Products of Composition, Multiplication and Differentiation between Hardy Spaces and Weighted Growth Spaces of the Upper-Half Plane

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Abstract

Let \( \psi \) be a holomorphic function of the upper-half plane \( \Lambda^+ \) and \( \varphi \) a holomorphic self-map of \( \Lambda^+ \). Let \( C_\varphi, M_\psi \) and \( D \) denote, respectively, the composition, multiplication and differentiation operators. In this paper, we characterize boundedness of the operators induced by products of these operators acting between Hardy and growth spaces of the upper-half plane.

Key words and phrases: Composition operator, Differentiation operator, Multiplication operator, Growth space, Hardy space, Upper-half plane.

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1. INTRODUCTION

Let \( G \) be a non-empty set, \( X \) a topological vector space, \( F(G, X) \) the topological vector space of functions from \( G \) to \( X \) with point-wise vector space operations and \( \varphi : G \rightarrow G \) be a function such that \( f \circ \varphi \in F(G, X) \) for all \( f \in F(G, X) \). Then the
linear transformation $C_\varphi : F(G, X) \rightarrow F(G, X)$, defined as $C_\varphi(f) = f \circ \varphi$ for all $f \in F(G, X)$, is known as the composition transformation induced by $\varphi$ on the space $F(G, X)$. If $C_\varphi$ is continuous, then it is called the composition operator or substitution operator induced by $\varphi$ on the space $F(G, X)$. Let $\Lambda^+ = \{x + iy : x, y \in \mathbb{R}, y > 0 \}$ be the upper half-plane and $1 \leq p < \infty$. Then the Hardy space $\mathcal{H}^p(\Lambda^+)$ is the collection of all analytic functions $f : \Lambda^+ \rightarrow \mathbb{C}$ such that

$$\sup_{y > 0} \int_{-\infty}^{+\infty} |f(x + iy)|^p dx < \infty.$$ 

It is well known that $\mathcal{H}^p(\Lambda^+)$ is a Banach space under the norm

$$\|f\|_{\mathcal{H}^p(\Lambda^+)} = \left(\sup_{y > 0} \int_{-\infty}^{+\infty} |f(x + iy)|^p dx\right)^{1/p},$$

and $\mathcal{H}^2(\Lambda^+)$ is a Hilbert space under the inner product:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f_\ast(x) \overline{g_\ast(x)} dx, \quad f, g \in \mathcal{H}^2(\Lambda^+),$$

where

$$f_\ast(x) = \lim_{y \to 0} f(x + iy),$$

which exists almost everywhere on $\mathbb{R}$. These Hardy spaces fall under the category of the functional Banach spaces which consist of bonafide functions with continuous evaluation functionals. For any positive real number $\alpha$, the growth space $\mathcal{A}^\alpha(\Lambda^+)$ consists of analytic functions $f : \Lambda^+ \rightarrow \mathbb{C}$ such that

$$\|f\|_{\mathcal{A}^\alpha(\Lambda^+)} = \sup\{(\text{Im}z)^\alpha |f(z)| : z \in \Lambda^+\} < \infty.$$ 

With the norm $\|\cdot\|_{\mathcal{A}^\alpha(\Lambda^+)}$, $\mathcal{A}^\alpha(\Lambda^+)$ is a Banach space. Note that $\mathcal{A}^1(\Lambda^+)$ is the usual growth space. For $\varphi \in \text{H}(\Lambda^+)$ the multiplication operator $M_\varphi$ is defined by $M_\varphi f = \varphi f$. The product of composition and multiplication operators, denoted by $W_\varphi \varphi$ and defined as $W_\varphi \varphi = M_\varphi \circ C_\varphi$, is known as weighted composition operator and has been studied intensively in recent times. The differentiation operator denoted by $D$ is defined by $Df = f'$. As a consequence of the Little - wood Subordination principle, it is known that every analytic self-map $\varphi$ of the open unit disk $\mathbb{D}$ induces a bounded composition operator on Hardy and weighted Bergman spaces of the open unit disk $\mathbb{D}$.
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(see [3] and [17]). However, if we move to Hardy and weighted Bergman spaces of the upper half-plane $\Lambda^+$, the situation is entirely different. In fact, there exist analytic self-maps of the upper half-plane which do not induce composition operators on the Hardy spaces and weighted Bergman spaces of the upper half-plane. Interesting work on composition operators on the spaces of upper half-plane have been done by many authors, to cite a few, Singh [10], Singh and Sharma [11, 12], Sharma [18], Matache [7, 8], Sharma, Sharma and Shabir [19, 20], Stevic and Sharma [22, 23, 24, 26], Sharma, Sharma and Abbas [16]. Recently, some attention have been paid to the study concrete operators and their products between spaces of holomorphic functions, for example, Sharma and Abbas [14], Sharma, Sharma and Abbas [15], Sharma and Abbas [13], Bhat, Abbas and Sharma[2], Kumar and Abbas [6], Abbas and Kumar[1], Kumar and Abbas [5] and [4, 9, 21, 23, 25, 27] and the related references therein.

We can define the products of composition, multiplication and differentiation operators in the following six ways.

\[
(M_\psi C_\varphi Df)(z) = \psi(z)f'(\varphi(z)),
\]
\[
(M_\psi DC_\varphi f)(z) = \psi(z)\varphi'(z)f'(\varphi(z)),
\]
\[
(C_\varphi M_\psi Df)(z) = \psi(\varphi(z))f'(\varphi(z)),
\]
\[
(DM_\psi C_\varphi f)(z) = \psi'(z)f(\varphi(z)) + \psi(z)\varphi'(z)f'(\varphi(z)),
\]
\[
(C_\varphi DM_\psi f)(z) = \psi'(\varphi(z))f(\varphi(z)) + \psi(\varphi(z))f'(\varphi(z)),
\]
\[
(DC_\varphi M_\psi f)(z) = \psi'(\varphi(z))\varphi'(z)f(\varphi(z)) + \psi(\varphi(z))\varphi'(z)f'(\varphi(z)),
\]
for $z \in \Lambda^+$ and $f \in H(\Lambda^+)$.

Note that the operator $M_\psi C_\varphi D$ induces many known operators. If $\psi = 1$, then $M_\psi C_\varphi D = C_\varphi D$, while when $\psi(z) = \varphi'(z)$, then we get the operator $DC_\varphi$. If we put $\varphi(z) = z$, then $M_\psi C_\varphi D = M_\psi D$, that is, the product of differentiation operator and multiplication operator. Also note that $M_\psi DC_\varphi = M_{\psi\varphi'} C_\varphi D$ and $C_\varphi M_\psi D = M_{\psi\varphi} C_\varphi D$. Thus the corresponding characterizations of boundedness and compactness of $M_\psi DC_\varphi$ and $C_\varphi M_\psi D$ can be obtained by replacing $\psi$, respectively by $\psi\varphi'$ and $\psi \circ \varphi$ in the results stated for $M_\psi C_\varphi D$. 

In order to treat these operators in a unified manner, we introduce the following operator

\[ T_{g,h,\varphi}(z) = g(z) f(\varphi(z)) + h(z) f'(\varphi(z)) \]

where \( g, h \in \mathcal{H}(\Lambda^+) \) and \( \varphi \) a holomorphic self-map of \( \Lambda^+ \). It is clear that composition, multiplication, differentiation operators and all the products of the composition, multiplication and differentiation operators defined above can be obtained from the operator \( T_{g,h,\varphi} \) by fixing \( g \) and \( h \). More specifically, we have

\[ C_{\varphi} T_{1,0,\varphi} = T_{\varphi,0,\varphi}, \quad M_{\varphi} T_{\varphi,0,\varphi} = T_{\varphi,0,\varphi}, \quad C_{\varphi} D T_{\varphi,0,\varphi} = T_{\varphi,0,\varphi}, \quad M_{\varphi} D T_{\varphi,0,\varphi} = T_{\varphi,0,\varphi}. \]

In this paper we characterize the boundedness of the operator \( T_{g,h,\varphi} \) acting between Hardy spaces and growth spaces of the upper-half plane. Throughout this paper, constants are denoted by \( C \), they are positive and not necessarily the same at each occurrence.

### 2. BOUNDEDNESS OF \( T_{g,h,\varphi} : \mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+) \)

In this section, we characterize boundedness of \( T_{g,h,\varphi} : \mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+) \).

**Theorem 2.1.** Let \( 1 \leq p < \infty \) and \( \varphi \) be a holomorphic self-map of \( \Lambda^+ \). Then \( T_{g,h,\varphi} : \mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+) \) is bounded if and only if

\[ (i) \quad M = \sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{(\text{Im}(\varphi(z)))^{\frac{1}{p}}} |g(z)| < \infty, \]

\[ (ii) \quad N = \sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{(\text{Im}(\varphi(z)))^{1+\frac{1}{p}}} |h(z)| < \infty. \]

Moreover if \( T_{g,h,\varphi} : \mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+) \) is bounded, then

\[ ||T_{g,h,\varphi}\|_{\mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+)} \sim M + N. \]

**Proof:** Firstly, suppose that (i) and (ii) hold, then

\[ ||T_{g,h,\varphi}f||_{\mathcal{A}^\alpha(\Lambda^+)} = \sup \{ \text{Im}(z) |(T_{g,h,\varphi}f)(z)| : z \in \Lambda^+ \}. \]
Now

\[ \text{Im}(z) | (T_{g,h,f}(z)) | = (\text{Im} z)\alpha | f(\varphi(z))g(z) + h(z) f'(\varphi(z)) | \]
\[ \leq (\text{Im} z)\alpha (| f(\varphi(z))|| g(z)| + | h(z)|| f'(\varphi(z))|) \]
\[ \leq C \| f \|_{H^p(\Lambda^+)} \left( \frac{(\text{Im} z)^{\alpha}}{(\text{Im} \varphi(z))^{\frac{1}{p}}} | g(z)| + \frac{(\text{Im} z)^{\alpha}}{(\text{Im} \varphi(z))^{\frac{1}{p}+\frac{1}{p}}} | h(z)| \right) \]
\[ \leq C (M + N) \| f \|_{H^p(\Lambda^+)} \]

Thus,

\[ \| (T_{g,h,f})f \|_{A^\alpha(\Lambda^+)} \leq C (M + N) \| f \|_{H^p(\Lambda^+)} \]

and so \( T_{g,h,f} : H^p(\Lambda^+) \to A^\alpha(\Lambda^+) \) is bounded and

\[ \| T_{g,h,f} \|_{H^p(\Lambda^+) \to A^\alpha(\Lambda^+)} \leq C (M + N). \]  \( (2.1) \)

Conversely, suppose that \( T_{g,h,f} : H^p(\Lambda^+) \to A^\alpha(\Lambda^+) \) is bounded. Let \( z_0 \in \Lambda^+ \) be fixed and let \( \omega = \varphi(z_0) \). Consider the function

\[ f_{\omega}(z) = (\text{Im} \omega)^{\frac{2}{p} - \frac{1}{2}} \frac{1}{\pi^p(z - \bar{\omega})^2} - 2i (\text{Im} \omega)^{\frac{3}{p} - \frac{1}{4}} \frac{1}{\pi^p(z - \bar{\omega})^3}. \]

Writing \( z = x + iy \) and \( \omega = u + iv \) and using the elementary inequality \( (x + iy)^a \leq 2^a (x^a + y^a) \) which holds for all \( x, y \geq 0 \) and \( a > 0 \), we have

\[ \| f \|_{H^p(\Lambda^+)} \leq 2^p \left[ \sup_{y > 0} \int_{-\infty}^{\infty} \frac{v^{2p-1}}{\pi (x + iy - (u - iv))^{2p}} dx \right] \]
\[ + \sup_{y > 0} \int_{-\infty}^{\infty} \frac{2v^{3p-1}}{\pi (x + iy - (u - iv))^{3p}} dx \]

Again using the inequalities

\[ |(x + iy) - (u + iv)|^{2p} \geq (v + y)^{2p-2}((x - u)^2 + (y + v)^2) \]
\[ |(x + iy) - (u + iv)|^3p \geq (v + y)^{3p-2}((x - u)^2 + (y + v)^2), \]

we get

\[
\|f\|_{\mathcal{H}^p(\Lambda^+)} \leq 2^p \left[ v^{2p-1} \sup_{y > 0} \frac{1}{(y + v)^2} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y + v}{(x - u)^2 + (y + v)^2} \, dx \right. \\
\left. + v^{3p-1} \sup_{y > 0} \frac{2}{(y + v)^{3p-1}} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y + v}{(x - u)^2 + (y + v)^2} \, dx \right]
\]

\[ = 2^p \left[ v^{2p-1} \sup_{y > 0} \frac{1}{(y + v)^2} + v^{3p-1} \sup_{y > 0} \frac{2}{(y + v)^{3p-1}} \right] \leq 2^{p+2}. \]

Also,

\[ f'_\omega(z) = \frac{-2(\text{Im}(\omega))^{2-\frac{1}{p}}}{\pi^{\frac{1}{p}}(z - \bar{\omega})^3} + 6i \frac{(\text{Im}(\omega))^{-\frac{1}{p}}}{\pi^{\frac{1}{p}}(z - \bar{\omega})^4} \]

Moreover,

\[ f'_\omega(\varphi(z_0)) = \left( \frac{1}{4i} + \frac{3i}{8} \right) \frac{1}{\pi^{\frac{1}{p}} (\text{Im}(\omega))^{1+\frac{1}{p}}} \]

and \( f'_\omega(\varphi(z_0)) = 0. \)

Since \( T_{g,h,\varphi} : \mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+) \) is bounded, we have

\[
\|T_{g,h,\varphi}f\|_{\mathcal{A}^\alpha(\Lambda^+)} \leq \|T_{g,h,\varphi}\|_{\mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+)} \|f\|_{\mathcal{H}^p(\Lambda^+)} \\
\leq 2^{p+2} \|T_{g,h,\varphi}\|_{\mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+)}.
\]

This implies for each \( z \in \Lambda^+ \), we have

\[
2^{p+2} \|T_{g,h,\varphi}\|_{\mathcal{H}^p(\Lambda^+) \to \mathcal{A}^\alpha(\Lambda^+)} \geq (\text{Im}(z))^{\alpha} \left| (T_{g,h,\varphi}f)'(z) \right| \\
= (\text{Im}z)^\alpha |f(\varphi(z))g(z) + h(z)f'(\varphi(z))|.
\]
In particular, take \( z = z_0 \), we get
\[
2^{p+2} \| T_{g,h,\varphi} \| \mathcal{H}^p(\Lambda^+) \rightarrow \mathcal{A}^\alpha(\Lambda^+) \geq \frac{(\operatorname{Im}(z_0))^\alpha |h(z)| \left| \frac{1}{4} + \frac{3i}{8} \right| \frac{1}{\pi^p}}{(\operatorname{Im}(\varphi(z_0)))^{1 + \frac{1}{p}}}.
\]

Since \( z_0 \in \Lambda^+ \) is arbitrary, we have
\[
N = \sup_{z \in \Lambda^+} \frac{(\operatorname{Im}(z))^\alpha}{(\operatorname{Im}(\varphi(z)))^{1 + \frac{1}{p}}} \left| \varphi''(z) \right| \leq 2^{p+2} \| T_{g,h,\varphi} \| \mathcal{H}^p(\Lambda^+) \rightarrow \mathcal{A}^\alpha(\Lambda^+). \tag{2.2}
\]

Again consider the function
\[
f_\omega(z) = \frac{3i(\operatorname{Im}(\omega))^{2-\frac{1}{p}}}{\pi^\frac{1}{p}(z - \overline{\omega})^2} + \frac{4(\operatorname{Im}(\omega))^{3-\frac{1}{p}}}{\pi^\frac{1}{p}(z - \overline{\omega})^3}, \quad \omega = \varphi(z_0).
\]

Once again it is easy to prove that \( \| f \| \mathcal{H}^p(\Lambda^+) \leq 2p \times 7 \). Also,
\[
f_\omega'(z) = -\frac{6i(\operatorname{Im}(\omega))^{2-\frac{1}{p}}}{\pi^\frac{1}{p}(z - \overline{\omega})^3} - \frac{12(\operatorname{Im}(\omega))^{3-\frac{1}{p}}}{\pi^\frac{1}{p}(z - \overline{\omega})^4}
\]

Thus \( f_\omega'\varphi(z_0)) = 0 \) and
\[
f_\omega(\varphi(z_0)) = \frac{1}{\pi^\frac{1}{p}} \left( \frac{i}{4} \right) \frac{1}{(\operatorname{Im}(\omega))^\frac{1}{p}}.
\]

Since \( T_{g,h,\varphi} : \mathcal{H}^p(\Lambda^+) \rightarrow \mathcal{A}^\alpha(\Lambda^+) \) is bounded, there exists a positive constant \( C \) such that
\[
7 \times 2^p \| T_{g,h,\varphi} \| \mathcal{H}^p(\Lambda^+) \rightarrow \mathcal{A}^\alpha(\Lambda^+) \geq \| T_{g,h,\varphi} f_\omega \| \mathcal{A}^\alpha(\Lambda^+)
\]
\[
\geq (\operatorname{Im}(z_0))^{\alpha} |f'(\varphi(z_0))g(z_0) + h(z_0)f'(\varphi(z_0))| \geq \frac{3}{8} \frac{(\operatorname{Im}(z_0))^{\alpha}}{(\operatorname{Im}(\varphi(z_0)))^{\frac{1}{p}}} |g(z_0)|.
\]
Since $z_0 \in \Lambda^+$ is arbitrary, we have

$$M = \sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{\left((\text{Im}(\varphi(z)))^{\frac{1}{p}}\right)^{\frac{1}{\alpha}}} |g(z)| \leq C \|T_{g,h,\varphi}\|_{\mathcal{H}^P(\Lambda^+) \rightarrow \mathcal{A}^\alpha(\Lambda^+)}.$$  \hspace{1cm} (2.3)

From (2.2) and (2.3), we have

$$M + N \leq C \|T_{g,h,\varphi}\|_{\mathcal{H}^P(\Lambda^+) \rightarrow \mathcal{A}^\alpha(\Lambda^+)}.$$  \hspace{1cm} (2.4)

From (2.1) and (2.4), we have

$$\|T_{g,h,\varphi}\|_{\mathcal{H}^P(\Lambda^+) \rightarrow \mathcal{A}^\alpha(\Lambda^+)} \sim M + N.$$

**Corollary 2.2.** Let $1 \leq p < \infty$ and $\varphi$ be a holomorphic self-map of the upper half-plane $\Lambda^+$. Then $C_\varphi : \mathcal{H}^P(\Lambda^+) \rightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{\left((\text{Im}(\varphi(z)))^{\frac{1}{p}}\right)^{\frac{1}{\alpha}}} < \infty.$$}

**Corollary 2.3.** Let $1 \leq p < \infty$ and $\psi \in H(\Lambda^+)$ and $\alpha \geq \frac{1}{p}$. Then $M_\psi : \mathcal{H}^P(\Lambda^+) \rightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded if and only if $\psi \in \mathcal{A}^{\alpha-\frac{1}{p}}(\Lambda^+)$ if $\alpha > \frac{1}{p}, \psi \in X$, where

$$X = \begin{cases} \mathcal{A}^{\alpha-\frac{1}{p}} & \text{if } \alpha > \frac{1}{p} \\ H^\infty & \text{if } \alpha = \frac{1}{p} \end{cases}.$$

**Corollary 2.4.** Let $1 \leq p < \infty$ and $\varphi$ be a holomorphic self-map of the upper half-plane $\Lambda^+$. Then $C_{\varphi}D : \mathcal{H}^P(\Lambda^+) \rightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{\left((\text{Im}(\varphi(z)))^{\frac{1}{p}}\right)^{\frac{1}{\alpha}}} < \infty.$$}

**Corollary 2.5.** Let $1 \leq p < \infty$ and $\varphi$ be a holomorphic self-map of the upper half-plane $\Lambda^+$. Then $DC_\varphi : \mathcal{H}^P(\Lambda^+) \rightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded if and only if
Corollary 2.6. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and $\varphi$ be a holomorphic self-map of the upper half-plane $\Lambda^+$. Then $M_\psi C_\varphi : H^p(\Lambda^+) \to A^a(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{(\text{Im}(\varphi(z)))^{\frac{1}{1+p}}} |\varphi'(z)| < \infty.$$ 

Corollary 2.7. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and $\varphi$ be a holomorphic self-map of the upper half-plane $\Lambda^+$. Then $C_\varphi M_\psi : H^p(\Lambda^+) \to A^a(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{(\text{Im}(\varphi(z)))^{\frac{1}{1+p}}} |\psi(z)| < \infty.$$ 

Corollary 2.8. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and $\varphi$ be a holomorphic self-map of the upper half-plane $\Lambda^+$. Then $M_\psi C_\varphi D : H^p(\Lambda^+) \to A^a(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{(\text{Im}(\varphi(z)))^{\frac{1}{1+p}}} |\varphi(z)| < \infty.$$ 

Corollary 2.9. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and $\varphi$ be a holomorphic self-map of the upper half-plane $\Lambda^+$. Then $M_\psi D C_\varphi : H^p(\Lambda^+) \to A^a(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{(\text{Im}(\varphi(z)))^{\frac{1}{1+p}}} |\psi(z)\varphi'(z)| < \infty.$$ 

Corollary 2.10. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and $\varphi$ be a holomorphic self-map of the upper half-plane $\Lambda^+$. Then $C_\varphi M_\psi D : H^p(\Lambda^+) \to A^a(\Lambda^+)$ is bounded if and only if

$$\sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{(\text{Im}(\varphi(z)))^{\frac{1}{1+p}}} |\psi(\varphi(z))| < \infty.$$
Corollary 2.11. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and $\varphi$ be a holomorphic self-map of the upper half-plane $\Lambda^+$. Then $DM_{\psi}C_{\varphi} : H^p(\Lambda^+) \rightarrow A^\alpha(\Lambda^+)$ is bounded if and only if

$$
\sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{(\text{Im}(\varphi(z)))^{1+\frac{1}{p}}} |\psi'(\varphi(z))| < \infty
$$

and

$$
\sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{(\text{Im}(\varphi(z)))^{1+\frac{1}{p}}} |\psi(z)\varphi'(z)| < \infty.
$$

Corollary 2.12. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and $\varphi$ be a holomorphic self-map of the upper half-plane $\Lambda^+$. Then $C_{\varphi}DM_{\psi} : H^p(\Lambda^+) \rightarrow A^\alpha(\Lambda^+)$ is bounded if and only if

$$
\sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{(\text{Im}(\varphi(z)))^{1+\frac{1}{p}}} |\psi'(\varphi(z))| < \infty
$$

and

$$
\sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{(\text{Im}(\varphi(z)))^{1+\frac{1}{p}}} |\psi(\varphi(z))| < \infty.
$$

Corollary 2.13. Let $1 \leq p < \infty$, $\psi \in H(\Lambda^+)$ and $\varphi$ be a holomorphic self-map of the upper half-plane $\Lambda^+$. Then $DC_{\varphi}M_{\psi} : H^p(\Lambda^+) \rightarrow A^\alpha(\Lambda^+)$ is bounded if and only if

$$
\sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{(\text{Im}(\varphi(z)))^{1+\frac{1}{p}}} |\psi'(\varphi(z))\varphi'(z)| < \infty
$$

and

$$
\sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{(\text{Im}(\varphi(z)))^{1+\frac{1}{p}}} |\psi(\varphi(z))\varphi'(z)| < \infty.
$$

Example 2.14. Let

$$
\varphi(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc > 0.
$$

Then $DC_{\varphi} : H^p(\Lambda^+) \rightarrow A^\alpha(\Lambda^+)$ is bounded if and only if $c = 0$ and $\alpha = 1 + 1/p$. 
Proof: First suppose that $c = 0$ and $\alpha = 1 + 1/p$. then
\[\sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{\left(\text{Im}(\varphi(z))\right)^{1+1/p}} |\varphi'(z)| = \sup_{z \in \Lambda^+} \frac{y^{2+1/p} a}{\left(\frac{a}{y}\right)^{1+1/p} d}, \quad z = x + iy\]

\[= \left(\frac{a}{d}\right)^{1+1/p} < \infty.\]

Thus $C_\varphi D : \mathcal{H}^p(\Lambda^+) \rightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded. Again suppose that $c \neq 0$ or $\alpha \neq 1 + 1/p$. Then
\[\text{Im}(\varphi(z)) = \frac{(ad - bc)y}{(cx + d)^2 + c^2 y^2} \quad \text{and} \quad |\varphi'(z)| = \frac{|ad - bc|}{(cx + d)^2 + c^2 y^2}.\]

Therefore,
\[\sup_{z \in \Lambda^+} \frac{(\text{Im}(z))^\alpha}{\left(\text{Im}(\varphi(z))\right)^{1+1/p}} |\varphi'(z)| = \sup_{z \in \Lambda^+} \frac{y^\alpha ((cx + d)^2 + c^2 y^2)^{1+1/p}}{(ad - bc)y^{1+1/p} ((cx + d)^2 + c^2 y^2)}\]
\[= \sup_{z \in \Lambda^+} \frac{y^{\alpha - (1+1/p)} ((cx + d)^2 + c^2 y^2)^{1/p}}{(ad - bc)^{1+1/p}}\]
\[= \infty,\]

and so $DC_\varphi : \mathcal{H}^p(\Lambda^+) \rightarrow \mathcal{A}^\alpha(\Lambda^+)$ is unbounded. Hence we are done.$\blacksquare$

Example 2.15. Let
\[\varphi(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc > 0.\]

Then $C_\varphi D : \mathcal{H}^p(\Lambda^+) \rightarrow \mathcal{A}^\alpha(\Lambda^+)$ is bounded if and only if $c = 0$ and $\alpha = 1 + 1/p$.

Proof: First suppose that $c = 0$ and $\alpha = 1 + 1/p$. then for $z = x + iy$, we have
Thus $C_φD : H^p(Λ^+) \to A^α(Λ^+)$ is bounded. Again suppose that $c \neq 0$ or $α \neq 1 + \frac{1}{p}$. Then

$$\text{Im}(φ(z)) = \frac{(ad - bc)y}{(cx + d)^2 + c^2y^2}.$$ 

Therefore,

$$\sup_{z \in Λ^+} \left( \frac{(lm(z))^α}{\left( \text{Im}(φ(z)) \right)^{1 + \frac{1}{p}}} \right) = \sup_{z \in Λ^+} \frac{y^{α - (1 + \frac{1}{p})}}{(ad - bc)^{1 + \frac{1}{p}}} \left((cx + d)^2 + c^2y^2\right)^{1 + \frac{1}{p}} \leq \frac{y^{α - (1 + \frac{1}{p})}}{\left( \frac{ad}{c} \right)^{1 + \frac{1}{p}}} < \infty,$$

and so $C_φD : H^p(Λ^+) \to A^α(Λ^+)$ is unbounded. Hence the proof.

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