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A New Approach of the Generalized $\left(\frac{G'}{G}\right)$ - Expansion Method to Construct Exact Solutions for the Generalized Fractional Modified Benjamin-Bona-Mahony (BBM) Equation With Variable Coefficients

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Abstract

The main objective of this paper is to introduce for the first time a newapproach for the generalized $\left(\frac{G'}{G}\right)$−expansion method to construct an explicit exact traveling wave solution involving parameters of the following fractional generalized modified Benjamin-Bona-Mahony equation (BBM)

$$D_t^{\alpha}u + D_x^{\beta}u + a(t)u^2 D_x^{\beta}u + D_x^{3\beta}u = 0, 0 < \alpha, \beta \le 1$$

As a result, new traveling wave solutions including hyperbolic function, trigonometric function and rational function are obtained. Our solutions can be viewed as a generalization to the results which found in some recent published papers. Our solutions can be written in the form of infinite series, which make our solutions are advanced more than the other solutions which found in some recent published papers.

Keywords: Bona-Mahony equation (BBM), Generalized $\left(\frac{G'}{G}\right)$- expansion method, Traveling wave solutions.

1. INTRODUCTION

Phenomena in physics and other fields are often described by nonlinear evolution equations (NLEEs).When we want to understand the physical mechanism of phenomena in nature, described by nonlinear evolution equations, exact solutions for the nonlinear evolution equations have to be explored. For example, the wave phenomena observed in fluid dynamics [1, 2], plasma and elastic media [3, 4] and optical fibers [5, 6], etc. In the past several decades, many effective methods for obtaining exact solutions of NLEEs have been proposed, such as Hirota's bilinear method [7], Backlund transformation [8], Painlevé expansion [9], sine-cosine method [10], homogeneous balance method [11], homotopy perturbation method [12-14], variational iteration method [15-18], asymptotic methods [19], non-perturbative methods [20], Adomian decomposition method [21], tanh-function method [22-26], algebraic method [27-30], Jacobi elliptic function expansion method [31-33], F-expansion method [34-36] and auxiliary equation method [37-40]. Recently, Wang et al. [41] introduced a new direct method called the $\left(\frac{G'}{G}\right)$-expansion method to look for travelling wave solutions of NLEEs. Consider the fractional generalized mKdV and KdV partial differential equation

$$D_t^{\alpha}u + D_x^{\beta}u + a\left(t\right)u^2 D_x^{\beta}u + D_x^{3\beta}u = 0, 0 < \alpha, \beta \le 1 \tag{1}$$

where $a\left(t\right)$ are functions of t.When $a(t)$ is constant has been widely used in many physical fields such as plasma physics, fluid physics,solid-state physics and quantum field theory. In this paper we try to solve the above equation using a new approach of the generalized $\left(\frac{G'}{G}\right)$-expansion method when $a\left(t\right)$ is afunctions of t.The $\left(\frac{G'}{G}\right)$-expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$, and that $G = G\left(\xi\right)$ satisfies a second order linear ordinary differential equation (LODE):

$$GG'' - \beta GG' - (\alpha + 1)(G')^2 - \mu G^2 = 0, \tag{2}$$

where prime denotesderivative with respect to ξ. The degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivative and nonlinear terms appearing in the given NLEE. The coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the method.By using the $\left(\frac{G'}{G}\right)$-expansion method, Wang et al. [41] successfully obtained more travelling wave solutions of four NLEEs. Very recently, Zhang et al. [42] proposed a generalized $\left(\frac{G'}{G}\right)$-expansion method to improve the work made in [41].The main objective of this paper is to introduce for the first time anew approach for the generalized $\left(\frac{G'}{G}\right)$−expansion method to construct an explicit exact traveling wave solution involving parameters of the following fractional generalized modified Benjamin-Bona-Mahony equation (BBM (1).As a result, new traveling wave solutions including hyperbolic function, trigonometric function and rational function are obtained. Our solutions can be viewed as a generalization to the results which found in some recent published papers. Our solutions can be written in the form of infinite series, which make our solutions are advanced more than the other solutions which found in some recent published papers.

The paper is organized as follows. In Section 2, we describe briefly the generalized $\left(\frac{G'}{G}\right)$-expansion method, where $G = G(\xi)$ satisfies the second order ordinary differential equation (2). In section 3, we give some basic definitions and properties of the fractional calculus theory which will be used further in this work. In section 4, we give the constructions of the fractal index method [43]. In Section 5, we apply this method to the fractional generalized modified Benjamin-Bona-Mahony equation (BBM). In section 6, some conclusions are given.

2. DESCRIPTION OF THE GENERALIZED $\left(\frac{G'}{G}\right)$-EXPANSION METHOD

Suppose that we have the following nonlinear partial differential equation

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0 \tag{3}$$

we suppose its solution can be expressed by a polynomial $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi)=\sum_{j=0}^{N} d_j\left(d+\frac{G'}{G}\right)^{j}+\sum_{j=1}^{N} e_j\left(d+\frac{G'}{G}\right)^{-j} \tag{4}$$

where either d_N or e_N may be zero, but both d_N an e_N cannot be zero at a time , $d_j\ (j=0,1,2,...,N)$ and $e_j\ (j=0,1,2,...,N)$ are arbitrary constants to be determined later and $G=G(\xi)$ satisfies nonlinear auxiliary ordinary differential equation (ODE):(2) ,where prime denotes derivative with respect to ξ,α,β and μ are all parameters.To determine $u(\zeta)$ explicitly we take the following four steps.

Step 1. Determine the integer N by balancing the highest order nonlinear term (s) and the highest order partial derivative of u in Eq. (3).

Step 2. Substitute Eq. (4) along with Eq. (2) into Eq. (3) and collect all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left hand side of Eq. (3) is converted into a polynomial in $\left(\frac{G'}{G}\right)$.

Step 3. Solve the system of all equations obtained in Step 2 by use of Maple.

Step 4. Use the results obtained in above steps to derive a series of fundamental solutions of Eq. (2) depending on $\left(\frac{G'}{G}\right)$, since the solutions of this equation have been well known for us, then we can obtain exact solutions of Eq. (3).

Step 5. Using the general solution of Eq. (4), we have the following solutions:

Family1. Hyperbolic function solution: When $\beta\neq 0, \Psi=-\alpha$ and $\Omega=\beta^2-4\alpha\mu>0$,

$$\left(\frac{G'}{G}\right)=\frac{\beta}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\,\frac{C_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+C_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+C_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)} \tag{5}$$

Family 2. Trigonometric function solution: When $\beta\neq 0, \Psi=-\alpha$ and $\Omega=\beta^2-4\alpha\mu<0$,

$$\left(\frac{G'}{G}\right)=\frac{\beta}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-C_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+C_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{C_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+C_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)} \tag{6}$$

Family 3. Rational form solution: When $\beta\neq 0, \Psi=-\alpha$ and $\Omega=\beta^2-4\alpha\mu=0$,

$$\left(\frac{G'}{G}\right)=\frac{\beta}{2\Psi}+\frac{C_2}{C_1+C_2\xi} \tag{7}$$

Family 4. Hyperbolic function solution: When $\beta=0, \Psi=-\alpha$ and $\Delta=\Psi\mu>0$,

$$\left(\frac{G'}{G}\right)=\frac{\sqrt{\Delta}}{\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{C_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+C_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+C_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)} \tag{8}$$

Family 5. Trigonometric function solution: When $\beta=0, \Psi=-\alpha$ and $\Delta=\Psi\mu<0$,

$$\left(\frac{G'}{G}\right)=\frac{\sqrt{-\Delta}}{\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{-C_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+C_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{C_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+C_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)} \tag{9}$$

3. PRELIMINARIES AND NOTATION

In this section, we give some basic definitions and properties of the fractional calculus theory which will be used further in this work. For more details see [1]. For the finite derivative in $[a,b]$, we define the following fractional integral and derivatives.

Definition 3.1 *A real function $f(x), x>0$, is said to be in the space $C_\mu, \mu\in\mathbb{R}$, if there exists a real number $(p>\mu)$ such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C(0,\infty)$, and it is said to be in the space C_μ^m if $f^m\in C_\mu, m\in N$.*

Definition 3.2 *The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$ of a function $f\in C_\mu$, $\mu\geq -1$, is defined as*

$$J^{\alpha}(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt,\alpha>0,x>0,J^0(x)=f(x)$$

Properties of the operator J^{α} can be found in [1]; we mention only the following:

For $f\in C_{\mu},\ \mu\geq -1,\ \alpha,\ \beta\geq 0$ and $\gamma>-1$:

(1) $J^{\alpha}J^{\beta}f(x)=J^{\alpha+\beta}f(x)$

(2) $J^{\alpha}J^{\beta}f(x)=J^{\beta}J^{\alpha}f(x)$

(3) $J^{\alpha}x^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}$

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^{α} proposed by Caputo in his work on the theory of viscoelasticity [1].

Definition 3.3 *For* m *to be the smallest integer that exceeds* α, *the Caputo time fractional derivative operator of order* $\alpha>0$ *is defined as*

$$D_t^{\alpha}f(x)=\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}f^{m}(t)dt, & \text{for } m-1<\alpha\leq m, m\in N\\ \frac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha=m\end{cases}$$

4. FRACTAL INDEX METHOD

To understanding the fractional complex transform consider a plane with fractal structure shown in Fig. 1. The shortest path between two points is not a line and we have

$$ds_E=kds^{\alpha}, \tag{10}$$

where ds_E is the actual distance between two points *A* and *B*(the green curve in Fig.1), ds is the line distance between two points (the red line in Fig.1), α is the fractal dimension and *k* is a constant.

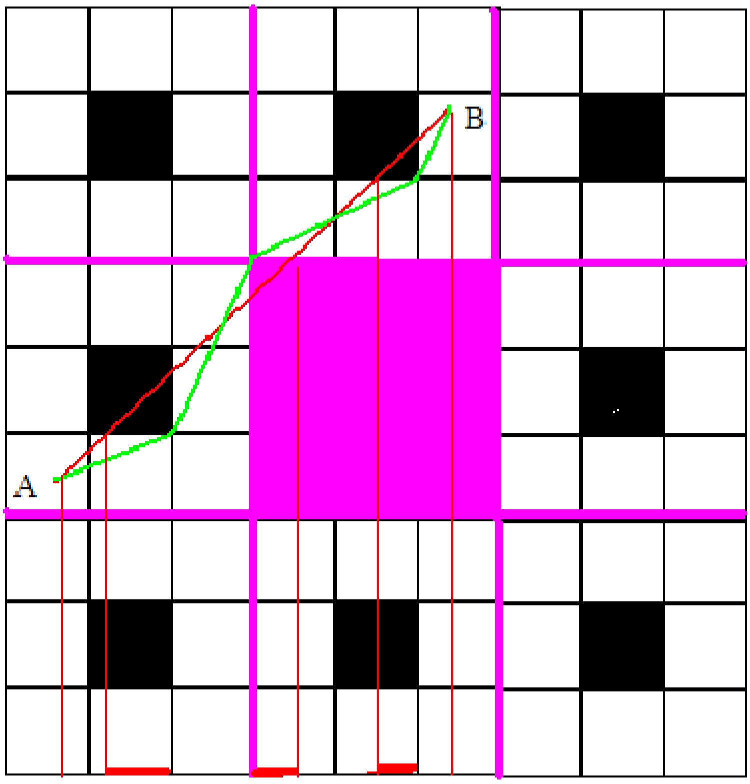

Fig.1. The distance between two points in a discontinuous space.

Projection the ds_E (the green curve) into horizontal direction yields Canton-like sets, andits length can be expressed as

$$\Delta_x AB = k_x x^{\alpha_x} \tag{11}$$

where α_x are the fractal dimensions of the Canton-like sets in the horizontal direction, k_x is a constant. Eq. (10) means the following transform $s_E = k s^{\alpha}$, this idea leads to the fractional complex transform,the fractal curve "*AB*" in Fig. 1 is projected to Cantor-like sets in horizontal direction. From Fig. 1, we have

$$\Delta_x AB = \cos\theta ds_E \tag{12}$$

or

$$\Delta_x AB = \frac{dx}{ds} ds_E \tag{13}$$

where θ is the slope angle of straight line *AB*. From the relations Eqs. (11) and (13), we have $k_x dx^{\alpha_x} = k \dfrac{dx}{ds} ds^{\alpha}$ or $dx^{\alpha_x} = \dfrac{k}{k_x}\dfrac{dx}{ds} ds^{\alpha} = \sigma \dfrac{dx}{ds} ds^{\alpha}$

where $\sigma = \dfrac{k}{k_x}$ and so called the fractal index, therefore, we have the following chain rule for fractional calculus $\dfrac{\partial^{\alpha} u}{\partial t^{\alpha}} = \sigma \dfrac{\partial u}{\partial s}\dfrac{\partial^{\alpha} s}{\partial t^{\alpha}}$.

5. THE GENERALIZED FRACTIONAL BONA-MAHONY EQUATION (BBM)PARTIAL DIFFERENTIONAL EQUATIONS

In this section, we apply the new approach of the generalized $\left(\dfrac{G'}{G}\right)$- expansion method to solve the generalized fractional Bona-Mahony equation (BBM) partial differentional equation, construct the traveling wave solutions for it as follows:

Let us first consider the following fractional generalized Bona-Mahony equation (BBM) partial differentional equation

$$D_t^{\alpha} u + D_x^{\beta} u + a(t) u^2 D_x^{\beta} u + D_x^{3\beta} u = 0, 0 < \alpha, \beta \le 1 \tag{14}$$

where $a(t)$ is a functions of t. There is no any method gave the exact solution of the above equation before. In order to look for the traveling wave solution of Eq. (14) we suppose that

$$u(x,t) = u(\xi), \xi(t) = \frac{x^{\alpha}}{\Gamma(1+\alpha)} - \omega t \tag{15}$$

By using the the chain rule $D_t^{\alpha} u = \sigma_t' \dfrac{du}{d\xi} D_t^{\alpha}\xi$ and $D_x^{\alpha} u = \sigma_x' \dfrac{du}{d\xi} D_t^{\alpha}\xi$, where σ_t' and σ_x' are called the fractal indexes (See section 3) for details see [16], without loss of generality we can take $\sigma_x' = \sigma_t' = l$, where l is a constant by using the definition of Capatu derivative and the above modefied chain rule, equation (14) after integrability convert to the ordinary differential equation

$$(1-\omega) u(\xi) + \frac{a(t)}{3} u^2 + k^2 \frac{d^2 u}{d\xi^2} = 0. \tag{16}$$

Suppose that the solution of Eq.(14) can be expressed by a polynomial in $\left(\dfrac{G'}{G}\right)$ as follows

$$u(\xi) = \sum_{k=0}^{N} d_j \left(p + \frac{G'}{G}\right)^k + \sum_{k=1}^{N} e_j \left(p + \frac{G'}{G}\right)^{-k} \tag{17}$$

considering the homogeneous balance between $u_{\zeta\zeta\zeta}$ and $u^2 u_\xi$ in Eq. (14) we required that $N + 3 = 2N + N + 1$, then $N = 1$. So we try to find a solution of the form

$$u(x,t) = u(\xi) = d_0 + d_1\left(p + \frac{G'}{G}\right) + e_1\left(p + \frac{G'}{G}\right)^{-1} \tag{18}$$

where d_0, d_1 and e_1 are arbitrary constants to be determined.

Substituting Eq.(18) together with Eq. (16) into Eq.(17), the left-hand side is converted into polynomials in $\left(\frac{G'}{G}\right)^N$, $(N = 0,1,2,...)$ and $\left(\frac{G'}{G}\right)^{-N}$, $(N = 1,2,3,...)$. We collect each coefficient of these resulted polynomials to zero, yield a set of simultaneous algebraic equations (for simplicity, which are not presented) for d_0, d_1, e_1, k and ω. Solving these algebraic equations with the help of algebraic software Maple, we obtain following:

Case 1

$$d_0 = d_1 = 0, e_1 = e_1, \omega = \frac{1}{3}\frac{3\mu - e_1^2 a(t)\alpha - 3p^2\alpha}{\mu - p^2\alpha}, k = \frac{\sqrt{-a(t)}}{\sqrt{6}\left(\mu - p^2\alpha\right)} e_1, \alpha = \alpha, \beta = 2p\alpha, \mu = \mu$$

so we have $u(\xi) = e_1\left(p + \frac{G'}{G}\right)^{-1}$, in this casse the are possibly five solutions for the equation and they are

Family 1. Hyperbolic function solution: when $\beta \neq 0, \Psi = -\alpha$ and $\Omega = \beta^2 - 4\alpha\mu > 0$,

$$u(\xi) = e_1\left(p + \frac{\beta}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi}\frac{C_1 \sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)^{-1}$$

Family 2. Trigonometric function solution: when $\beta \neq 0, \Psi = -\alpha$ and $\Omega = \beta^2 - 4\alpha\mu < 0$,

$$u(\xi)=e_1\left(p+\frac{\beta}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-C_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+C_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{C_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+C_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-1}$$

Family 3. Rational form solution: when $\beta\neq 0,\Psi=-\alpha$ and $\Omega=\beta^2-4\alpha\mu=0,$

$$u(\xi)=e_1\left(p+\frac{\beta}{2\Psi}+\frac{C_2}{C_1+C_2\xi}\right)^{-1}$$

Family 4. Hyperbolic function solution: when $B=0,\Psi=-\alpha$ and $\Delta=\Psi\mu>0,$

$$u(\xi)=e_1\left(p+\frac{\sqrt{\Delta}}{\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{C_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+C_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+C_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^{-1}$$

Family 5. Trigonometric function solution: when $\beta=0,\Psi=-\alpha$ and $\Delta=\Psi\mu<0,$

$$u(\xi)=e_1\left(p+\frac{\sqrt{-\Delta}}{\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{-C_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+C_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{C_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+C_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-1}$$

Case 2

$$d_0=d_1=0,e_1=e_1,\omega=1,k=\frac{\sqrt{-a(t)}}{\sqrt{6\mu}}e_1,\alpha=0,\beta=0,\mu=\mu$$

so we have $u(\xi)=e_1\left(p+\frac{G'}{G}\right)^{-1}$, in this casse the is one and only one rational form solution: $u(\xi)=e_1\left(p+\frac{\beta}{2\Psi}+\frac{C_2}{C_1+C_2\xi}\right)^{-1}$.

Case 3

$$d_0 = d_0, d_1 = 0, e_1 = \frac{2d_0(\mu - \beta p)}{\beta}, \omega = 1 + \frac{1}{3}a(t)d_0^{\,2}, k = \frac{\sqrt{-a(t)}}{\sqrt{3}\beta}d_0, \alpha = 0, \beta = \beta, \mu = \mu$$

so we have $u(\xi) = d_0 + e_1\left(p + \frac{G'}{G}\right)^{-1} = d_0 + \frac{2d_0(\mu - \beta p)}{\beta}\left(p + \frac{G'}{G}\right)^{-1}$, in this casse the is one only Hyperbolic function solution which take the form:

$$u(\xi) = d_0 + \frac{2d_0(\mu - \beta p)}{\beta}\left(p + \frac{\beta}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi}\frac{C_1 \sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)^{-1}$$

Case 4

$$d_0 = d_0, d_1 = 0, e_1 = e_1, \omega = \frac{1}{3}\frac{1}{2p\alpha - \beta}\left(6p\alpha + 2a(t)d_0^{\,2}p\alpha + 2d_0 a(t)e_1\alpha - 3\beta - \beta a(t)d_0^{\,2}\right),$$

$$k = \frac{\sqrt{-a(t)}}{\sqrt{6}(2p\alpha - \beta)}d_0, \alpha = \alpha, \beta = \beta, \mu = -\frac{1}{2}\frac{-2d_0 p\beta + 2d_0 p^2\alpha - e_1\beta + 2e_1\alpha p}{d_0}$$

so we have $u(\xi) = d_0 + e_1\left(p + \frac{G'}{G}\right)^{-1}$, in this casse the are possibly five solutions for the equation and they are

Family 1. Hyperbolic function solution: when $\beta \neq 0, \Psi = -\alpha$ and $\Omega = \beta^2 - 4\alpha\mu > 0$,

$$u(\xi) = d_0 + e_1\left(p + \frac{\beta}{2\Psi} + \frac{\sqrt{\Omega}}{2\Psi}\frac{C_1 \sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)^{-1}$$

Family 2. Trigonometric function solution: when $\beta \neq 0, \Psi = -\alpha$ and $\Omega = \beta^2 - 4\alpha\mu < 0$,

$$u(\xi)=d_0+e_1\left(p+\frac{\beta}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-C_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+C_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{C_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+C_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-1}$$

Family 3. Rational form solution: when $\beta\neq 0,\Psi=-\alpha$ and $\Omega=\beta^2-4\alpha\mu=0$,

$$u(\xi)=d_0+e_1\left(p+\frac{\beta}{2\Psi}+\frac{C_2}{C_1+C_2\xi}\right)^{-1}$$

Family4. Hyperbolic function solution: when $B=0,\Psi=-\alpha$ and $\Delta=\Psi\mu>0$,

$$u(\xi)=e_1\left(p+\frac{\sqrt{\Delta}}{\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{C_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+C_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+C_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^{-1}$$

Family 5. Trigonometric function solution: when $\beta=0,\Psi=-\alpha$ and $\Delta=\Psi\mu<0$,

$$u(\xi)=d_0+e_1\left(p+\frac{\sqrt{-\Delta}}{\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{-C_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+C_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{C_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+C_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-1}$$

6. CONCLUSIONS

This study shows that the new approach of the generalized $\left(\frac{G'}{G}\right)$-expansion method is quite efficient and practically will suited for use in finding exact solutions for the problem considered here. New and more general excat solutions for any arbitray function $a(t)$ are obtained, there is no any method before, gave any exact solution for this equation. Also we construct an innovative explicit traveling wave solutions involving parameters of the modified fractional Benjamin-Bona-Mahony equation (BBM).

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