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Notions of Convergence and Limits in Sequence Spaces

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Abstract

In this article we generalise some notions in context of Kothe-Toeplitz dual of sequence spaces such as c-limit, p-cgt, p-limit etc. for η-dual of sequence spaces as introduced in the paper of Dr. P. Chandra and Dr. B.C. Tripathi entitled "on generalized Kothe-Toeplitz duals of some sequence spaces" published in Indian journal of Pure and Applied Mathematics by Indian National Sc. Academy, New Delhi 33 (8), pp. 1301-1306 in August 2002.

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1. INTRODUCTION

The notion of projective convergence and coordinate convergence of sequences is quite old. It is found in Cooke [3]. Throughout the article $\omega, c, c_0, l_1, l_p, l_\infty, \phi$ denote the spaces of all, convergent, null, absolutely summable, p-absolutely summable, bounded and finite sequence respectively.

2. DEFINITIONS AND PRELIMINARIES

By σ we denote the space of all eventually alternating sequences i.e. if $(x_k) \in \sigma,$ then there exists $k_0 \in N$ such that $x_k = -x_{k+1}$ for all $k > k_0$.

Definition — Let E be a non empty subset of w and $r \geq 1,$ then the η-dual of E is defined as

$$E^{\eta} = \{ (y_k) \in \omega : (x_k\, y_k) \in l_r \text{ for all } (x_k) \in E \}.$$

The sequence of points $x^{(n)}$ is said to be co-ordinate convergent (c-cgt) when $\lim_{n\to\infty} x_k^{(n)}$ exists for every K. If this limit is x_k, the point $x = (x_k)$ is called the coordinate limit of $x^{(n)}$ and we write $c - \lim x^{(n)} = x$.

We denote sequence spaces by $\alpha, \beta, \ldots\ldots$.

If $\phi \leq \beta \leq \alpha^{\eta}$ and if for sequence $x^{(n)}$ in α, the sequence $u'_n = \sum_{k=1}^{\infty} [x_k^{(n)} u_k]^r$ converges for every u in β, we say that $x^{(n)}$ is projective convergent (p-cgt) relative to β or $\alpha\beta$-cgt.

When $\beta = \alpha^{\eta}$, we say that $x^{(n)}$ is p-cgt in α or α-cgt.

A sequence x in α or outside α is called the projective limit (p-limit) of $x^{(n)}$ in α relative to β or $\alpha\beta$-lim $x^{(n)}$, when

(i) $(x_k u_k) \in l_r$ for every (u_k) in β i.e. $x \in \beta^{\eta}$

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} [x_k^{(n)} u_k]^r = \sum_{k=1}^{\infty} x_k^r u_k^r$ for every u in β

When $\beta = \alpha^{\eta}$, x is called the p-limit of $x^{(n)}$ in α or $\alpha - \lim x^{(n)}$.

As given in Cooke[3], it can be shown that if $\alpha\beta - \lim x^{(n)} = x$, then $c - \lim x^{(n)} = x$, provided r is not an even integer.

3. MAIN RESULTS

A sequence may be $\alpha\beta$ -cgt and c-limit may not satisfy (i) or (ii) or both.

Ex.1

For example, let $x_k^{(n)} = \left(\frac{n}{n+1}\right)^k$

Then $\sum_{k=1}^{\infty} |x_k^{(n)}|^r = \Sigma\left(\frac{n}{n+1}\right)^{kr} = \left(\frac{n}{n+1}\right)^r + \left(\frac{n}{n+1}\right)^{2r} + ...$

Which is a geometric series with common ratio $\left(\frac{n}{n+1}\right)^r < 1$ since $r > 0$.

So $x^{(n)} \in l_r$. We take r such that r is not an even integer.

Let u be a sequence in σ.

Then $\sum_{k=1}^{\infty} [u_k x_k^{(n)}]^r = \sum_{k=1}^{k_0} + \sum_{k_0+1}^{\infty}$

Now

$$\sum_{k_0+1}^{\infty} [u_k x_k^{(n)}]^r = u_{k_0+1}^r [x_{k_0+1}^{(n)}]^r + u_{k_0+1}^r (-1)^r [x_{k_0+2}^{(n)}]^r + u_{k_0+1}^r (-1)^{2r} [x_{k_0+3}^{(n)}]^r + ...$$

$$= u_{k_0+1}^r [\{x_{k_0+1}^{(n)}\}^r + (-1)^r \{x_{k_0+2}^{(n)}\}^r + (-1)^{2r} \{x_{k_0+3}^{(n)}\}^r + ...]$$

$$= u_{k_0+1}^r \left[\sum_{s=1}^{\infty} (-1)^{r(s-1)} \{x_{k_0+s}^{(n)}\}^r\right] = u_{k_0+1}^r \left[\sum_{s=1}^{\infty} (-1)^{r(s-1)} \left(\frac{n}{n+1}\right)^{(k_0+s)r}\right]$$

$$= u_{k_0+1}^r (-1)^{2n-r} \left|\sum_{s=1}^{\infty} (-1)^{rs} \left(\frac{n}{n+1}\right)^{rs}\right| \left(\frac{n}{n+1}\right)^{k_0 r} \text{ since } (-1)^r = (-1)^{2n-r}$$

$$= u_{k_0+1}^r (-1)^{2n-r} \left(\frac{n}{n+1}\right)^{k_0 r} \left\{(-1)^r \left(\frac{n}{n+1}\right)^r + (-1)^{2r} \left(\frac{n}{n+1}\right)^{2r} + ...\right\}$$

$$= u_{k_0+1}^r (-1)^{2n-r} \frac{\left(\frac{n}{n+1}\right)^{k_0 r} (-1)^r \left(\frac{n}{n+1}\right)^r}{1 - (-1)^r \left(\frac{n}{n+1}\right)^r}$$

$$= (-1)^{2n} \frac{u^r_{k_0+1}\left(\frac{n}{n+1}\right)^{r(k_0+1)}}{1-(-1)^r\left(\frac{n}{n+1}\right)^r} = \frac{u^r_{k_0+1}\left(\frac{n}{n+1}\right)^{r(k_0+1)}}{1-(-1)^r\left(\frac{n}{n+1}\right)^r}$$

Hence $\sum\limits_{k=1}^{\infty}[u_k x_k^{(n)}]^r = \sum\limits_{k=1}^{k_0} u_k^r\left(\frac{n}{n+1}\right)^{kr} + \frac{u^r_{k_0+1}\left(\frac{n}{n+1}\right)^{r(k_0+1)}}{1-(-1)^r\left(\frac{n}{n+1}\right)^r}$

As $n \to \infty$, $\left(\frac{n}{n+1}\right)^{kr} = \frac{1}{\left(1+\frac{1}{n}\right)^{kr}} \to 1$, for all k.

$\therefore \lim\limits_{n\to\infty}\sum\limits_{k=1}^{\infty}[u_k x_k^{(n)}]^r = \sum\limits_{1}^{k_0} u_k^r + \frac{u^r_{k_0+1}}{1-(-1)^r}$

Which exists for every *u* in β, provided $1-(-1)^r \neq 0$

$\Rightarrow$ $(-1)^r \neq 1$ i.e. r is not an even integer.

Thus if *r* is not an even integer, then $x^{(n)}$ is $l_r\sigma$-cgt.

$(\sigma \leq l_r^{\eta} = l_\infty)$

Now $\lim\limits_{n\to\infty} x_k^{(n)} = \lim\limits_{n}\left(\frac{n}{n+1}\right)^k = 1$

So $c-\lim x^{(n)} = x$ where $x_k = 1$ for every k.

In that case $\sum\limits_{k=1}^{\infty}|u_k x_k|^r = \sum\limits_{k=1}^{\infty}|u_k|^r$ is not convergent since u is in σ. So condition (i) is not satisfied by $c-\lim x^{(n)}$.

So *x* is not $l_r\sigma-\lim x^{(n)}$.

Ex.2

This example shows that condition (ii) may not be satisfied by $c-\lim x^{(n)}$.

Consider the sequence $x^{(n)} = e^{(n)}$ in l_r (Thus $\alpha = l_r$)

Take $\beta = c$.

Let $u \in \beta$ i.e.c.

Then $\sum_{k=1}^{\infty} [u_k x_k^{(n)}]^r = \sum_{k=1}^{\infty} [u_k e_k^{(n)}]^r = u_n^r$ since $e_n^{(n)} = 1$, $e_p^{(n)} = 0 (p \neq n)$

Since $u \in c$, $\lim_n u_n$ exists and $\{u_k\}$ is bounded.

Hence $\sum_{k=1}^{\infty} [u_k x_k^{(n)}]^r$ converges for every u in c.

Therefore $x^{(n)}$ is $l_r c$ -cgt $(c \leq l_r^{\eta} = l_\infty)$

Also $\lim_{n\to\infty} x_k^{(n)} = \lim_n e_k^{(n)} = 0 \Rightarrow x_k = 0, \forall_k$

Thus c-limit is zero.

Hence $\sum_{k=1}^{\infty} x_k^r u_k^r = 0$

But $\lim_{n\to\infty} \sum_{k=1}^{\infty} [u_k x_k^{(n)}]^r = \lim_{n\to\infty} u_n^r$

If we take $u_k = 1$ for every k, $u \in c$ but then above limit is not 0 as it is 1. Hence condition (ii) is not satisfied.

When $\beta = \phi$ $(\phi \leq \alpha^{\eta}$, for any α) we have the following simple result, in case r is not even.

Theorem 1

$\alpha\phi$ -convergence coincides with c-convergence and $\alpha\phi$ -limit is coextensive with the c-limits of c-cgt sequences in α , in case r is not an even integer.

Proof :

Let $x^{(n)}$ in α be c-cgt. Let $x_k^{(n)} \to x_k$ as $n \to \infty$.

Then $c - \lim x^{(n)} = x$.

Let $u \in \phi$ and $u_k = 0$ for $k > p$.

Then $\lim_{n} \sum_{k=1}^{\infty} \{x_k^{(n)}u_k\}^r = \lim_{n} \sum_{k=1}^{p} \{x_k^{(n)}u_k\}^r = \sum_{k=1}^{p} x_k^r u_k^r$

So limit exists for every u in ϕ.

Hence $x^{(n)}$ is $\alpha\phi$-convergent.

So every c-cgt sequence in α is $\alpha\phi$-cgt.

We know that $\alpha\phi$-convergence implies c-convergence and so the two convergences coincide.

Now $\sum_{k=1}^{\infty} |u_k^r x_k^r| = \sum_{k=1}^{p} |u_k^r x_k^r|$ which converges for all u (in ϕ)

$\lim_{n} \sum_{k=1}^{\infty} [x_k^{(n)}u_k]^r = \sum_{k=1}^{\infty} x_k^r u_k^r$ for every u in β.

Thus c-limit of c-cgt sequence in α is its $\alpha\phi$-limit.

Also $\alpha\phi$-limit is c-limit as r is not even. Hence the result is proved.

Definition

A sequence space α is said to be normal, if whenever x is in α and $|y_k| \le |x_k|$ for every *k*, then *y* is in α.

Ex.

σ and *c* are not normal while ϕ and c_0 are normal.

As given in Cooke (10.2.I) we can show that

(i) A necessary and sufficient condition for the $\alpha\beta$-convergence of $x^{(n)}$ in α is that to every u in β and to every $\in > 0,$ there corresponds a + ve number $N(\in, u)$ s.t. for every $p, q \ge N$.

$$\left| \sum_{k=1}^{\infty} u_k^r [\{x_k^{(p)}\}^r - \{x_k^{(q)}\}^r] \right| \le \in$$

(ii) When β is normal, the necessary and sufficient condition that $x^{(n)}$ in α should be $\alpha\beta$-cgt is that to every u in $\beta,$ and to every $\in > 0,$ there corresponds a + ve number $N(\in, u)$ s.t. for every $p, q \ge N$,

$$\sum_{k=1}^{\infty} | u_k^r [\{x_k^{(p)}\}^r - \{x_k^{(q)}\}^r]| \leq \in$$

We have seen that $x^{(n)}$ may be $\alpha\beta$-cgt and its coordinate limit may not satisfy the conditions necessary for $\alpha\beta$-limit. In the above examples σ and c are not normal. When β is normal we obtain the following existence theorem for $\alpha\beta$-limits.

Theorem 2 :

When β is normal, the c-limit of every $\alpha\beta$-cgt sequence is the $\alpha\beta$-limit of that sequence.

Proof : An analogy to Cooke (10.2.I), we have in case $x^{(n)}$ in α is $\alpha\beta$-cgt sequence is that given any u in β and any $\in > 0,$ there corresponds a + ve number $N(\in, u)$ s.t. for $p, q \geq N$,

$$\sum_{k=1}^{\infty} | u_k^r [\{x_k^{(p)}\}^r - \{x_k^{(q)}\}^r]| \leq \in$$

Thus for every m and for $p, q \geq N$,

$$\sum_{k=1}^{m} | u_k^r [\{x_k^{(p)}\}^r - \{x_k^{(q)}\}^r]| \leq \in$$

If q is fixed and p increased, since $\underset{p \to \infty}{lt} x_k^{(p)} = x_k$, due to c-convergence,

We have

$$\sum_{k=1}^{m} | u_k^r [x_k^r - \{x_k^{(q)}\}^r]| \leq \in \qquad \ldots(1)$$

For $q \geq N$ and every m. Let $m \to \infty,$ then for $q \geq N$,

$$\sum_{k=1}^{\infty} | u_k^r [x_k^r - \{x_k^{(q)}\}^r]| \leq \in \qquad \ldots (2)$$

From (1), we obtain

$$\in \geq \sum_{k=1}^{m} | u_k^r x_k^r - \{u_k x_k^{(q)}\}^r | \geq \sum_{k=1}^{m} | u_k x_k |^r - \sum_{k=1}^{m} | u_k x_k^{(q)} |^r$$

(as $|a-b| \geq |a|-|b|$)

Hence $\sum_{k=1}^{m} |u_k x_k|^r \leq \in + \sum_{k=1}^{m} |u_k x_k^{(q)}|^r$

But since $x^{(q)}$ is in α and u is in $\beta(\leq \alpha^{\eta})$, $\sum_{k=1}^{\infty} |u_k x_k^{(q)}|^r$

Converges, and thus $\sum_{k=1}^{\infty} |u_k x_k|^r$ converges, so that x is in β^{η} and condition (i) of projective limit is satisfied.

Also by (2),

$$|\sum_{k=1}^{\infty} u_k^r [\{x_k^r\} - \{x_k^{(q)}\}^r]| \leq \sum_{k=1}^{\infty} |u_k^r [x_k^r - \{x_k^{(q)}\}^r]| \leq \in \text{for}\, q \geq N.$$

Therefore $\lim_{q \to \infty} \sum_{k=1}^{\infty} u_k^r \{x_k^{(q)}\}^r = \sum_{k=1}^{\infty} u_k^r x_k^r$

And condition (ii) is satisfied.

Hence $\alpha\beta - \lim x^{(n)} = x.$

This proves the theorem

From the condition (i) of $\alpha\beta$-limit, it is in β^{η}. Next result enables us to determine the set of $\alpha\beta$-limits when $\alpha \geq \phi$.

If $x = \{x_k\}$ and $x^{(n)} = \{x_1, x_2, \ldots x_n, 0, 0, \ldots\}$

Then $x^{(n)}$ is called a section of x.

Theorem 3

If $\alpha \geq \phi$, every sequence in β^{η} is the $\alpha\beta$-limit of its sections.

Proof

Let $x = \{x_k\}$ and $x_k^{(n)} = x_k (1 \leq k \leq n)$

$$= 0 (k > n).$$

Then $x^{(n)}$ is a section of x and is in ϕ and hence in α .

If $x \in \beta^{\eta}$, then $\sum_{k=1}^{\infty} |u_k x_k|^r$ converges for every u in β, and

$$\lim_n \sum_{k=1}^{\infty} [u_k x_k^{(n)}]^r = \lim_n \sum_{k=1}^{n} u_k^r x_k^r = \sum_{k=1}^{\infty} u_k^r x_k^r$$

Hence $\alpha\beta - \lim x^{(n)} = x$ which proves the result.

We next observe that the c-limit of an $\alpha\beta$-sequence may or may not be in α. We have seen before that $\left\{\left(\frac{n}{n+1}\right)^k\right\}$ is $l_r\sigma$-cgt, but its c-limit $x(x_k = 1, \forall\, k)$ is not in l_r.

Also $\alpha\beta$-limits are not always in α. For example the sequences $x^{(n)}$ in ϕ where $x_k^{(n)} = 1$ for $1 \le k \le n$ and $x_k^{(n)} = 0$ for $k > n$ are ϕl_r-cgt $(l_r^{\eta} = l_\infty \le \omega = \phi^{\eta})$ because

$$\sum_{k=1}^{\infty} [x_k^{(n)} u_k]^r = \sum_{k=1}^{n} u_k^r \text{ converges for every } u \text{ in } l_r.$$

If ϕl_r-$\lim x^{(n)} = x$ such that $x_k = 1, \forall\, k.$ Then

(i) $\sum_{k=1}^{\infty} |u_k^r x_k^r| = \sum_{k=1}^{\infty} |u_k^r|$ which converges.

(ii) $\lim \sum_{k=1}^{\infty} \{x_k^{(n)} u_k\}^r = \lim_n \sum_{k=1}^{n} u_k^r = \sum_{k=1}^{\infty} u_k^r = \sum_{k=1}^{\infty} u_k^r x_k^r$

Hence $\phi l_r - \lim x^{(n)} = x$ such that $x_k = 1$ for every k. But x is not in ϕ.

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