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Solving the Fractional Convection-Diffusion Equation by Means of the Optimal q-Homotopy Analysis Method (Oq-HAM)

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Abstract

The main aim of this paper is to propose a new and simple algorithm namely optimal q- homotopy analysis method (Oq-HAM), to obtain approximate analytical solutions of fractional Convection Diffusion (FCD) equation. Comparison of Oq- HAM with the homotopy analysis method (HAM) and the homotopy perturbation method (HPM) are made. The results reveal that the Oq-HAM has more accuracy than the others. Finally, numerical example is given to illustrate the accuracy and stability of this method. Comparison of the approximate solution with the exact solutions also we show that the proposed method is very efficient and computationally attractive. A new efficient approach is proposed to obtain the optimal value of convergence controller parameter $\hbar$ to guarantee the convergence of the obtained series solution.

Keywords: Optimal q-homotopy analysis method; Convergence-control parameter; Fractional Convection Diffusion equation.

1. INTRODUCTION

Fractional differential equation is a fascinating area of study in mathematics. Over the past decades, many methods have been developed to handle these differential equations. Several methods have been suggested to solve nonlinear equations. These methods include the Homotopy perturbation method (HPM) [1], Luapanov's artificial small parameter method[2], Adomian decomposition method [3,4], variation iterative method [5,6] and so on. Homotopy analysis method (HAM), first proposed by Liao in his Ph.D dissertation[7], is an elegant method which has proved its effectiveness and efficiency in solving many types of nonlinear equations [8-15]. The HAM contains a certain auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution [16]. In 2005 Liao [17] has pointed out that the HPM is only a special case of the HAM (The case of h=−1). El-Tawil and Huseen [18] proposed a method namely q-homotopy analysis method (q-HAM) which is more general method of homotopy analysis method (HAM) , The q-HAM contains an auxiliary parameter n as well as h such that the cases of (q-HAM ; n=1) the standard homotopy analysis method (HAM) can be reached. The q-HAM has been successfully applied to solve many types of nonlinear problems [18-25].

In [26] the CD equation was solved with finite differences approximation, the Adomian decomposition method was used to solve CD equation and in the homotopy perturbation method was applied to find the solution of CD equation. In this work, we consider the following linear CD equation and the Oq-HAM is applied to solve it.

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + c\frac{\partial u}{\partial x} = \gamma \frac{\partial^2 u}{\partial x^2}, \mathbf{0} < \alpha \leq \mathbf{1},\ \mathrm{t} \geq 0, \tag{1}$$

subject to the initial condition

$$u(x,0) = f(x), \mathbf{0} \leq \mathbf{x} \leq \mathbf{l}, \tag{2}$$

where c and γ are arbitrary constants.

This paper is organized as follows: We begin by introducing some necessary definitions, mathematical preliminaries of the fractional calculus theory. In section 3, we introduce the basic definitions of Optimal q-homotopy analysis method. In section 4, example is solved to show the importance of the proposed method and in the last section the conclusion is stated.

2. PRELIMINARIES AND NOTATIONS

In this section, we give some basic definitions and properties of fractional calculus theory which are further used in this paper.

Definition 2.1. A real function $h(t), t>0$, is said to be in the space $C_\mu, \mu \in \mathrm{R}$ if there exists a real number $p>\mu,$ such that $h(t)=t^p h_1(t),$ where $h_1(t)\in C[0,\infty),$ and it is said to be in the space C_μ^n if and only if $h^{(n)}\in C_\mu, n\in \mathrm{N}.$

Definition 2.2. The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0,$ of a function $h \in C_\mu, \mu \geq -1,$ is defined as

$$J^\alpha h(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}h(\tau)d\tau \quad (\alpha>0),$$
$$J^0 h(t)=h(t),$$

$\Gamma(\alpha)$ is the well- known Gamma function. Some of the properties of the operator J^α , which we will need here, are as follows:

(1) $J^\alpha J^\beta h(t)=J^{\alpha+\beta}h(t),$

(2) $J^\alpha J^\beta h(t)=J^\beta J^\alpha h(t),$

(3) $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}t^{\alpha+\gamma}$, where $\beta \geq 0,$ and $\gamma \geq -1.$

Definition 2.3. The fractional derivative (D^α) of $h(t)$ in the Caputo's sense is defined as

$$D^\alpha h(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}h^{(n)}(\tau)d\tau \quad ,$$

for $n-1<\alpha\leq n, \quad n\in \mathrm{N}, t>0, \quad h\in C_{-1}^n.$

The following are two basic properties of Caputo's fractional Derivative [12]:

(1) Let $h\in C_{-1}^n, n\in \mathrm{N}.$ Then $D^\alpha h, 0\leq \alpha \leq n$ is well defined and $D^\alpha h \in C_{-1}.$

(2) Let $n-1<\alpha\leq n$, $n\in \mathrm{N}$ and $h\in C_\mu^n, \mu\geq -1.$ Then

$$(J^\alpha D^\alpha)h(t)=h(t)-\sum_{k=0}^{n-1}h^{(k)}(0^+)\frac{t^k}{k!}.$$

Definition 2.4: The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t)=J^{m-\alpha}D^m f(t),$$

For $m-1<\alpha\leq m, m\in N, t>0$ and $f\in C_{-1}^{n}$.

2. BASIC IDEA OF THE OPTIMAL q-HOMOTOPY ANALYSIS METHOD (Oq-HAM)

To describe the basic ideas of the optimal Oq-HAM for nonlinear partial differential equations. Let us consider the following nonlinear partial differential equation:

$$N[D_t^{\alpha}u(x,t)]-f(x,t)=0, \tag{3}$$

where N is linear and nonlinear operator for this problem, x and t denote the independent variables, $D_t^{\alpha}u(x,t)$ denotes the Caputo fractional derivative, $u(x,t)$ is an unknown function and f is a known function. We first construct the zero-order deformation equation as follows:-

$$(1-nq)L[\phi(x,t;q)-u_0(x,t)]=qhH(x,t)N[\mathrm{D}_t^{\alpha}\phi(x,t;q)-f(x,t)], \tag{4}$$

where $n>1, q\in[0,\frac{1}{n}]$ is the embedding parameter, $h\neq 0$ is an auxiliary parameter, $H(x,t)\neq 0$ is an auxiliary function, L is an auxiliary linear operator and $u_0(x,t)$ is an initial guess. Clearly, when $q=0$ and $q=\frac{1}{n}$, equation (4) becomes:

$$\phi(x,t;0)=u_0(x,t), \qquad \phi(x,t;\frac{1}{n})=u(x,t). \tag{5}$$

respectively. so, as q increases from 0 to $\frac{1}{n}$ the solution $\phi(x,t,q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. If $u_0(x,t)$, $L, h, H(x,t)$ are chosen appropriately, solution of equation (5) exists for $q\in[0,\frac{1}{n}]$.

Taylor series expression of $\phi(x,t,q)$ with respect to q in the form

$$\phi(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m, \tag{6}$$

where

$$\phi_m(x,t)=\frac{1}{m!}\frac{\partial^m\phi(x,t;q)}{\partial q^m}|_{q=0}. \tag{7}$$

We assume that the auxiliary linear operator, the initial guess, the auxiliary parameter h and the auxiliary function $H(x,t)$ is selected such that the series (7) is convergent when $q\to\frac{1}{n}$, then the approximate solution (6) takes the form:-

$$u(x,t) = \phi(x,t;\frac{1}{n}) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)(\frac{1}{n})^m. \tag{8}$$

Let us define the vector

$\vec{u_n}(t) = \{u_0(x,t), u_1(x,t), u_2(x,t), ..., u_n(x,t)\}$.

Differentiating (4) m times with respect to q, then setting $q = 0$ and dividing then by $m!$, we have the m^{th}-order deformation equation (Lioa [7-8]) as

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hH(x,t)\mathbf{R}_m(\vec{u}_{m-1}(x,t)), \tag{9}$$

with initial conditions

$$u_m^{(k)}(x,t) = 0, \qquad k = 0,1,2,3,...,m-1$$

where

$$\mathbf{R}_m(\vec{u}_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathbf{N}[\mathrm{D}_t^{\alpha}\phi(x,t;q) - f(x,t)]}{\partial q^{m-1}}\Big|_{q=0}, \tag{10}$$

and

$$\chi_m = \begin{cases} 0 & m \le 1, \\ n & m > 1. \end{cases} \tag{11}$$

It should be emphasized that $u_m(x,t)$ for $m \geq 1$ is governed by the linear equation (9) with linear boundary conditions that come from the original problem. Due to the existence of the factor $(\frac{1}{n})^m$, more chances for convergence may occur or even much faster convergence can be obtained better than the standard HAM. It should be noted that the cases of n=1 in equation (4), standard HAM can be reached.

The h-curves cannot tell us the best convergence-control parameter, which corresponds to the fastest convergent series. In 2007, Yabushita et al. [27] applied the HAM to solve two coupled nonlinear ODEs. They suggested the so-called optimization method to find out the two optimal convergence-control parameters by means of the minimum of the squared residual error of governing equations. In 2008, Akyildiz and Vajravelu [28] gained optimal convergence-control parameter by the minimum of squared residual of governing equation, and found that the corresponding homotopy-series solution converges very quickly.

S.J. Liao [29] and Mohamed S. Mohamed et al. [13, 14] they have discussed the optimization method to find out the optimal convergence control parameters by minimum of the square residual error integerated in the whole region having physical meaning. Their approach is based on the square residual error.

Let $\Delta(h)$ denote the square residual error of the governing equation (3) and express as:

$$\Delta(h) = \int_{\Omega} (N[u_n(t)])^2 \, d\Omega, \tag{12}$$

where

$$u_m(t) = u_0(t) + \sum_{k=1}^{m} u_k(t). \tag{13}$$

The optimal value of the auxiliary parameter h is given by solving the following nonlinear algebraic equation

$$\frac{d\Delta(h)}{dh} = 0. \tag{14}$$

3. NUMERICAL RESULTS

To demonstrate the effectiveness of the Oq-HAM algorithm discussed above, example of variation problems will be studied in this section. In this section, example is solved according to the mentioned algorithm in previous section. The results have been provided by Mathematica.

Consider the FCD equation:

$$\mathrm{D}_t^{\alpha} u - \mathbf{0.02}\mathrm{u}_{xx} + 0.1\mathrm{u}_x = 0 \quad 0 < \alpha \leq 1, \tag{15}$$

with the initial condition

$$\mathbf{u(x,0)} = \mathrm{e}^{1.1771243x},$$

with the exact solution at $\alpha = 1$

$$\mathbf{u(x,t)} = \mathrm{e}^{1.1771243x-0.09t}. \tag{16}$$

This problem solved by HAM [26]. For Oq- HAM solution we choose the linear operator

$$L[\phi(x,t;q)] = \frac{\partial^\alpha \phi(x,t;q)}{\partial t^\alpha},$$

with the property that

$L[c] = 0,$ where c is constant.

We define a nonlinear operator as

$$\mathbf{N}[\phi(x,t;q)] = \frac{\partial^\alpha \phi(x,t;q)}{\partial t^\alpha} - \mathbf{0.02}\frac{\partial \phi^2(x,t;q)}{\partial x^2} + \mathbf{0.1}\frac{\partial \phi(x,t;q)}{\partial x}. \tag{17}$$

We construct the zero order deformation equation

$(1-nq)L[\phi(x,t;q) - u_0(x,t)] = qhH(x,t)N[\mathrm{D}_\mathrm{t}^\alpha \phi(x,t;q)].$

For $q=0$ and $q=1$, we can write

$\phi(x,t;0) = u_0(x,t) = u(x,0),$
$\phi(x,t;1) = u(x,t).$

We can take $H(x,t) = 1,$ and the m^{th} -order deformation equation is

$$L(u_m(x,t) - \chi_m u_{m-1}(x,t)) = h\mathbf{R}_m(u_{\overrightarrow{m-1}}(x,t)), \tag{18}$$

with the initial conditions for $m \geq 1$

$$\mathbf{u_m}(\mathbf{x},\mathbf{0}) = 0, \tag{19}$$

where χ_m as defined by (11) and

$$\mathbf{R}_m(u_{\overrightarrow{m-1}}) = D_t^\alpha u_{m-1} - 0.02\frac{\partial^2}{\partial x^2}u_{m-1} + 0.1\frac{\partial}{\partial x}u_{m-1}. \tag{20}$$

Now the solution of the m^{th} - order deformation equations for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h\, j_t^\alpha(\mathbf{R}_m(u_{\overrightarrow{m-1}})) + c_1 \tag{21}$$

where the constant of integration c_1 is determined by the initial conditions (19). Then, the components of the solution using Oq- HAM are

$$u(x,0) = e^{1.1771243446770x},$$
$$u_1(x,t) = \frac{0.09\, e^{1.1771243446770x}}{\alpha\Gamma(\alpha)} h\, n t^{\alpha},$$
$$u_2(x,t) = \frac{0.09\, e^{1.1771243446770x}}{\alpha\Gamma(\alpha)} h\, n t^{\alpha} + h^2 \Big(\frac{0.09\, e^{1.1771243446770x}}{\alpha\Gamma(\alpha)} t^{\alpha} + \ldots\Big), \tag{22}$$
...

...

According to the optimal q-homotopy analysis, we can conclude that

$$u(x,t;n;h) \simeq U_m(x,t;n;h) = \sum_{i=0}^{M} u_i(x,t;n;h)\Big(\frac{1}{n}\Big)^i. \tag{23}$$

Equation (23) is an approximate solution to the problem (15) in terms of convergence parameter h and n. Then we have at $\alpha = 1$,

$$u_{app} = u_0(x,t) + \Big(\frac{1}{n}\Big)u_1(x,t) + \Big(\frac{1}{n}\Big)^2 u_2(x,t) + \Big(\frac{1}{n}\Big)^3 u_3(x,t) + \Big(\frac{1}{n}\Big)^4 u_4(x,t) + \Big(\frac{1}{n}\Big)^5 u_5(x,t) + \ldots$$
$$= e^{1.1771243446770x} + 0.09\, e^{1.1771243446770x} h\, t + h\, t(0.09h + 0.09n) + 0.00405 e^{1.1771243446770x} h^2\, t^2 + \ldots \tag{24}$$

As special case if n=1 and h=−1, then we obtain the same result in [26].

Equation (24) is an approximate solution to the problem (15) in terms of the convergence parameters h *and* n. To find the valid region of h, the h-curves given by the 5th order q-HAM approximation at different values of x, t, $\alpha = 1$, *and* n are drawn in figures (1−9). These figures show the interval of h at which the value of U_5 $(x,t;n)$ is constant at certain values of x, t *and* n. We choose the horizontal line parallel to $x-axis$ (h) as a valid region which provides us with a simple way to adjust and control the convergence region of the series solution. From these figures, the valid intersection region of h for the values of x, t *and* n in the curves becomes larger as n increase.

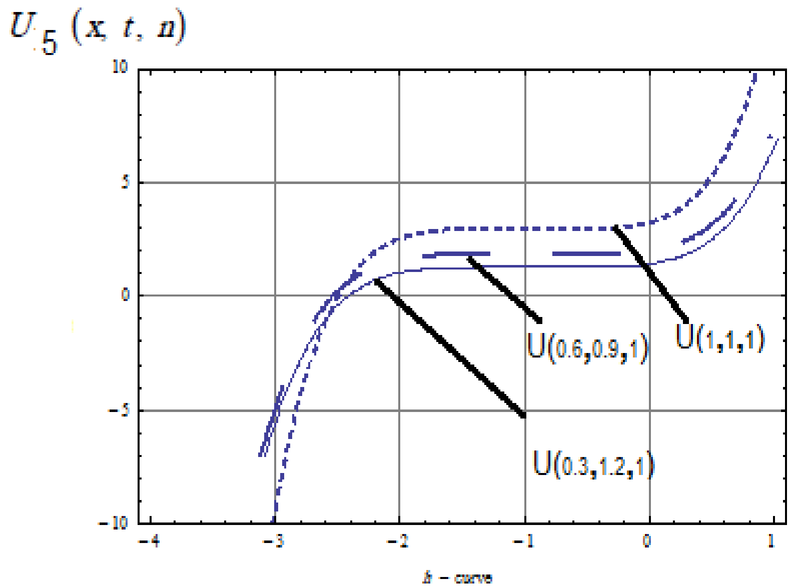


Figure (1) : *h* - curve for the HAM (q-HAM; *n* = 1) approximation solution U_5 (x, t; 1) of problem (15) at different values of x, t and h_{optmal} = -0.97.

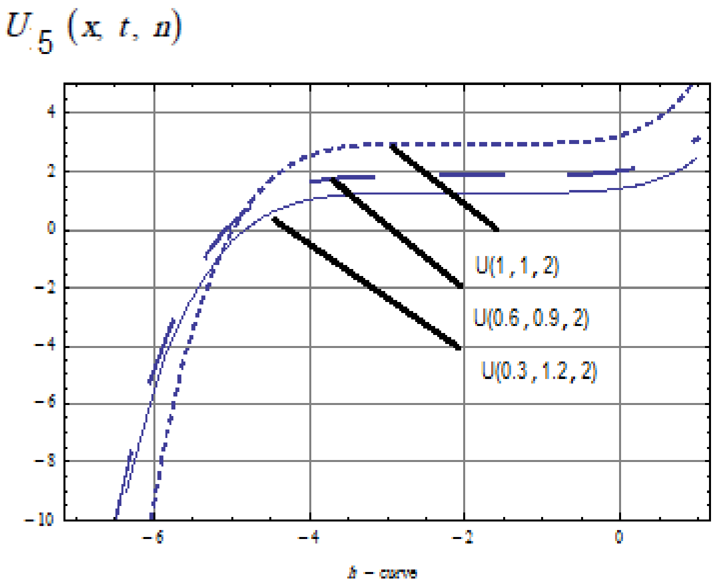


Figure (2) : ***h*** - curve for the HAM (q-HAM; n = 2) approximation solution U_5 (x, t; 2) of problem (15) at different values of x , t and h_{optmal} = -1.99.

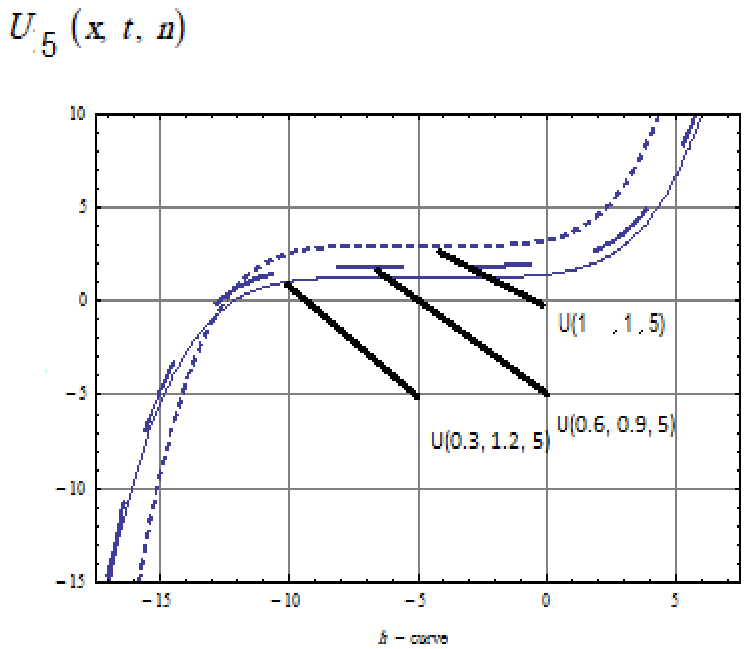


Figure (3) : $\boldsymbol{h}$ - curve for the HAM (q-HAM; $\boldsymbol{n} = 5$) approximation solution $\boldsymbol{U}_5$ ($\boldsymbol{x}$, $\boldsymbol{t}$; 5) of problem (15) at different values of $\boldsymbol{x}$, $\boldsymbol{t}$ *and* $\boldsymbol{h}_{\text{optmal}} = -4.5$.

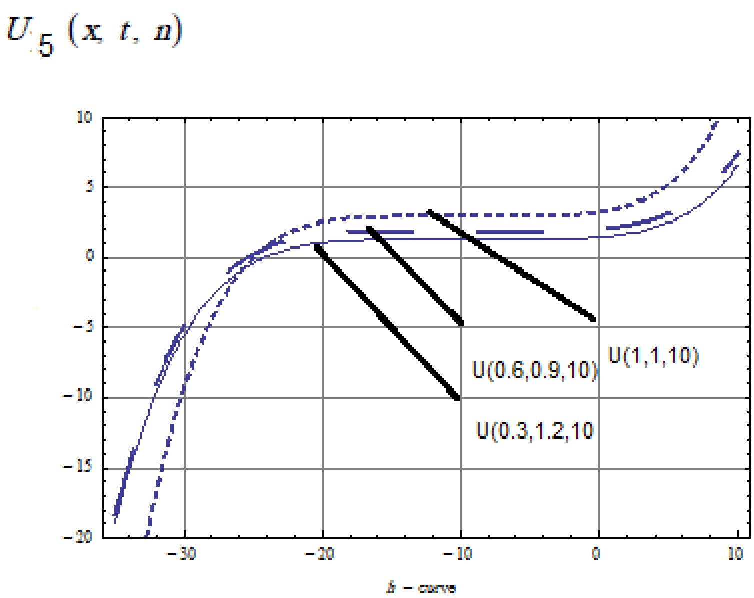


Figure (4) : $\boldsymbol{h}$ - curve for the HAM (q-HAM; $\boldsymbol{n} = 10$) approximation solution $\boldsymbol{U}_5$ ($\boldsymbol{x}$, $\boldsymbol{t}$; 10) of problem (15) at different values of $\boldsymbol{x}$, $\boldsymbol{t}$ and $\boldsymbol{h}_{\text{optmal}} = -11.45$.

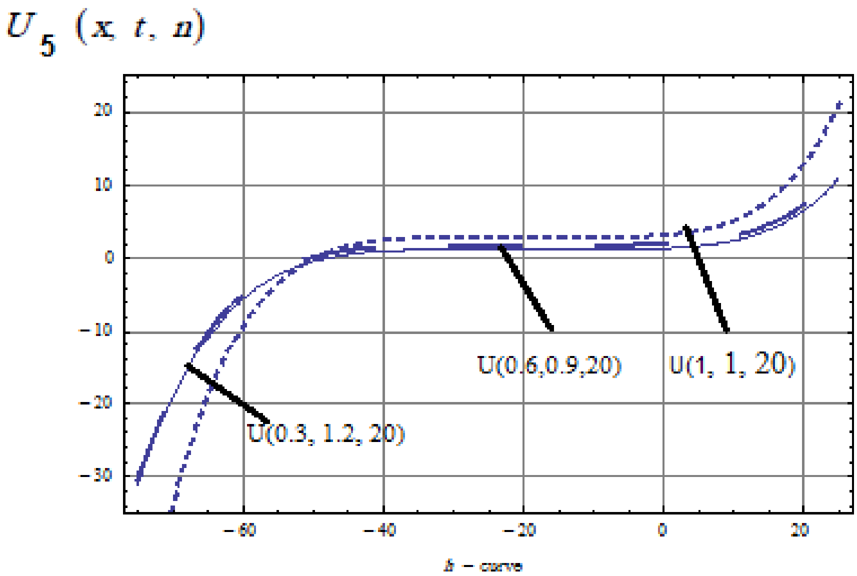


Figure (5) : $\boldsymbol{h}$ - curve for the HAM (q-HAM; $\boldsymbol{n}$ = 20) approximation solution $\boldsymbol{U}_5$ ($\boldsymbol{x}$, $\boldsymbol{t}$; 20) of problem (15) at different values of $\boldsymbol{x}$, $\boldsymbol{t}$ and $\boldsymbol{h}_{\text{optmal}} = -16.75$.

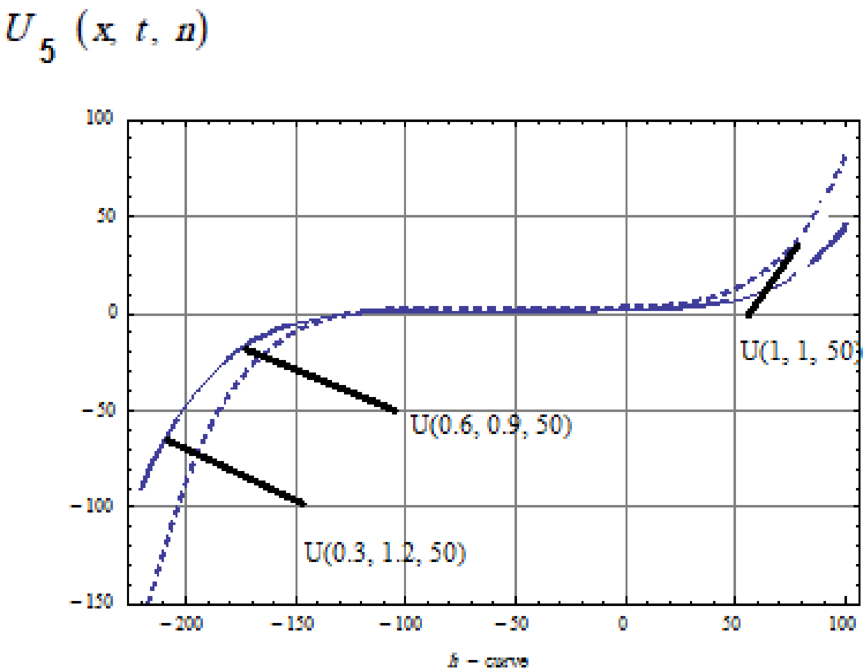


Figure (6) : $\boldsymbol{h}$ - curve for the HAM (q-HAM; $\boldsymbol{n}$ = 50) approximation solution $\boldsymbol{U}_5$ ($\boldsymbol{x}$, $\boldsymbol{t}$; 50) of problem (15) at different values of $\boldsymbol{x}$, $\boldsymbol{t}$ and $\boldsymbol{h}_{\text{optmal}} = -22.05$.

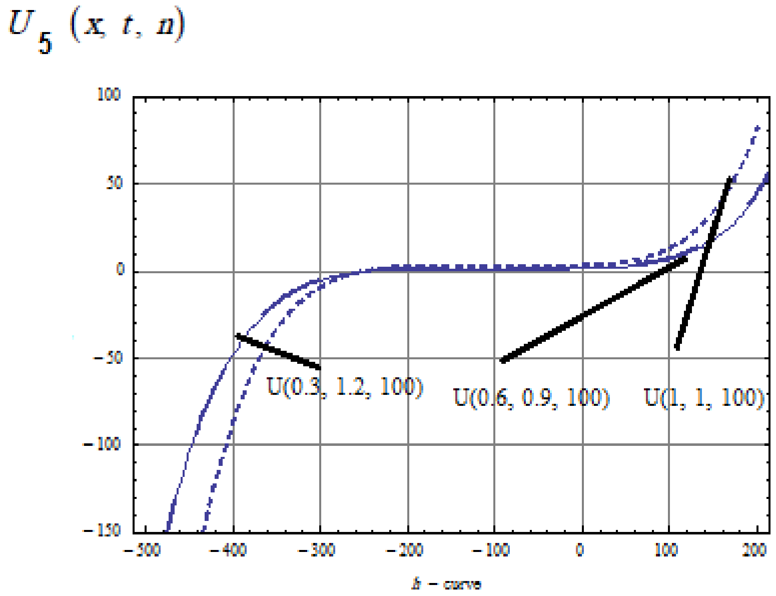


Figure (7) : ***h*** - curve for the HAM (q-HAM; ***n*** = 100) approximation solution $\boldsymbol{U}_5$ ($\boldsymbol{x}$, $\boldsymbol{t}$; 100) of problem (15) at different values of $\boldsymbol{x}$, $\boldsymbol{t}$ and $\boldsymbol{h}_{\text{optmal}}$ = -55.65.

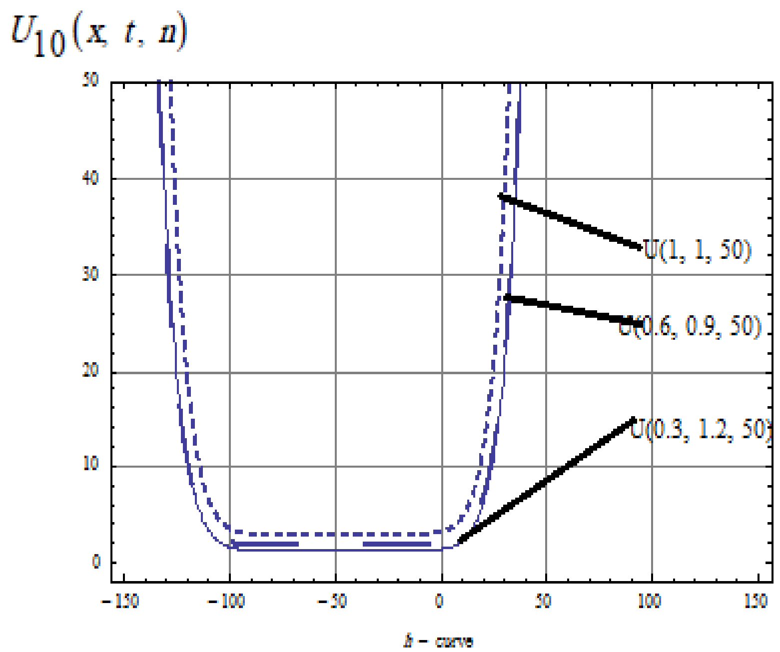


Figure (8) : ***h*** - curve for the HAM (q-HAM; ***n*** = 50) approximation solution $\boldsymbol{U}_{10}$ ($\boldsymbol{x}$, $\boldsymbol{t}$; 50) of problem (15) at different values of $\boldsymbol{x}$, $\boldsymbol{t}$ and $\boldsymbol{h}_{\text{optmal}}$ = -10.25.

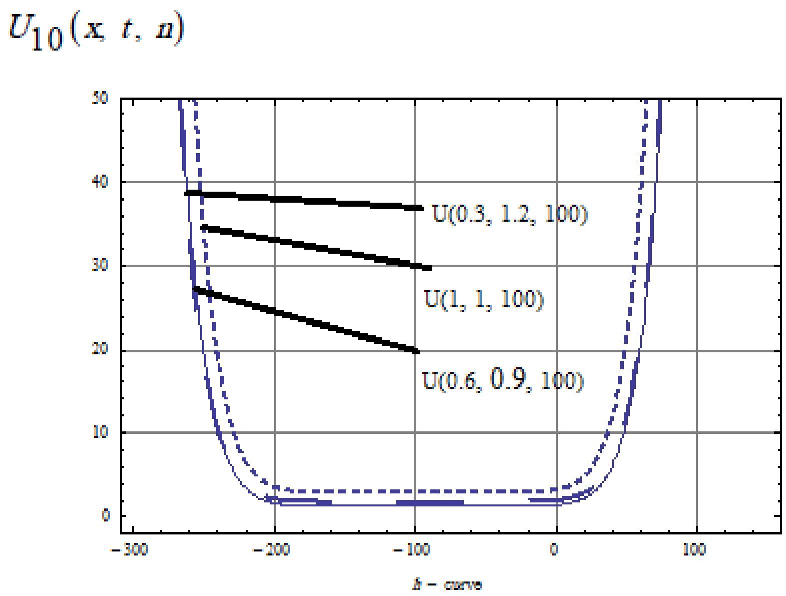
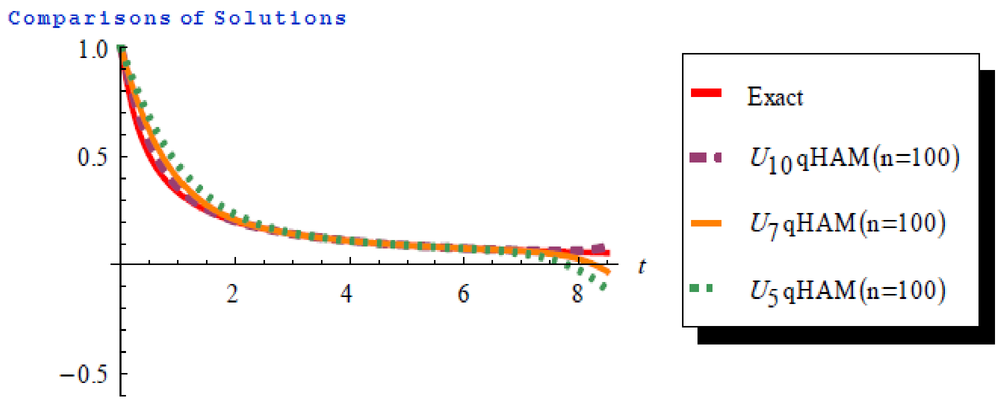
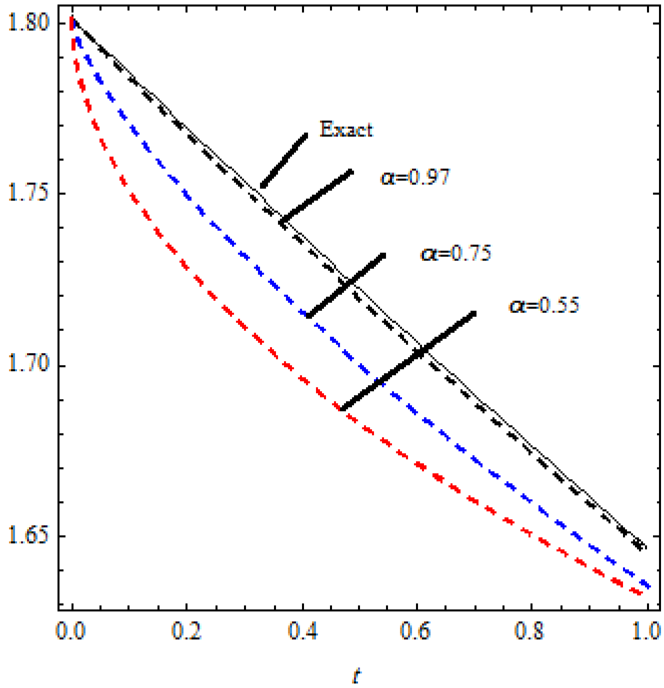


Figure (9) : *h* - curve for the HAM (q-HAM; n = 100) approximation solution U_{10} (x, t; 100) of problem (15) at different values of x, t and h_{optmal} = -15.43.



Figure(10): Comparison between $U5$, $U7$, $U10$ of (q-HAM ; n = **100** and α =1) and exact solution of (16) at x = **1** with h_{optmal} = -15.43 , $0 < t \leq 8$.

Figure 10 show the comparison between U_5,U_7 *and* U_{10} using different values of n with the exact solution (16). Therefore, based on these present results, we can say that q-HAM is more effective than HAM and HPM.



Figure(11): Comparison between $\boldsymbol{U5}$(α =0.97, 0.75 and 0.55) of (q-HAM ; $\boldsymbol{n} = \mathbf{1}$) and exact solution of (16) at $\boldsymbol{x} = 0.5$ with $\boldsymbol{h}_{\text{optmal}} = -1.03$, $0 < t \leq 1$.

4. CONCLUSIONS

An approximate solution of FCD equation was found by using the Oq-homotopy analysis method (q-HAM). The results show that the convergence region of series solutions obtained by Oq-HAM is increasing as q is decreased. The comparison of Oq-HAM with the HAM and HPM [26] was made. It was shown that the convergence of Oq-HAM is faster than the convergence of HAM and HPM. The results show that the method is powerful and efficient techniques in finding exact and approximate solutions for equations and also**,** this method uses simple computation with acceptable solution.

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