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Hyper Algebraic Structure Associated to Convex Function and Chemical Equilibrium

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Abstract

In this Paper, we prove important results on convex function using hyperstructure .we prove theorem on constrained optimization. The purpose of this paper is to provide examples of hyperstructures associated with chemical equilibrium using convex constrained optimization.

Key words: Hyper structure, Convex function, Optimization, Chemical equilibrium.

Mathematics Subject Classifications: 20N20, 06A11, 80A50.

1. INTRODUCTION

Hyperstructures were introduced in 1934 by a French mathematician, Marty 1934 at 8^{th} congress of Scandinavian mathematics [4] and plays a central role in the theory of algebraic hyperstructures .Since then this theory has enjoined a rapid development [1,2,6,10].There are many applications of hyperstructures in various areas. In [3], Davvaz defined convex function on hyperoperation and proved some results on convex function .We obtain proof of theorem in [3] by contradiction method. In this paper we provide some theorems on convex function and convex optimization. Also we investigate convex optimization of chemical equilibrium of decomposition of $CaCO_3$ when heated.

In the second section, we prove the theorem by Davvaz [3] by contradiction. We define point wise maximum and composition of two convex functions and prove it as convex function.

In third section we prove that optimal solution set of the optimization problem is convex.

Results on convex sets and convex functions are useful in understanding some important aspects of optimization theory.

Chemical equilibrium for heterogeneous function is described in [7].In [5], Chemical equilibrium is expressed in linear programming problem. In the fourth section, we give example of convex function using chemical equilibrium associated to hyperstructure.

2. PRELIMINARIES:

Following definitions are from [3]

Definition 2.1: a hyperstructure is defined as,

$$\star : H\times H \rightarrow H\otimes H \subseteq P^{*}(H) \text{ where } H \text{ is non empty.}$$

$$+: H \times H \rightarrow H$$

$$\cdot : F\times H \rightarrow H,$$

where $H \neq \varphi$, $\star$ is a commutative hyperoperation such that $\star(H\times H)=H\otimes H$, $\cdot$ and $+$ are commutative binary operations and F is a field.

Let us define $P^*(H)=\{ x\star y \in H\otimes H \mid x ,y \in X\}$, where X is a non empty subset in H.

Definition 2.2: Let $f: H\otimes H\rightarrow R$. If $\hat{x} \star \hat{y} \in P^*(X)$ and $f(\hat{x}\star \hat{y}) \leq f(x\star y)$ for each $x\star y \in P^*(X)$, then $\hat{x} \star \hat{y}$ is called a global minima. If $\bar{x} \star \bar{y} \in P^*(X)$ and there exists an ε − neighborhood $N_\varepsilon^* (\bar{x},\bar{y}) = \{(x\star y) \in H \otimes H \mid x \in N_\varepsilon^*(\bar{x}), y \in N_\varepsilon^*(\bar{y})\}$ such that $f(\bar{x}\star \bar{y}) \leq f(x\star y)$ for each $x\star y \in N_\varepsilon^* (\bar{x},\bar{y}) \cap P^\star(X)$, then $(\bar{x}\star \bar{y})$ is called a local minima.

3. CONVEX FUNCTION:

Definition 3.1: *Let f: P*(X) →R,* where X is non-empty convex subset in H .The function f is called a convex function $P^*(X)$ if

$$f([\lambda \hat{x} +(1-\lambda) \bar{x}] \star[\lambda \hat{y} +(1-\lambda) \bar{y}]) \leq \lambda f(\hat{x}\star \hat{y})+(1-\lambda)f(\bar{x}\star \bar{y})$$

for each $\hat{x}$, $\hat{y}$, $\bar{x}$, $\bar{y} \in X$ $\hat{x} \star \hat{y}$, $\bar{x} \star \bar{y} \in P^*(X)$ and for all $0 \leq \lambda \leq 1$. The function is called strictly convex on $P^*(X)$ if the inequality is satisfied as strict inequality for each distinct $\hat{x} \star \hat{y}$, $\bar{x} \star \bar{y} \in P^*(X)$ and $0 < \lambda < 1$. The function f is called concave on X if $-f$ is convex on X.

Definition 3.2: Let $f: P^*(X) \to R$, *where* X *is* a non-empty subset in H. The *epigraph* of f denoted by $epi^* f$ and *hypograph* is denoted by $hypo^* f$ is a subset of $H \times H \times R$ defined by

$$epi^* f = \{(x, y, z): x, y \in X, z \in R, f(x \star y) \leq z\}$$

$$hypo^* f = \{(x, y, z): x, y \in X, z \in R, f(x \star y) \geq z\}$$

Following theorem is by Davvaz [3], we prove the following theorem by contradiction.

Theorem 3.3: *Let* $f: P^*(X) \to R$, *where* X *is a non-empty convex subset in H.* f *is a convex function if and only if* $epi^* f$ *is a convex set.*

Proof: *Let* f *be convex.* Suppose $epi^* f$ is not convex.

There exist $(\hat{x}, \hat{y}, f(\hat{x} \star \hat{y})), (\bar{x}, \bar{y}, f(\bar{x} \star \bar{y}))$ and $\lambda \in [0,1]$ such that

$f(\hat{x} \star \hat{y}) \leq z_1$, $f(\bar{x} \star \bar{y}) \leq z_2$ *for all* $z_1, z_2 \in R$ and

$$([\lambda \hat{x} + (1-\lambda)\bar{x}], [\lambda \hat{y} + (1-\lambda)\bar{y}], \lambda z_1 + (1-\lambda) z_2) \notin epi^* f.$$

This implies

$$f([\lambda \hat{x} + (1-\lambda)\bar{x}] \star [\lambda \hat{y} + (1-\lambda)\bar{y}]) > \lambda z_1 + (1-\lambda) z_2 \geq \lambda f(\hat{x} \star \hat{y}) + (1-\lambda) f(\bar{x} \star \bar{y})$$

It gives f is not convex. Hence the contradiction.

Therefore $epi^* f$ is convex.

Let us suppose $epi^{*}f$ is convex. Let f is not convex. there exist , $\hat{x}\star\hat{y}$, $\bar{x}\star\bar{y}$ $\in dom(f)$, $\lambda \in [0,1]$ such that

$$f([\lambda\hat{x}+(1-\lambda)\ \bar{x}\,]\star[\lambda\hat{y}+(1-\lambda)\ \bar{y}\,]) > [\lambda f(\hat{x}\star\hat{y})+(1-\lambda)f(\bar{x}\star\bar{y}\,]). \qquad (1)$$

If $(\hat{x},\hat{y},f(\hat{x}\star\hat{y})),(\bar{x},\bar{y},f(\bar{x}\star\bar{y}))\in epi^{*}f$ then

(1) implies $([\lambda\hat{x}+(1-\lambda)\ \bar{x}\,],[\lambda\hat{y}+(1-\lambda)\ \bar{y}\,]\ ,\lambda f(\hat{x}\star\hat{y})+(1-\lambda)f(\bar{x}\star\bar{y}\ ))\notin epi^{*}f$

A contradiction.

Therefore f is convex.

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Following theorem is from [3],

Theorem (Jensen Inequality) 3.4: *Let f: P*(X) → R , where X is a non-empty convex subset in H. The function f is convex if and if*

$$f([\sum_{i=1}^{n}\lambda_i x_i\,]\star[\sum_{i=1}^{n}\lambda_i y_i\,] \le \sum_{i=1}^{n}\lambda_i f(x_i * y_i)$$

for each $x_i \star y_i$ $,i=1,\ldots..k$ *and* $\sum_{i=1}^{n}\lambda_i = 1,\ \lambda_i \ge 0$.

Definition 3.5: If f_1 and f_2 are convex functions then their point wise maximum f defined by

$f(\hat{x}\star\hat{y}) = Max\{f_1(\hat{x}\star\hat{y}),\ f_2(\hat{x}\star\hat{y})\}$ with dom $f = dom\ f_1 \cap dom\ f_2$ *with* $dom\ f_1 \cap dom\ f_2 \ne \emptyset$.

Now we discuss convexity of point wise maximum

If $0 \le \lambda \le 1$, $(\hat{x}\star\hat{y})$, $(\bar{x}\star\bar{y}) \in dom\ f$.

$f([\lambda \hat{x} + (1-\lambda)\ \bar{x}] \star [\lambda \hat{y} + (1-\lambda)\ \bar{y}])$

$= Max\ \{f_1([\lambda \hat{x} + (1-\lambda)\ \bar{x}] \star [\lambda \hat{y} + (1-\lambda)\ \bar{y}]), f_2([\lambda \hat{x} + (1-\lambda)\ \bar{x}] \star [\lambda \hat{y} + (1-\lambda)\ \bar{y}])\}$

$\leq Max\ \{(\lambda f_1(\hat{x} \star \hat{y}) + (1-\lambda) f_1(\bar{x} \star \bar{y})), (\lambda f_2(\hat{x} \star \hat{y}) + (1-\lambda) f_2(\bar{x} \star \bar{y}))\}$

$\leq \lambda\ Max\ \{f_1(\hat{x} \star \hat{y}), f_2(\hat{x} \star \hat{y})\} + (1-\lambda)\ Max\{f_1(\bar{x} \star \bar{y}), f_2(\bar{x} \star \bar{y})\}$

$= \lambda f(x \star y) + (1-\lambda) f(x \star y)$

Which establish convexity of *f*.

It is easily shown that $f_1, f_2 \ldots\ldots\ldots f_m$ are convex functions then their point wise maximum is

$f(x \star y) = max\ \{f_1(x \star y), f_2(x \star y) \ldots\ldots\ldots\ldots f_m(x \star y)\}$.

Definition 3.6: If $f_1, f_2, \ldots\ldots f_n$ are convex functions and $w_1, w_2, \ldots\ldots w_n \geq 0$ then

$f(x \star y) = w_1 f_1(x \star y) + w_2 f_2(x \star y) + \ldots\ldots\ldots\ldots w_2 f_2(x \star y)$ *is also* .

Now we discuss convexity of *f*,

Let f_1, f_2 be convex function . $w_1, w_2 \geq 0$, $(\hat{x}, \hat{y})$,$(\bar{x}, \bar{y}) \in R \times R$ *and* $\lambda \in [0,1]$

$f([\lambda \hat{x} + (1-\lambda)\ \bar{x}] \star [\lambda \hat{y} + (1-\lambda)\ \bar{y}])$

$= w_1 f_1([\lambda \hat{x} + (1-\lambda)\ \bar{x}] \star [\lambda \hat{y} + (1-\lambda)\ \bar{y}]) + w_2 f_2([\lambda \hat{x} + (1-\lambda)\ \bar{x}] \star [\lambda \hat{y} + (1-\lambda)\ \bar{y}])$

$\leq w_1(\lambda f_1(\hat{x} \star \hat{y}) + (1-\lambda) f_1(\bar{x} \star \bar{y})) + w_2(\lambda f_n(\hat{x} \star \hat{y}) + (1-\lambda) f_n(\bar{x} \star \bar{y}))$

$= \lambda (w_1 f_1(\hat{x} \star \hat{y}) + w_2 f_n(\hat{x} \star \hat{y})) + (1-\lambda)(w_1 f_1(\bar{x} \star \bar{y}) + w_2 f_n(\bar{x} \star \bar{y}))$.

∎

Definition 3.7:

In this section we examine conditions on $h: P^*(X) \to R$ and $g: P^*(X) \to P^*(X)$ that guarantee convexity or concavity of their composition $f = h \circ g : H \otimes H \to R$, defined by $f(x \star y) = h(g(x \star y))$, $dom\, f = \{ x \star y \in dom\, g \,/\, g(x \star y) \in dom\, h\}$.

The composition can be proved directly, as an example, we will prove the following composition theorem.

Theorem 3.8: *If g is convex, h is convex, and h is non decreasing, then $f = h \circ g$ is convex.*

Proof:

Assume that $\hat{x}, \hat{y}, \bar{x}, \bar{y} \in X$, and $0 \le \lambda \le 1$.

Since $\hat{x}, \hat{y}, \bar{x}, \bar{y} \in X$, we have that $\hat{x} \star \hat{y}, \bar{x} \star \bar{y} \in dom\, g$ and $g(\hat{x} \star \hat{y})$, $g(\bar{x} \star \bar{y}) \in dom\, h$. Since *dom g* is convex,

we conclude that $([\lambda \hat{x} + (1-\lambda) \bar{x}] \star [\lambda \hat{y} + (1-\lambda) \bar{y}]) \in dom\, g$, and from convexity of *g*,

we have

$$g([\lambda \hat{x} + (1-\lambda) \bar{x}] \star [\lambda \hat{y} + (1-\lambda) \bar{y}]) \le \lambda g(\hat{x} \star \hat{y}) + (1-\lambda) g(\bar{x} \star \bar{y}).$$

Since $g(\hat{x} \star \hat{y})$, $g(\bar{x} \star \bar{y}) \in dom\, h$,

we conclude that ,

$$\lambda g(\hat{x} \star \hat{y}) + (1-\lambda) g(\bar{x} \star \bar{y}) \in dom\, h,$$

Now we use the assumption that *h* is non decreasing,

$$h(g([\lambda \hat{x} + (1-\lambda) \bar{x}] \star [\lambda \hat{y} + (1-\lambda) \bar{y}]) \le h(\lambda g(\hat{x} \star \hat{y}) + (1-\lambda) g(\bar{x} \star \bar{y})) \qquad (1)$$

From convexity of h, we have

$$h(\lambda g(\hat{x} \star \hat{y}) + (1-\lambda) g(\bar{x} \star \bar{y})) \le \lambda h(g(\hat{x} \star \hat{y})) + (1-\lambda) h(g(\bar{x} \star \bar{y})) \qquad (2)$$

Therefore from the equations (1) and (2),we have

$$h(g([\lambda \hat{x} + (1-\lambda) \bar{x}] \star [\lambda \hat{y} + (1-\lambda) \bar{y}]) \le \lambda h(g(\hat{x} \star \hat{y})) + (1-\lambda) h(g(\bar{x} \star \bar{y}))$$

4. CONSTRAINED OPTIMIZATION:

A convex optimization problem (or just a convex problem) is a problem consisting of minimizing a convex function over a closed and convex set. More explicitly, a convex problem on hyperstructure is of the form

$$\min \; f(x \star y)$$

$$\text{s. t. } x, y \in X, \qquad (3)$$

where $f : H \otimes H \to R$, X is a closed and convex non –empty set. we will often consider more explicit formulations of convex problems such as convex optimization problems in functional form, which are convex problems on hyperstructures

$$\min \; f(x \star y)$$

$$\text{s. t. } g_i(x \star y) \leq 0, \; I = 1,2,\ldots, m,$$

where $f, g_1,\ldots, g_m : H \otimes H \to R$ are convex functions .

Following theorem is from [3].

Theorem 4.1: *Let $f : P^*(X) \to R$ where X is non empty convex function in H and $\star(A, B)=\{ x \star y \in H \otimes H \mid x \in A,\; y \in B \}$ be a convex function defined over the convex set in $H \otimes H$, for all convex subsets A and B in H. Consider the (3) and $\bar{x} \star \bar{y} \in P^*(X)$ is a local minimum, so we have*

(1) If f is convex , then $\bar{x} \star \bar{y}$ is a global minimum

(2) If f is strictly convex , then $\bar{x} \star \bar{y}$ is the unique global minimum

■

Theorem 4.2: Let $f : P^*(X) \to R$ where X is non empty convex function in H and $\star(A, B)=\{ x \star y \in H \otimes H \mid x \in A,\; y \in B \}$ be a convex function defined over the convex set in $H \otimes H$ Then the set of optimal solutions of the problem

$$min \{ f(x \star y) : x \star y \in P^*(X) \},$$

which we denote by X^*, is convex. If, in addition, f is strictly convex over $P^*(X)$, then there exists at most one optimal solution of the problem .

Proof:

Let the optimal value by f^*. Let $\hat{x}$, $\hat{y}$, $\bar{x}$, $\bar{y} \in X^*$ and $\lambda \in [0, 1]$.

Then by Jensen's inequality,

$$f([\lambda \hat{x} + (1-\lambda) \bar{x}] \star [\lambda \hat{y} + (1-\lambda) \bar{y}]) \leq \lambda f^* + (1-\lambda) f^* = f^*,$$

and hence

$[\lambda \hat{x} + (1-\lambda) \bar{x}] \star [\lambda \hat{y} + (1-\lambda) \bar{y}])$ is also optimal, i.e., belongs to X^*,

This gives the convexity of X^*.

Suppose now that f is strictly convex and $P^*(X^*)$ is nonempty; to show that $P^*(X^*)$ is a singleton, suppose in contradiction that,

There exist $\hat{x}$, $\hat{y}$, $\bar{x}$, $\bar{y} \in X^*$ such that $(\hat{x} \star \hat{y}) \neq (\bar{x} \star \bar{y})$.

Then $(\hat{x} \star \hat{y})/2 + (\bar{x} \star \bar{y})/2 \in P^*(X)$ and by the strict convexity of f we have

$$f((\hat{x} \star \hat{y})/2 + (\bar{x} \star \bar{y})/2) = 1/2 f(\hat{x} \star \hat{y}) + 1/2(\bar{x} \star \bar{y})$$

$$= 1/2 f^* + 1/2 f^*$$

$$= f^*.$$

Which is a contradiction to the fact that f^* is the optimal value.

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5. CHEMICAL EQUILIBRIUM: A CONVEX HYPERSTRUCTURE

Sometimes a series of chemical equilibria are related .That is two or more reactions add up to find a reaction.

In this situation , the equilibrium constats are related.

If equilibrium reactions can be added to give a total reaction, then the equilibrium constant for the total reaction is equal to the product of the equilibrium constants for the contributing reactions·

Linyi Gao [9], studied about convex optimization for chemical reaction using algebraic structures.

In this section, we formulate chemical equilibrium of phases of $CaCO_3$ as convex optimization problem using hyper algebraic structure.

A state in which the rates of the forward and reverse reactions are equal and the concentrations of the reactants and products remain constant.

Equilibrium is a dynamic process – the conversions of reactants to products and products to reactants are still going on, although there is no net change in the number of reactant and product molecules.

At equilibrium, the mass distribution may be heterogeneous. Consider a mixture of CaO with CO_2 .Suppose the Phases of the system contains n distinct types of species, and

let $\hat{x} = (x_1, x_2, \ldots\ldots\ldots x_n)^T \in R_+^n$

$\hat{y} = (y_1, y_2, \ldots\ldots\ldots y_n)^T \in R_+^n$

Let x_i, y_i be the amount of species i and $\hat{x} \star \hat{y}$ hyperoperation between elements $\hat{x}$ and $\hat{y}$ then

$$(\hat{x} \star \hat{y}) = (x_1 \star y_1, x_2 \star y_2 \ldots\ldots\ldots\ldots\ldots x_n \star y_n \} \in P^*(X),\ 0 \le j \le k.$$

and let x^j be the j^{th} phase ,then

$$(\hat{x} \star \hat{y}) = (\hat{x} \star \hat{y})^{(1)} + (\hat{x} \star \hat{y})^{(2)} + \ldots\ldots\ldots\ldots + (\hat{x} \star \hat{y})^{(k)} \text{ for some } \hat{x}, \hat{y}$$

Consider heterogeneous chemical system in equilibrium,

Consider the following example from the chapter 6 , which is a hyperstructure defined on a hyperoperation .

Calcium carbonate is strongly heated until it undergoes thermal decomposition to form calcium oxide and carbon dioxide.

It is a three phase and two component system

$$\underset{\text{(Solid Phase)}}{CaCO_3} \overset{\Delta}{\leftrightarrow} \underset{\text{(Solid Phase)}}{CaO} + \underset{\text{(Gases phase)}}{CO_2 \uparrow}$$

Composition of any one phase can be expressed only in terms of any two phase composition as if we consider $CaO + CO_2$ as components.

$$CaCO_3 \rightarrow CaO + CO_2 \text{ or } CaCO_3 - CaO - CO_2 \rightarrow 0$$

Phase can be considered by using there components.

Phase CaO composition can be expressed as

$$\text{CaO} \rightarrow CaO + 0.\ CO_2 \ \ or\ \text{CaO} - CaO - 0.\ CO_2 \rightarrow 0$$

Phase CO_2 Composition

$$CO_2 \rightarrow 0.\ \text{CaO} + CO_2\ or\ \ CO_2 - 0.\ \text{CaO} - CO_2 \rightarrow 0$$

Hence it is two component system.

If for each phase $CaCO_3$ contains three elements Ca , C,O .CaO contains Ca and O as elements and CO_2 contains C and O as elements .

Chemical reactions:

Chemical reaction process for the above heterogonous system of n species is

$$a_{11}X_1 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{1n}X_n \leftrightarrow 0$$

.
.
.

$$a_{1m}X_1 + + \ldots\ldots\ldots\ldots + a_{nm} X_n \leftrightarrow 0$$

where a_{ij} is constant .For each reaction j, $j= 1,2,3\ldots..n$ and $X_1= CaCO_3$, $X_2 = CaO$, $X_3= CO_2$.

Let A be the coefficient matrix where $A_{ij} = a_{ij}$.

By the conservation of mass,

(i.e. the mass of the products in a chemical reaction must equal the mass of the reactants)

$$\widehat{x \star \hat{y}} = (\hat{x} \star \hat{y})_{init} + A\ (\overline{x \star y}) \qquad (4)$$

for some $\hat{x} * \hat{y} \in P^*(X)$ and $(\hat{x} \star \hat{y})_{init} \in P^*(X)$

The overall composition of the system is the sum of the initial composition $(\hat{x} \star \hat{y})_{init}$ and changes caused by chemical reactions $A\ (\overline{x \star y})$.

Physically $\hat{y}$ represents the quantity of reaction. The reactions are assumed to be reversible.

So $\hat{y}$ can be either positive or negative.

Optimization:

$$\text{Minimize } f(x \star y)$$

$$\text{Subject to } \widehat{x \star \hat{y}} = (\hat{x} \star \hat{y})_{init} + A\ (\overline{x \star y}) \qquad (5)$$

$$x \geq 0 .$$

Where $f : H \otimes H \rightarrow R$ is a real function and X is any non-empty subset in H.

Given a composition,

$$\text{Minimize } f_1((x \star y)^{(1)}) + f_2((x \star y)^{(2)}) + \ldots\ldots + f_k((x \star y)^{(k)})$$

$$\textit{Subject to } (x \star y)^{(1)} + (x \star y)^{(2)} + \ldots\ldots + (x \star y)^{(k)} = \hat{x} \star \hat{y} \qquad (6)$$

$$\text{and } (x \star y)^{(i)} \geq 0 .$$

Equation (4) along with the condition (5) is convex problem of chemical equilibrium reaction process using hyperoperation.

Any optimal point $(x \star y)'$ of (5) is an equilibrium composition of the system

At equilibrium, the composition of each phase of the system is given by solving the problems (6).

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REFERENCES

[1] A. D. Lokhande Aryani Gangadhara, “Congruences in Hypersemilattices”, International Mathematical Forum, Vol. 7, 2012, no. 55, 2735 - 2742.

[2] A. D. Lokhande, Aryani Gangadhara, On Poset of Subhypergroup of a hypergroup and Hyper Lattices”, Int. J. Contemp. Math. Sciences, Vol. 8, 2013, no. 12, 559 – 564.

[3] Ali Delavar Khala, Bijan Davvaz, Algebraic hyper-structures associated to Convex analysis and applications, Faculty of Sciences and Mathematics, University of Nis, Serbia 26:1 (2012), 55.

[4] F.Marty, “Surune generalization de la notion de group”, the 8th Congress Math, Scandinavas, Stockholm, 1934.

[5] George Dantzlg, Selmer Johnson, Wayne White A linear programming approach to the Chemical equilibrium problem, (1957), P-1060, 1-11.

[6] G. Massouros, On the Hypergroup theory, F.S.A.I, 4 (2) (1995),13-25.

[7] J. W. Gibbs. First Part, Art. V. On the equilibrium of heterogeneous substances. Transactions Connecticut academy of Arts and sciences. vol iii.

[8] Stephen Boyd and Lieven Vandenberg he Convex Optimization, book, Cambridge University Press.

[9] Linyi Gao, Chemical Equilibrium: A Convex Optimization Problem, June 4, 2014.

[10] Z.Guo and X.L.Xin, ”Hyperlattices” Pure and applied Mathematics 20(1) 40-43, 2004.