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A study of $*$-*frames* in Hilbert Spaces

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Abstract

Relations between frames and *-frames are established. The operators associated to $*$-frames in Hilbert space and Hilbert and C*-modules are studied.

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1. INTRODUCTION

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaefer in 1952 to study some problems in nonharmonic Fourier series. K.Amir and BehroozKhosravi[1] are studied frames for tensor product of Hilbert C*-modules and Hilbert spaces. Alijani and Dehghan[2] introduced the *-frames , as a generalization of frames in Hilbert C*-modules. They studied the operators associated to given *-frame for Hilbert C*-modules over commutative unitary C*-algebras.

Peter G. Casazza [3] presented a tutorial on frame theory and he suggested the major directions of research in frame theory. The generalization of K-frames are introduced and some of their properties are obtained by Bahram Dastourian, Mohammad Janfada [4]. D. Han and D.R. Larson [5] have developed a number of basic aspects of operator-theoretic approach to frame theory in Hilbert space. M.Frank and D.R.Larson[6] are introduced a general module frame theory in C*-algebras and Hilbert C*-modules.

In 2012, K-frames were introduced by Gavrufa [7] to study the atomic systems with respect to a bounded linear operator K in Hilbert Spaces. Hilbert C*-modules are generalization of Hilbert spaces by allowing the inner product to take values in a C*-algebra rather than in the field of complex numbers.Problem about frames and *-frames for Hilbert C*-modules are more complicated than those for Hilbert Spaces. This makes the study of the *-frames for Hilbert C*-modules important and interesting.

In this paper relations between frames and *-frames are established. The operators associated to $*$-frames in Hilbert space and Hilbert and C*-modules are studied.

2. PRELIMINARIES

Definition 2.1. *AC* Algebra is a Banach algebra equipped with an involution* $a \to a^*$ satisfying the condition $\|aa^*\| = \|a\|^2$.

Definition 2.2. The standard Hilbert A-module $l_2(A)$ defined by $l_2(A) = \left\{ \{a_j\}_{j \in J} \subseteq \mathrm{A}, \sum_{j \in J} a_j a_j^* \textit{ converges in } \mathrm{A} \right\}$

Definition 2.3. A be a C*-algebra and H be a A-module. Suppose that the linear structures given on A and H are compatible, i.e. $\lambda(ax) = (\lambda a)x = a(\lambda x)$ for every $\lambda \in C, a \in A$ and $x \in H$.

If there exists a mapping $< \cdot, \cdot >: H \times H \to A$ with the properties

(i) $< x, x > \geq 0$ *for every* $x \in H$

(ii) $< x, x > = 0$ *if and only if* $x = 0$

(iii) $< x, y > = < y, x >^*$ *for every* $x, y \in H$

(iv) $< ax, y > = a < x, y >$ *for every* $a \in A$, *every* $x, y \in H$

(iv) $< x + y, z > = < x, z > + < y, z >$ *for every* $x, y, z \in H$

Then the pair $\{H, < \cdot, \cdot >\}$ is called a pre-Hilbert A-module. The map $< \cdot, \cdot >$ is said to be an A-valued inner product.If the pre-Hilbert module $\{H, < \cdot, \cdot >\}$ is complete with respect to the norm $\|x\| = \|< x, x >\|^{1/2}$ then it is called a Hilbert A-module.

The following Lemma will illustrate lower and upper bounds of operators corresponding to a given operator T with respect to A-valued inner products.

Lemma .2.4[2]: Let H and K be a two Hilbert A-modules and $T \in B(H,K)$. Then

(i) If T is injective and T has closed range, then the adjointable map T^*T is invertible and $\|(T^*T)^{-1}\|^{-1} \le T^*T \le \|T\|^2$

(ii) If T is surjective, then the adjointable map TT^* is invertible and $\|(TT^*)^{-1}\|^{-1} \le TT^* \le \|T\|^2$.

Let H and K be two Hilbert A -modules A mapping $T: H \to K$ is called adjointable if there exists a mapping $S: H \to H$ such that $< Tx, y > = < x, Sy >$ *for all* $x \in H,\ y \in K$. The unique mapping S is denoted by T^* and is called the adjoint of T.

The set of all adjointable operators from H to K is denoted by $Hom_{A}^{*}(H,K)$.

The following definitions from [3 ,5] are useful in our discussion.

Definition2.5. *A sequence* $\{x_j\}_{j\in J}$ *of vectors in a Hilbert space H is called a frame if there exist constants 0 < A ≤ B <∝ such that*

$$A\ \|x\|^2 \le \sum_{i=1}^{\infty} \left|\langle x, x_i\rangle\right|^2 \le B\ \|x\|^2 \ \ \textit{for all } x \in \text{H}.$$

The above inequality is called the frame inequality. The numbers A and B are called lower and upper frame bounds respectively.

Definition2.6. A synthesis operator $T: l_2 \to H$ is defined as $Te_j = x_j$ where $\{e_j\}$ is an orthonormal basis for l_2.

Definition2.7. Let $\{x_j\}_{j\in J}$ be a frame for H and $\{e_j\}$be an orthonormal basis for l_2. Then, the analysis operator $T^*: H \to l_2$ is the adjoint of synthesis operator T and is defined as $T^*x = \sum_{j\in J}\langle x, x_j\rangle$ for all x ∈H.

Definition2.8. Let $\{x_j\}_{j\in J}$ be a frame for the Hilbert space H. A frame operator $S = TT^*: H \to H$ is defined as $Sx = \sum_{j\in J}\langle x, x_j\rangle x_j$ for all x ∈H.

3. *-Frames

$*-frames$ are C^*-algebra version of frames. In this section we extend the concept of Hilbert space frames to $*-frames$ in Hilbert C^*-modules with A-valued bounds.

Definition3.1. Let H be a Hilbert A-module. A family $\{x_j\}_{j\in J}$ of elements of H is a frame for H, if there exist constants $0 < A \le B < \infty$, such that for all $x \in H$

$$A < x,x > \le \sum_{j\in J} \langle x, x_j \rangle \langle x_j, x \rangle \le B < x,x >$$

The numbers A and B are called lower and upper bounds of the frame, respectively.

$If\ A = B \Rightarrow \{x_j\} is\ a\ tight\ frame$. $If\ A = B = \lambda \Rightarrow \{x_j\} is\ a\ \lambda - tight\ frame$.
$If\ A = B = 1 \Rightarrow \{x_j\} is\ a\ normalized\ tight\ frame\ or\ a\ parseval\ frame$.

If $\sum_{j\in J} < x, x_j > < x_j, x >$ is convergent in norm, the frame is called standard.

Definition3.2. Let A be a C^*-algebra and J be a finite or countable index set. A sequence $\{x_j\}_{j\in J}$ of elements in a Hilbert A-module H is said to be a $*-frame$ for H is there exists strictly non-zero elements A and B of A such that

$$A < x,x > A^* \le \sum_{j\in J} < x, x_j > < x_j, x > \le B < x,x > B^*, \ x \in H$$

Where the sum in the middle of the inequality is convergent in norm.

Then elements A and B are called lower and upper $*-frame$ bounds respectively. We note that every frame for a Hilbert module is a $*-frame$. If A=C then the $*-frame$ $\{x_j\}_{j\in J}$ is indeed a frame a frame for the Hilbert space H.If A=B. Then the $*-frame$ $\{x_j\}_{j\in J}$ is tight $*-frame$.If A=B= λ then the $*-frame$ $\{x_j\}_{j\in J}$ is λ -tight $*-frame$.If A=B=1 then the $*-frame$ $\{x_j\}_{j\in J}$ is normalized $*-frame$ or Perseval $*-frame$

Note that in a Hilbert A-module, the set of all normalized $*-frames$ and set of all normalized frames are the same but this is not true in the tight case.

Definition3.3. Let $\{x_j\}_{j\in J}$ be a $*-frame$ for H. The pre $*-frame$ operator $T: H \to l_2(A)$ defined by $T(x) = \{< x, x_j >\}_{j\in J}$ is an injective and closed range ajointable A -module map.

Definition3.4. Let $\{x_j\}_{j\in J}$ be a $*-frame$ for H. The adjoint operator of T is $T^*: l_2(\mathrm{A}) \to H$ which is surjective and defined as $T^*(e_j) = x_j \ for \ j \in J$ where $\{e_j\}_{j\in J}$ is the standard basis for $l_2(\mathrm{A})$.

Definition 3.5. Let $\{x_j\}_{j\in J}$ be a $*-frame$ for H. The $*-frame$ operator $S: H \to H$ is defined as $Sx = T^*Tx = \sum_{j\in J} < x, x_j > x_j$. The $*-frame$ operator has some similar properties with frame operator in ordinary frames S is positive and invertible.

Theorem 3.6. Let $\{x_j\}_{j\in J}$ be a $*-frame$ for H with $*-frame$ operator S. Then $\|A^{-1}\|^{-2} \le \|S\| \le \|B\|^2$.

Proof: Given that $\{x_j\}_{j\in J}$ is a $*-frame$ for H by definition

$$A < x, x > A^* \le \sum_{j\in J} < x, x_j > < x_j, x > \le B < x, x > B^*, \ \forall x \in H$$

$$\Rightarrow A < x, x > A^* \le < Sx, x > \le B < x, x > B^*, \forall x \in H$$

$$\Rightarrow A < x, x > A^* \le Sx, x > and < Sx, x > \le B < x, x > B^*$$

$$\Rightarrow < x, x > \le A^{-1} < Sx, x > A^{*-1} \ and < Sx, x > \le B < x, x > B^*$$

$$\|< x, x >\| \le \|A^{-1}\| \ \|< Sx, x >\| \ \|A^{*-1}\| and \| < Sx, x >\| \le \|B\| \|< x, x >\| \|B^*\|$$

$$\Rightarrow \|A^{-1}\|^{-2} \|< x, x >\| \le \|< Sx, x >\| \ and \ \|< Sx, x >\| \le \|B\|^2 \ \|< x, x >\|$$

$$\Rightarrow \|A^{-1}\|^{-2} \|< x, x >\| \le \|< Sx, x >\| \le \|B\|^2 \ \|< x, x >\|$$

By taking Sup over all $x \in H$, *with* $\|x\| \le 1$ *we get*

$$\|A^{-1}\|^{-2} \le \|S\| \le \|B\|^2$$

Theorem 3.7.[2] Let $\{x_j\}_{j\in J}$ be a $*-frame$ for H with pre $*-frame$ operator T. Then $\{x_j\}_{j\in J}$ is a frame for H.

Proposition3.8. [2] Let A be a C*-module over itself every $*-frame$ $\{x_j\}_{j\in J}$ is a tight *-frame for A.

Proof. Suppose that $\{x_j\}_{j\in J}$ is a $*-frame$ for A with $*-frame$ operator S.

Consider $I_A = SS^{-1}I_A = \sum_{j\in J} < S^{-1}I_A, x_j > x_j = S^{-1}I_A \sum_{j\in J} |x_j|^2$

The above equality shows that $\sum_{j\in J} |x_j|^2$ is an invertible element in A and

$\sum_{j\in J} |x_j|^2$ is a

strictly positive element of A.

So $\sum_{j\in J} < x, x_j >< x_j, x > = \sum_{j\in J} |x_j|^2 < x, x >, \forall x \in A$

Then $\{x_j\}_{j\in J}$ is tight $*-frame$ for A.

4. $*-K-frame$

Definition4.1. A sequence $\{x_j\}_{j\in J}$ in Hilbert space H is said to be a K-frame for H if there exists positive real numbers λ, μ such that $\lambda \|K^* x\|^2 \le \sum_{j\in J} |< x, x_j >|^2 \le \mu \|x\|^2, (x \in H)$. Frames are a special case of K-frames when K is the identity operator.

Throughout this section H is a finitely or countably generated Hilbert C-modulues over a unital*

C-algebra* A *and* $K \in Hom_{\mathcal{A}}^*(H, K)$

Definition4.2. A sequence $\{x_j\}_{j\in J} \subseteq H$ is called a $*-frame$ for the operator K $(*-K-frame)$, if there exists strictly non-zero $A, B \in$ A such that $A < K^* x, K^* x > A^* \le \sum_{j\in J} < x, x_j >< x_j, x > \le B < x, x > B^*$, where the sum in the middle of the inequality is convergent in norm.

The element A and B are called the lower and upper $*-K-frame$ bounds respectively. The operator $T: l^2(\mathrm{A}) \to H$ defined by $T(\{a_j\}_{j\in J}) = \sum_{j\in J} a_j x_j$ is called the synthesis operator. The operator $T^*: H \leftarrow l^2(\mathrm{A})$ defined by $T^*(x) = \{< x, x_j >\}_{j\in J}$ is called the analysis operator. The operator $S: H \to H$ defined by $Sx = TT^*(x) = \sum_{j\in J} < x, x_j > x_j$ is called the $*-K-frame$ operator of $\{x_j\}_{j\in J}$ .Notethat if K=I, S invertible and $\{S^{-1}x_j\}_{j\in J}$ is a $*-frame$.

The inequality $< K^* x, K^* x > \leq \|K\|^2 < x, x >, \ \forall x \in H$ holds. Note that for any $A, B \in \mathrm{A}$ the inequality $A \leq B \Rightarrow CAC^* \leq CBC^*$ for any $C \in \mathrm{A}$.

Lemma 4.3 [4]. If $\{x_j\}_{j\in J}$ is a $*-K-frame$ with $*-frame$ bounds A and B then.

$$\|AK^* x\|^2 \leq \sum_{j\in J} \|< x, x_j > < x_j, x >\| \leq \|Bx\|^2, x \in H.$$

Lemma 4.4.[4] Let $\{x_j\}_{j\in J}$ be a frame for Hilbert A -module H over a unital C*-algebra A with frame bounds A, B respectively if and only if

$$A\|x\|^2 \leq \left\| \sum_{j\in J} < x, x_j > < x_j, x > \right\| \leq B\|x\|^2, x \in H.$$

Lemma4.5. Let $\{x_j\}_{j\in J}$ be a $*-frame$ with bounds A and B then it is a $*-K-frame$.

Proof: Suppose $\{x_j\}_{j\in J}$ is a $*-frame$ for H by definition, we have

$$A < x, x > A^* \leq \sum_{j\in J} < x, x_j > < x_j, x > \leq B < x, x > B^*, \forall x \in H$$

$$\Rightarrow \left(A\|K\|^{-1}\right) < K^* x, K^* x > \left(A\|K\|^{-1}\right)^* \leq A < x, x > A^* \leq \sum_{j\in J} < x, x_j > < x_j, x > \leq B < x, x > B^* \ \forall x \in H$$

$\Rightarrow \{x_j\}_{j\in J}$ is a $*-K-frame$ with frame bounds $A\|K\|^{-1}$ and B.

i.e. every $*-frame$ is a $*-K-frame$

Theorem4.6. Let $K, L \in Hom_{\mathcal{A}}^{*}(H)$ and $\{x_j\}_{j\in J}$ be a $*-K-frame$ with the $*-K-frame$ bounds A,B then

(i) *If* $V: H \to H$ *is a co-isometry such that KV=VK then* $\{Vx_j\}_{j\in J}$ is a $*-K-frame$ with the same $*-K-frame$ bounds.

(ii) $\{Lx_j\}_{j\in J}$ is a $*-LK-frame$ with the $*-frame$ bounds A and $B\|L\|$ respectively.

Proof: *Suppose* $\{x_j\}_{j\in J}$ is a $*-K-frame$ by definition we have

$$A < K^*x, K^*x > A^* \leq \sum_{j\in J} < x, x_j >< x_j, x > \leq B < x, x > B^*, \forall\ x \in H \text{ ----(1)}$$

(i) By using equation(1) we get

$$\sum_{j\in J} < x, Vx_j >< Vx_j, x > = \sum_{j\in J} < V^*x, x_j >< x_j, V^*x >$$
$$\leq B < V^*x, V^*x > B^* = B < x, x > B^*, \forall x \in H$$

V is Co-isometry so for any $x \in H$,

$$\sum_{j\in J} < x, Vx_j >< Vx_j, x > \geq A < K^*V^*x, K^*V^*x > A^*$$
$$= A < V^*K^*x, V^*K^*x > A^*$$
$$= A < K^*x, K^*x > A^*$$

Hence we have

$$A < K^*x, K^*x > A^* \leq \sum_{j\in J} < x, Vx_j >< Vx_j, x > \leq B < x, x > B^*, \forall\ x \in H$$

$\Rightarrow \{Vx_j\}_{j\in J}$ is a $*-K-frame$

(ii) For any $x \in H$,by using equation (1), we have

$$A < (LK)^*x, (LK)^*x > A^* = A < K^*L^*x, K^*L^*x > A^*$$
$$\leq \sum_{j\in J} < L^*x, x_j >< x_j, L^*x >$$
$$\leq B < L^*x, L^*x > B^* = (B\|L\|) < x, x > (B\|L\|)^*$$

$$A < (LK)^*x, (LK)^*x > A^* \leq \sum_{j\in J} < x, Lx_j >< Lxj, x > \leq (B\|L\|) < x, x > (B\|L\|)^*\ \forall x \in H$$

$\{Lx_j\}_{j\in J}$ is a $*-LK-frame$ with the $*-frame$ bounds A and $B\|L\|$.

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