphi(z))\right)^{1+\frac{1}{p}}} |\psi(\varphi(z))\varphi'(z)| < \infty.$$

Example 2.14. Let

$$\varphi(z) = \frac{az + b}{cz + d}, a, b, c, d \in \mathbb{R}, \qquad ad - bc > 0.$$

Then $DC_\varphi : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded if and only if $c = 0$ and $\alpha = 1 + 1/p$.

Proof: First suppose that $c = 0$ and $\alpha = 1 + 1/p$. then

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^{\alpha}}{\left(Im(\varphi(z))\right)^{1+\frac{1}{p}}} |\varphi'(z)| = \sup_{z \in \Lambda^+} \frac{y^{2+\frac{1}{p}}}{\left(\frac{a}{d}y\right)^{1+\frac{1}{p}}} \frac{a}{d}, \qquad z = x + iy$$

$$= \left(\frac{a}{d}\right)^{1+\frac{1}{p}} < \infty.$$

Thus $C_{\varphi}D : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^{\alpha}(\Lambda^+)$ is bounded. Again suppose that $c \neq 0$ or $\alpha \neq 1 + 1/p$. Then

$$Im(\varphi(z)) = \frac{(ad - bc)y}{(cx + d)^2 + c^2y^2} \quad and \quad |\varphi'(z)| = \frac{|ad - bc|}{(cx + d)^2 + c^2y^2}.$$

Therefore ,

$$\sup_{z \in \Lambda^+} \frac{\left(Im(z)\right)^{\alpha}}{\left(Im(\varphi(z))\right)^{1+\frac{1}{p}}} |\varphi'(z)|$$

$$= \sup_{z \in \Lambda^+} \frac{y^{\alpha}((cx + d)^2 + c^2y^2)^{1+\frac{1}{p}}}{\left((ad - bc)y\right)^{1+\frac{1}{p}}} \frac{\mathrm{ad} - \mathrm{bc}}{(cx + d)^2 + c^2y^2}$$

$$= \sup_{z \in \Lambda^+} \frac{y^{\alpha - \left(1+\frac{1}{p}\right)}((cx + d)^2 + c^2y^2)^{\frac{1}{p}}}{(ad - bc)^{1+\frac{1}{p}}}$$

$$= \infty,$$

and so$DC_{\varphi} : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^{\alpha}(\Lambda^+)$ is unbounded. Hence we are done ■

Example 2.15. Let

$$\varphi(z) = \frac{az + b}{cz + d}, a, b, c, d \in \mathbb{R}, \qquad ad - bc > 0.$$

Then $C_{\varphi}D : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^{\alpha}(\Lambda^+)$ is bounded if and only if $c = 0$ and $\alpha = 1 + 1/p$.

Proof: First suppose that $c = 0$ and $\alpha = 1 + 1/p$. then for $z = x + iy$, we have

$$\sup_{z \in \Lambda^+} \frac{(Im(z))^{\alpha}}{\left(Im(\varphi(z))\right)^{1+\frac{1}{p}}} = \sup_{z \in \Lambda^+} \frac{y^{2+\frac{1}{p}}}{\left(\frac{a}{d}y\right)^{1+\frac{1}{p}}} = \left(\frac{a}{d}\right)^{1+\frac{1}{p}} < \infty,$$

Thus $C_{\varphi}D : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^{\alpha}(\Lambda^+)$ is bounded. Again suppose that $c \neq 0$ or $\alpha \neq 1 + 1/p$. Then

$$Im(\varphi(z)) = \frac{(ad - bc)y}{(cx + d)^2 + c^2y^2}.$$

Therefore,

$$\sup_{z \in \Lambda^+} \frac{(Im(z))^{\alpha}}{\left(Im(\varphi(z))\right)^{1+\frac{1}{p}}} = \sup_{z \in \Lambda^+} \frac{y^{\alpha}((cx + d)^2 + c^2y^2)^{1+\frac{1}{p}}}{\left((ad - bc)y\right)^{1+\frac{1}{p}}}$$

$$= \sup_{z \in \Lambda^+} \frac{y^{\alpha-\left(1+\frac{1}{p}\right)}((cx + d)^2 + c^2y^2)^{1+\frac{1}{p}}}{(ad - bc)^{1+\frac{1}{p}}}$$

$$= \infty,$$

and so $C_{\varphi}D : \mathcal{H}^p(\Lambda^+) \longrightarrow \mathcal{A}^{\alpha}(\Lambda^+)$ is unbounded. Hence the proof. ■

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