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Einstein-Cartan Relativity in 2-Dimensional Non-Riemannian Space

L. N. Katkar[1] and D. R. Phadatare[2]

[1]*Department of Mathematics,Shivaji University, Kolhapur 416004.*

[2]*Balasaheb Desai College, Patan,Tal: Patan, Dist: Satara 415206.*

Abstract

We describe the Einstein-Cartan theory of gravity in a 2- dimensional non-Riemannian space. To explore the implications and features of Einstein-Cartan field equations in a 2-dimensional non-Riemannian space, we have introduced two real null vector formalism. We here after referred to it as a dyad formalism. This formalism facilitates the computational complexity and will serve as an instructional tool to simplify mathematics. The results are derived by two different methods one based on the dyad formalism and one based on the techniques of differential forms introduced by the author by introducing a new derivative operator d_* defined with respect to the asymmetric connections. Both the methods will serve as an "amazingly useful" technique to reduce the complexity of mathematics. We have proved that the Einstein tensor vanishes identically yet the Riemann Curvature of the non-Riemannian 2- space is influenced by the torsion.

Key words: Non-Riemann space, Riemannian Curvature, Exterior calculus, Einstein-Cartan theory of gravitation.

1. INTRODUCTION:

Einstein's theory of general relativity is one of the cornerstones of modern theoretical physics and has been considered as one of the most beautiful structures of theoretical physics not just in its conceptual ingenuity and mathematical elegance but also in its ability to explain real physical phenomena. It is the most successful theory of gravitation in which the gravitation as a universal force can be described by a curvature of space-time consisting of three spatial dimensions and one time that has led Einstein to formulate his famous field equations of general relativity which are non-linear second order partial differential equations. General relativity has considered as one of the most difficult subject due to a great deal of complex mathematics. The complexity of the mathematics reflects the complexity of describing space-time curvature and some conceptual issues which are present and even more opaque in the physical 4- dimensions world. Hence in order to gain insight in to these difficult conceptual issues Deseret. al (1984) in a series of papers, Giddings et.al (1984), and Gott et. al. (1984, 1986) have examined general relativity in lower dimensional spaces and explored some solutions. Studies of general relativity in lower dimensional space-times have proved that solving Einstein's field equations of general relativity in a space-time of reduced dimensionality is rather simple but yields some amusing results that are pedagogical and scientific interests and yet are apparently unfamiliar to most physicists.

A.D. Boozer (2008) and R. D. Mellinger Jr. (2012) have examined the general relativity in (1+1) dimensions. Einstein-Cartan theory of gravitation is one of the extensions of the general theory of relativity developed by Cartan (1923) in a non-Riemannian space-time. It is only in the last couple of decades, the Einstein-Cartan theory has caught the imagination of researchers for constructing models with spin for the primary purpose of overcoming singularities. In this paper we intend to study the Einstein-Cartan theory of relativity in a 2-dimensional non-Riemannian space.

The material of the paper is organized as follows. In the Section 2, we give a brief introduction to a non-Riemannian space. An exposition of a new dyad formalism, consisting of two real null vector fields is given in Section 3. We have employed this dyad formalism and constructed a 2-dimensional non-Riemannian space and shown that the 2-dimensional non-Riemannian space contains no matter at all, so that there is no gravitational field either but torsion influences the curvature of the 2- dimensional non-Riemannian space.

In the section 4, the results obtained in the Section 3 are corroborated by employing the techniques of differential form developed by Katkar in (2015). Some conclusions are drawn in the last section.

2. NON-RIEMANNIAN SPACE:

A non-Riemannian space is achieved by taking a space of n dimensions endowed with a Riemannian metric in which the connections are asymmetric. Due to the asymmetric connections the geometry of the space of Einstein-Cartan theory of gravity does not remain Riemannian but it becomes non-Riemannian. The non-Riemannian character of the space is introduced through the asymmetric connections defined by

$$\Gamma^i{}_{jk} = \left\{ {i \atop jk} \right\} - K_{jk}{}^i , \tag{2.1}$$

where $K_{jk}{}^i$ is the contortion tensor skew symmetric in the last two indices, and $\left\{ {i \atop jk} \right\}$ are the usual symmetric Christoffel symbols of the Riemannian space. We denote $Q_{ij}{}^k$ to represent torsion tensor, which is skew symmetric in the first two indices, and is defined by

$$Q_{ij}{}^k = -\frac{1}{2}\left(K_{ij}{}^k - K_{ji}{}^k\right). \tag{2.2}$$

The field equations of the Einstein-Cartan theory of gravitation, in a 4-dimensional space-time, are given by Hehl. et. al. (1974) as

$$R_{ij} - \frac{R}{2} g_{ij} = Kt_{ij} ,$$

$$Q_{ij}{}^k + \delta^k{}_i Q_{jl}{}^l - Q_{il}{}^l \delta^k{}_j = KS_{ij}{}^k , \tag{2.3}$$

where R_{ij} is the Ricci tensor which is no longer symmetric but instead contains the information about the torsion tensor. The right hand side tensor t_{ij} cannot be symmetric either, which also contains the information about the spin tensor. $S_{ij}{}^k$ is the spin angular momentum tensor. In general, the spin angular momentum tensor can be decomposed (Hehl et.al (1974)) in to the spin tensor S_{ij} as

$$S_{ij}{}^k = S_{ij} u^k . \tag{2.4}$$

The Riemann curvature tensor of a non-Riemannian space R_{kjih} satisfies the following properties (Katkar (2015)):

$$R_{kjih} = -R_{kjhi} = -R_{jkih} , \ R_{kjih} \neq R_{ihkj} ,$$

$$R_{kji}{}^h + R_{jik}{}^h + R_{ikj}{}^h = 2\left(Q_{ij}{}^h{}_{;k} + Q_{jk}{}^h{}_{;i} + Q_{ki}{}^h{}_{;j}\right) - 4\left(Q_{ij}{}^l Q_{kl}{}^h + Q_{jk}{}^l Q_{il}{}^h + Q_{ki}{}^i Q_{jl}{}^h\right),$$

$$R_{kji}{}^l{}_{;h} + R_{jhi}{}^l{}_{;k} + R_{hki}{}^l{}_{;j} = -2\left(R_{jpi}{}^l Q_{kh}{}^p + R_{pki}{}^l Q_{jh}{}^p + R_{hpi}{}^l Q_{jk}{}^p\right). \tag{2.5}$$

The Riemann curvature tensors of a non-Riemannian space and a Riemannian space are related by the equation

$$R_{kji}{}^h = \hat{R}_{kji}{}^h + K_{ji}{}^h{}_{;k} - K_{ki}{}^h{}_{;j} + 2K_{li}{}^h Q_{kj}{}^l + K_{ji}{}^l K_{kl}{}^h - K_{ki}{}^l K_{jl}{}^h , \tag{2.6}$$

where $\hat{R}_{kji}{}^h$ is the Riemann curvature tensor of the Riemannian space. However, in the

2- dimensional space there exist only one non-vanishing component of the Riemann curvature tensor viz., R_{1212} whose symmetry in the pair of indices is inbuilt in its structure.

3. DYAD FORMALISM:

Consider a 2-dimensional space characterized by an indefinite metric

$$ds^2 = f^2(x,t)\,dx^2 - h^2(x,t)\,dt^2 \ , \tag{3.1}$$

where

$$g_{11} = f^2 \,, g_{22} = -h^2 \,, g = -f^2 h^2 \,,$$
$$g^{11} = f^{-2} \,, g^{22} = -h^{-2} \,. \tag{3.2}$$

We define a basis 1-form as

$$\theta^1 = \frac{1}{\sqrt{2}}\left[f(x,t)dx + h(x,t)dt \right] \quad ,\theta^2 = \frac{1}{\sqrt{2}}\left[f(x,t)dx - h(x,t)dt \right]. \tag{3.3}$$

In terms the basis 1-forms the metric (3.1) becomes

$$ds^2 = 2\theta^1\theta^2 \,. \tag{3.4}$$

In order to construct a 2- dimensional non-Riemann space, we introduce, in the following two null vector formalism. This formalism facilitates to introduce torsion in to the space and the space becomes non- Riemannian.

Consider a curve in a space. At each point of the curve, we define a dyad of basis vectors as

$$e_{(\alpha)i} = (l_i \,, n_i) \,, \tag{3.5}$$

where l_i and n_i are real null vector fields satisfying the ortho-normality conditions

$$l_i l^i = n_i n^i = 0 \,,$$
$$l_i n^i = 1 \,. \tag{3.6}$$

Here the Latin indices are used to denote the tensor indices while the Greek indices are used to denote the dyad indices. Any vector (or tensor) can always be expressed in terms of the dyad components of the vector (tensor) and vice versa. Thus we express

$$A_\alpha = A_i e_{(\alpha)}{}^i \,, A_{\alpha\beta} = A_{ij} e_{(\alpha)}{}^i e_{(\beta)}{}^j \quad ,$$
$$A_i = A_\alpha e^{(\alpha)}{}_i \,, A_{ij} = A_{\alpha\beta} e^{(\alpha)}{}_i e^{(\beta)}{}_j \quad , \tag{3.7}$$

Where $e^{(\alpha)}{}_i$ is the dyad of the dual basis vectors satisfying the conditions

$$e^{(\alpha)}{}_i \, e_{(\alpha)}{}^k = \delta^k{}_i \quad and \quad e^{(\alpha)}{}_i \, e_{(\beta)}{}^i = \delta^\alpha{}_\beta . \tag{3.8}$$

This gives

$$e^{(\alpha)}{}_i = (n_i \; , l_i) . \tag{3.9}$$

Consequently, we express the dyad components of the metric tensor g_{ij} as

$$\eta_{\alpha\beta} = g_{ij} e_{(\alpha)}{}^i e_{(\beta)}{}^j . \tag{3.10}$$

This gives

$$\eta_{\alpha\beta} = \eta^{\alpha\beta} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{3.11}$$

Hence the metric tensor in terms of the basis vectors is defined as

$$g_{ij} = l_i n_j + n_i l_j . \tag{3.12}$$

The tetrad indices can be raised and lowered by the dyad components of the metric tensor $\eta_{\alpha\beta}$, while the tensor indices are raised and lowered by the metric tensor g_{ij} .

The equation $\theta^\alpha = e_i{}^{(\alpha)} dx^i$,

yields

$$l_i = \frac{1}{\sqrt{2}}(f \; , -h) \quad , n_i = \frac{1}{\sqrt{2}}(f \; , h) ,$$

$$l^i = \frac{1}{\sqrt{2}}(f^{-1} \; , h^{-1}) \quad , n^i = \frac{1}{\sqrt{2}}(f^{-1} \; , -h^{-1}) . \tag{3.13}$$

The spin tensor is anti-symmetric; hence it has just one independent component in the 2-dimension space. We express the spin tensor as a linear combination of the basis vectors of the dyad as

$$S_{ij} = S_{\alpha\beta} e^{(\alpha)}{}_i e^{(\beta)}{}_j ,$$

$$S_{ij} = (-S_{12})_d (l_i n_j - n_i l_j) ,$$

$$S_{ij} = S_d (l_i n_j - n_i l_j) , \tag{3.14}$$

where $S_d = (-S_{12})_d$ is the dyad component of spin tensor. In general it is a function of coordinates. The tensor component of the spin tensor is obtain from equations (3.13) and (3.14) as

$$S_d = \frac{1}{fh} S_t \ . \tag{3.15}$$

Similarly, we express the spin angular momentum tensor in terms of the basis vectors of the dyad as

$$S_{ij}{}^{k} = -[(S_{12}{}^{1})_d l^k + (S_{12}{}^{2})_d n^k](l_i n_j - n_i l_j)\,. \tag{3.16}$$

For the choice of the time like vector field $u^i = \frac{1}{\sqrt{2}}(l^i + n^i)$ such that $u^i u_i = 1$, we have from the equations (3.14)

$$S_{ij}u^k = [(\frac{S_d}{\sqrt{2}})l^k + (\frac{S_d}{\sqrt{2}})n^k](l_i n_j - n_i l_j)\,. \tag{3.17}$$

It follows from the equations (2.4), (3.16) and (3.17) that

$$(S_{12}{}^{1})_d = (S_{12}{}^{2})_d = -(\frac{S_d}{\sqrt{2}})\,. \tag{3.18}$$

Hence we have from equations (3.16), (3.17) and (3.18)

$$S_{ij}{}^{k} = S_{ij}u^k = \frac{1}{\sqrt{2}} S_d (l_i n_j - n_i l_j)(l^k + n^k). \tag{3.19}$$

We express the torsion tensor $Q_{ij}{}^{k}$ in terms of its dyed components as

$$Q_{ij}{}^{k} = Q_{\alpha\beta}{}^{\gamma} e_i{}^{(\alpha)} e_j{}^{(\beta)} e^k{}_{(\gamma)} \ . \tag{3.20}$$

This yields

$$Q_{ij}{}^{k} = [-(Q_{12}{}^{1})_d l^k - (Q_{12}{}^{2})_d n^k](l_i n_j - n_i l_j)\,. \tag{3.21}$$

We approximate the values of the dyad components of torsion tensor to the dyad components of the spin tensor as

$$(Q_{12}{}^{1})_d = (Q_{122})_d = -\frac{KS_d}{2\sqrt{2}}\,, (Q_{12}{}^{2})_d = (Q_{121})_d = -\frac{KS_d}{2\sqrt{2}} \quad . \tag{3.22}$$

Consequently, the equation (3.21) becomes

$$Q_{ij}{}^{k} = \frac{KS_d}{2\sqrt{2}}(l_i n_j - n_i l_j)(l^k + n^k) \quad . \tag{3.23}$$

Similarly, the tensor components of Contortion tensor are obtain from the equations (3.13) and (3.21) as

$$(Q_{12}{}^{1})_t = (Q_{122})_t = \frac{KS_t}{2f}, \ (Q_{12}{}^{2})_t = (Q_{121})_t = 0. \tag{3.24}$$

We now express the contortion tensor $K_{ij}{}^{k}$ as the linear combinations of the basis vectors of the dyad as

$$K_{ijk} = -(K_{112})_d\, n_i(l_j n_k - n_j l_k) - (K_{212})_d\, l_i(l_j n_k - n_j l_k). \tag{3.25}$$

From the relation

$$K_{\alpha\beta\gamma} = -Q_{\alpha\beta\gamma} + Q_{\beta\gamma\alpha} - Q_{\gamma\alpha\beta} \quad , \tag{3.26}$$

we obtain

$$(K_{212})_d = (K_{112})_d = \frac{KS_d}{\sqrt{2}}. \tag{3.27}$$

Hence the equation (3.25) becomes

$$K_{ijk} = \frac{KS_d}{\sqrt{2}}(l_j n_k - n_j l_k)(l_i + n_i). \tag{3.28}$$

The equations (3.13) and (3.28) yield the tensor components of the Contortion tensor and are given by

$$(K_{11}{}^{2})_t = (K_{12}{}^{1})_t = -K\frac{f}{h^2}S_t. \tag{3.29}$$

For the given metric, the non vanishing components of the symmetric Christoffel symbols are given by

$$\left\{ {1 \atop 11} \right\} = \frac{f_{,1}}{f}, \left\{ {2 \atop 11} \right\} = \frac{f}{h^2} f_{,2}, \left\{ {1 \atop 12} \right\} = \frac{f_{,2}}{f},$$

$$\left\{ {1 \atop 22} \right\} = \frac{h}{f^2} h_{,1}, \left\{ {2 \atop 12} \right\} = \frac{h_{,1}}{h}, \left\{ {2 \atop 22} \right\} = \frac{h_{,2}}{h} \quad . \tag{3.30}$$

Thus the tensor components of the asymmetric connections becomes

$$(\Gamma_{11}{}^{1})_t = \frac{f_{,1}}{f}, (\Gamma_{22}{}^{1})_t = \frac{h}{f^2} h_{,1}, (\Gamma_{12}{}^{2})_t = (\Gamma_{21}{}^{2})_t = \frac{h_{,1}}{h},$$

$$(\Gamma_{12}{}^{1})_t = \frac{f_{,2}}{f} - \frac{K}{f} S_t, (\Gamma_{11}{}^{2})_t = \frac{f}{h^2} f_{,2} + K\frac{f}{h^2} S_t,$$

$$(\Gamma_{21}{}^{1})_t = \frac{f_{,2}}{f}\,,\; (\Gamma_{22}{}^{2})_t = \frac{h_{,2}}{h} \quad . \tag{3.31}$$

Due to equation (3.31), the expression for the Riemann curvature tensor becomes

$$(R_{121}{}^{2})_t = -\frac{h_{,11}}{h} + \frac{f}{h^2} f_{,22} + (\frac{h_{,1}}{h})(\frac{f_{,1}}{f}) - \frac{f}{h^3} f_{,2} h_{,2} + \frac{Kf}{h^3}(hS_{t,2} - S_t h_{,2})\,.$$

From this equation, we obtain the covariant components of the Riemann curvature tensor of a non-Riemannian space as

$$(R_{1212})_t = hh_{,11} - ff_{,22} - \frac{h}{f} f_{,1} h_{,1} + \frac{f}{h} f_{,2} h_{,2} - \frac{Kf}{h}(hS_{t,2} - S_t h_{,2})\,. \tag{3.32}$$

This equation can also be written as

$$(R_{1212})_t = (\hat{R}_{1212})_t - \frac{K}{t^2}(S_t + tS_{t,t})\,, \tag{3.33}$$

where

$$(\hat{R}_{1212})_t = hh_{,11} - ff_{,22} - \frac{h}{f} f_{,1} h_{,1} + \frac{f}{h} f_{,2} h_{,2}\,.$$

The tensor components of the Ricci tensor and the Ricci scalar are given by

$$(R_{11})_t = h^{-2}(R_{1212})_t\,,\; (R_{12})_t = 0\,,\; (R_{22})_t = -f^{-2}(R_{1212})_t\,, \tag{3.34}$$

$$(R_{11})_t = \frac{h_{,11}}{h} - \frac{f}{h^2} f_{,22} - \frac{1}{fh} f_{,1} h_{,1} + \frac{f}{h^3} f_{,2} h_{,2} - \frac{Kf}{h^3}(hS_{t,2} - S_t h_{,2})\,,$$

$$(R_{22})_t = -\frac{h}{f^2} h_{,11} + \frac{1}{f} f_{,22} + \frac{h}{f^3} f_{,1} h_{,1} - \frac{1}{fh} f_{,2} h_{,2} + \frac{K}{fh}(hS_{t,2} - S_t h_{,2})\,, \tag{3.35}$$

and

$$R = 2[\frac{1}{hf^2} h_{,11} - \frac{1}{fh^2} f_{,22} - \frac{1}{hf^3} f_{,1} h_{,1} + \frac{1}{fh^3} f_{,2} h_{,2} - \frac{K}{fh^3}(hS_{t,2} - S_t h_{,2})]\,. \tag{3.36}$$

We see from equations (3.35), (3.36) that

$$R_{ij} = \frac{R}{2} g_{ij}\,. \tag{3.37}$$

This is true for any 2-space. This shows that the Ricci tensor and the Ricci scalar terms cancel in the field equation of the Einstein-Cartan theory of gravitation. In other words, in 2- dimensions space, the Einstein tensor vanishes identically and from Einstein-Cartan field equations, we get $t_{ij} = 0$.

Curvature of a non-Riemannian Space:

Katkar (2015) has obtained the formula for the Riemann curvature of a non-Riemannian space in the form

$$\kappa = \kappa_1 + \frac{1}{b}[\frac{\partial}{\partial u^1}(K_{212})_t - \frac{\partial}{\partial u^2}(K_{112})_t], \tag{3.38}$$

where $b = (g_{hj}g_{ik} - g_{ij}g_{hk})\, p^h q^i p^j q^k$ is the determinant of the metric tensor of the 2-dimensional surface determined by the orientations of the two unit vectors p^i and q^i, and

$$\kappa_1 = \frac{\hat{R}_{hijk} p^h q^i p^j q^k}{(g_{hj}g_{ik} - g_{ij}g_{hk})\, p^h q^i p^j q^k}, \tag{3.39}$$

is the Riemann Curvature of Riemannian space, at a point, for the orientations determined by the two unit vectors p^i and q^i . The formula (3.39) gives the curvature of a Riemannian space as

$$\kappa_1 = -\frac{1}{hf^2}h_{,11} + \frac{1}{fh^2}f_{,22} + \frac{1}{hf^3}f_{,1}h_{,1} - \frac{1}{fh^3}f_{,2}h_{,2} \quad . \tag{3.40}$$

Consequently, the curvature of a non-Riemannian space becomes

$$\kappa = -\frac{1}{hf^2}h_{,11} + \frac{1}{fh^2}f_{,22} + \frac{1}{hf^3}f_{,1}h_{,1} - \frac{1}{fh^3}f_{,2}h_{,2} + \frac{K}{f^2h^2}(f\,S_{t,2} + S_t f_{,2}). \tag{3.41}$$

We see that curvature of the non-Riemannian 2-space is influenced by the torsion. In the absence of torsion, we see from the equations (3.40) and (3.41) that $\kappa = \kappa_1$. We also observe from the equations (3.36) and (3.41) that

$$\kappa \neq -\frac{R}{2}. \tag{3.42}$$

If the components of the spin tensor are zero, then the results (3.36) and (3.42) reduce to the results of Riemann space.

From the tetrad components of the Riemann curvature tensor $R_{\alpha\beta\gamma\delta} = R_{hijk}e^h{}_{(\alpha)}e^i{}_{(\beta)}e^j{}_{(\gamma)}e^k{}_{(\delta)}$,

we obtain

$$(R_{1212})_d = \frac{1}{f^2h^2}(R_{1212})_t \ . \tag{3.43}$$

Consequently, from the equation $R_{hijk} = R_{\alpha\beta\gamma\delta}e_h{}^{(\alpha)}e_i{}^{(\beta)}e_j{}^{(\gamma)}e_k{}^{(\delta)}$, we obtain the expression for the Riemannian curvature tensor of a non-Riemannian space as

$$R_{hijk} = [-\frac{1}{hf^2} h_{,11} + \frac{1}{fh^2} f_{,22} + \frac{1}{hf^3} f_{,1} h_{,1} - \frac{1}{fh^3} f_{,2} h_{,2} + + \frac{K}{fh^3} (hS_{t,2} - S_t h_{,2})](g_{hj} g_{ik} - g_{ij} g_{hk}) . \quad (3.44)$$

This equation, due to the equation (3.36), becomes

$$R_{hijk} = -\frac{R}{2} (g_{hj} g_{ik} - g_{ij} g_{hk}) . \quad (3.45)$$

If the Riemann curvature tensor R_{hijk} of any non-Riemann space V_n , $n > 2$, satisfies the Bianchi identities (2.5), then we obtain

$$R = c \exp[(\frac{4}{n-1}) \int Q_{hi}{}^{h} dx^i] , \quad (3.46)$$

where c is a constant of integration. If $Q_{hi}{}^{h} = 0 \Rightarrow c = \hat{R}$. Hence

$$R = \hat{R} \exp[(\frac{4}{n-1}) \int Q_{hi}{}^{h} dx^i] . \quad (3.47)$$

Where as in the case of Riemannian space, we have

$$\hat{R}_{hijk} = \kappa_1 (g_{hj} g_{ik} - g_{ij} g_{hk}) . \quad (3.48)$$

This gives $\hat{R} = n(1-n)\kappa_1$, where κ_1 is the constant Riemann curvature of a Riemannian space. Hence we have finally,

$$R = n(1-n)\kappa_1 \exp[(\frac{4}{n-1}) \int Q_{hi}{}^{h} dx^i] . \quad (3.49)$$

Contracting the index h with k in the equation (3.44) we get

$$R_{ij} = [\frac{1}{hf^2} h_{,11} - \frac{1}{fh^2} f_{,22} - \frac{1}{hf^3} f_{,1} h_{,1} + \frac{1}{fh^3} f_{,2} h_{,2} - \frac{K}{fh^3} (hS_{t,2} - S_t h_{,2})] g_{ij} . \quad (3.50)$$

This is nothing but $R_{ij} = \frac{R}{2} g_{ij}$. This shows that the Einstein tensor $G_{ij} = R_{ij} - \frac{R}{2} g_{ij}$ vanishes identically.

4. TECHNIQUES OF DIFFERENTIAL FORMS:

The author (2015) has introduced a new operator d_* on a non-Riemannian space and applied to a form of any degree. It converts p- form to p + 1- form and is obtained by taking the covariant derivative of an associated p^{th} ordered skew symmetric tensor with respect to the asymmetric connections. We note here that unlike the exterior derivative operator in a Riemannian space, the repetition of the new derivative operator d_* on any form ϕ of any degree is not zero. i. e.,

$$d_*^{\ 2}\phi \neq 0 .$$

However, the operator d_* satisfies all other properties of the exterior derivative. For the scalar function ϕ, the operator d_* gives

$$d_*\phi = \phi_{;i} d_* x^i \ . \tag{4.1}$$

Where for scalar function ϕ ,we have

$$\phi_{;i} = \phi_{/i} = \phi_{,i} \ . \tag{4.2}$$

Hence we have $d_*\phi = d\phi$ and $d_* x^i = dx^i$, where d is the usual exterior derivative defined in a Riemannian space in which the connections are the symmetric Christoffel symbols. However, the action of the repeated operator d_* on the scalar function ϕ gives

$$d_*(d_*\phi) = -\phi_{;l} Q_{ij}^{\ \ l} d_* x^i \Lambda d_* x^j - \phi_{;k} d_*^{\ 2} x^k \ .$$

For the coordinate functions $\phi = x^i$, this equation becomes

$$d_*^{\ 2} x^k = -\frac{1}{2} Q_{ij}^{\ \ k} d_* x^i \Lambda d_* x^j \ . \tag{4.3}$$

Consequently, the above equation yields

$$d_*(d_*\phi) = -\frac{1}{2}\phi_{;k} Q_{ij}^{\ \ k} d_* x^i \Lambda d_* x^j \tag{4.4}$$

The dyad equivalent of this equation is given by

$$d_*(d_*\phi) = -\frac{1}{2}\phi_{;\gamma} Q_{\alpha\beta}^{\ \ \ \gamma} \theta^\alpha \Lambda \theta^\beta \ , \tag{4.5}$$

where

$$\phi_{;\gamma} = \phi_{;i} e^i{}_{(\gamma)} \ ,$$

$$i.e.\ (\phi_{;1})_d = \frac{1}{\sqrt{2}}(f^{-1}\frac{\partial\phi}{\partial \mathrm{x}} + h^{-1}\frac{\partial\phi}{\partial t}),\ (\phi_{;2})_d = \frac{1}{\sqrt{2}}(f^{-1}\frac{\partial\phi}{\partial \mathrm{x}} - h^{-1}\frac{\partial\phi}{\partial t}).$$

Consequently, we obtain

$$d_*^{\,2}\phi = -\frac{1}{2\sqrt{2}}[(f^{-1}\frac{\partial\phi}{\partial x} + h^{-1}\frac{\partial\phi}{\partial t})Q_{\alpha\beta}{}^{1} + (f^{-1}\frac{\partial\phi}{\partial x} - h^{-1}\frac{\partial\phi}{\partial t})Q_{\alpha\beta}{}^{2}]\theta^{\alpha}\Lambda\theta^{\beta} \ . \quad (4.6)$$

From this equation, we readily find

$$d_*^{\,2}x = \frac{K}{2f^2h}S_t\theta^1\Lambda\theta^2 \ , \text{and} \ \ d_*^{\,2}t = 0 . \quad (4.7)$$

Now operating the new exterior derivative operator d_* to the basis 1-form defined in the equation (3.3), we obtain

$$d_*\theta^1 = \frac{1}{2\sqrt{2}fh}[2(f_{,2} - h_{,1}) + KS_t]\theta^1\Lambda\theta^2 ,$$

$$d_*\theta^2 = \frac{1}{2\sqrt{2}fh}[2(f_{,2} + h_{,1}) + KS_t]\theta^1\Lambda\theta^2 . \quad (4.8)$$

From the Cartan's first equation of structure of the non-Riemannian space we have

$$d_*\theta^{(\alpha)} = -\omega^{\alpha}{}_{\beta}\Lambda\theta^{\beta} + \frac{1}{2}Q_{\sigma\beta}{}^{\alpha}\theta^{\sigma}\Lambda\theta^{\beta} , \quad (4.9)$$

where

$$\omega^{\alpha}{}_{\beta} = \gamma^{\alpha}{}_{\beta\gamma}\theta^{\gamma} \quad (4.10)$$

and

$$\gamma^{\alpha}{}_{\beta\gamma} = \gamma^{0\alpha}{}_{\beta\gamma} - K_{\gamma\beta}{}^{\alpha} \ , \quad (4.11)$$

where $\gamma^{\alpha}{}_{\beta\gamma}$ are Ricci's coefficients of rotation and are defined by

$$\gamma^{\alpha}{}_{\beta\gamma} = -e^{(\alpha)}{}_{i;j}e_{(\beta)}{}^{i}e_{(\gamma)}{}^{j} \ ,$$

$$\gamma^{\alpha}{}_{\beta\gamma} = -e^{(\alpha)}{}_{i/j}e_{(\beta)}{}^{i}e_{(\gamma)}{}^{j} - e^{(\alpha)}{}_{k}K_{ji}{}^{k}e_{(\beta)}{}^{i}e_{(\gamma)}{}^{j} \ ,$$

where

$$\gamma^{0\alpha}{}_{\beta\gamma} = -e^{(\alpha)}{}_{i/j}e_{(\beta)}{}^{i}e_{(\gamma)}{}^{j} \ , \quad (4.12)$$

are the Ricci's rotation coefficients in the Riemannian space. From the equation (4.12) we find

$$\gamma^{01}{}_{11} = -e^{(1)}{}_{i/j}e_{(1)}{}^{i}e_{(1)}{}^{j} ,$$

$$\gamma^{01}{}_{11} = l_{i/j}n^i l^j \ , \text{and} \ \ \gamma^{01}{}_{12} = l_{i/j}n^i n^j \ .$$

We define

$$l_{i/j}n^i l^j = \kappa^0 \ , \ l_{i/j}n^i n^j = \nu^0 \ , \tag{4.13}$$

where κ^0 and ν^0 are the spin components. The components of the Ricci's coefficients of rotation are given by

$$\gamma^1{}_{11} = \gamma^{01}{}_{11} - (K_{11}{}^1)_d \ , \ \gamma^1{}_{12} = \gamma^{01}{}_{12} - (K_{21}{}^1)_d \ .$$

$$\Rightarrow \gamma^1{}_{11} = -\gamma^2{}_{21} = (\kappa^0 + \frac{K}{\sqrt{2}fh}S_t) \ , \ \gamma^1{}_{12} = -\gamma^2{}_{22} = (\nu^0 + \frac{K}{\sqrt{2}fh}S_t) \ . \tag{4.14}$$

Using the equations (4.14), we obtain the expression of the covariant derivative of a basis vector of the dyad as

$$l_{i;j} = (\nu^0 + \frac{K}{\sqrt{2}fh}S_t)l_i l_j + (\kappa^0 + \frac{K}{\sqrt{2}fh}S_t)l_i n_j \ . \tag{4.15}$$

The equations (4.10) and (4.14) yield the components of connection 1-form as

$$\omega^1{}_1 = -\omega^2{}_2 = (\kappa^0 + \frac{K}{\sqrt{2}fh}S_t)\theta^1 + (\nu^0 + \frac{Kt^2}{\sqrt{2}fh}S_t)\theta^2 \ . \tag{4.16}$$

Also from the Cartan's first equation of the structure (3.9), we obtain

$$d_*\theta^1 = (\nu^0 + \frac{K}{2\sqrt{2}fh}S_t)\theta^1\Lambda\theta^2 \ , d_*\theta^2 = (\kappa^0 + \frac{K}{2\sqrt{2}fh}S_t)\theta^1\Lambda\theta^2 \ . \tag{4.17}$$

Comparing the equations (4.8) and (4.17), we readily get

$$\kappa^0 = \frac{1}{\sqrt{2}fh}(f_{,2} + h_{,1}) \quad , \nu^0 = \frac{1}{\sqrt{2}fh}(f_{,2} - h_{,1}) \ . \tag{4.18}$$

Hence the equation (4.16) becomes

$$\omega^1{}_1 = -\omega^2{}_2 = \frac{1}{\sqrt{2}fh}[(h_{,1} + f_{,2} + KS_t)\theta^1 + (f_{,2} - h_{,1} + KS_t)\theta^2] \ . \tag{4.19}$$

The Cartan's second equation of structure in the non-Riemannian space, when the spin tensor is not u-orthogonal is given by Katkar (2015)

$$\Omega^\alpha{}_\beta = d_*\omega^\alpha{}_\beta + \omega^\alpha{}_\sigma\Lambda\omega^\sigma{}_\beta + \frac{K}{4}[2\gamma^\alpha{}_{\beta\sigma}S_{\delta\gamma}u^\sigma + \gamma^\alpha{}_{\beta\delta}S_{\gamma\sigma}u^\sigma - \gamma^\alpha{}_{\beta\gamma}S_{\delta\sigma}u^\sigma]\theta^\delta\Lambda\theta^\gamma \ . \tag{4.20}$$

From this we obtain

$$\Omega^1{}_1 = -\Omega^2{}_2 = d_*\omega^1{}_1 - \frac{KS_t}{2\sqrt{2}fh}[(\kappa^0 + \nu^0) + \frac{2}{\sqrt{2}fh}KS_t]\theta^1\Lambda\theta^2 \ .$$

On using equation (4.18) we get

$$\Omega^1{}_1 = -\Omega^2{}_2 = d_*\omega^1{}_1 - \frac{KS_t}{2f^2h^2}[f_{,2} + KS_t]\theta^1\Lambda\theta^2 . \qquad (4.21)$$

Operating the new exterior derivative operator d_* to the equation (4.19) we find

$$d_*\omega^1{}_1 = [\frac{1}{hf^3}f_{,1}h_{,1} - \frac{1}{hf^2}h_{,11} + \frac{1}{fh^2}f_{,22} - \frac{1}{fh^3}f_{,2}h_{,2} - \frac{K}{fh^3}S_th_{,2} +$$

$$+\frac{K}{fh^2}S_{t,2} + \frac{K}{2f^2h^2}S_t(f_{,2} + KS_t)]\theta^1\Lambda\theta^2 . \qquad (4.22)$$

Consequently, the equation (4.21) becomes

$$\Omega^1{}_1 = -\Omega^2{}_2 = [\frac{1}{hf^3}f_{,1}h_{,1} - \frac{1}{hf^2}h_{,11} + \frac{1}{fh^2}f_{,22} - \frac{1}{fh^3}f_{,2}h_{,2} +$$

$$+\frac{K}{fh^3}(hS_{t,2} - S_th_{,2})]\theta^1\Lambda\theta^2 . \qquad (4.23)$$

The components of the curvature 2- form are defined by

$$\Omega^1{}_1 = -\frac{1}{2}R_{\alpha\beta 1}{}^1\theta^\alpha\Lambda\theta^\beta , \qquad (4.24)$$

$$\Omega^1{}_1 = -(R_{121}{}^1)_d\theta^1\Lambda\theta^2 . \qquad (4.25)$$

Comparing the corresponding coefficients of the equations (4.23) and (4.25), we obtain the dyad component of the Riemannian curvature tensor as

$$(R_{1212})_d = (R_{121}{}^1)_d = -[\frac{1}{hf^3}f_{,1}h_{,1} - \frac{1}{hf^2}h_{,11} + \frac{1}{fh^2}f_{,22} - \frac{1}{fh^3}f_{,2}h_{,2} +$$

$$+\frac{K}{fh^3}(hS_{t,2} - S_th_{,2})]. \qquad (4.26)$$

Hence, the Riemann Curvature tensor of the Non-Riemannian 2-space becomes

$$R_{hijk} = [\frac{1}{hf^3}f_{,1}h_{,1} - \frac{1}{hf^2}h_{,11} + \frac{1}{fh^2}f_{,22} - \frac{1}{fh^3}f_{,2}h_{,2} +$$

$$+\frac{K}{fh^3}(hS_{t,2} - S_th_{,2})](g_{hj}g_{ik} - g_{ij}g_{hk}) . \qquad (4.27)$$

This on using the equation (3.36) we get

$$R_{hijk} = -\frac{R}{2}(g_{hj}g_{ik} - g_{ij}g_{hk}) . \tag{4.28}$$

The result is equivalent to (3.45). We see from the equation (3.35) that

$$\frac{R_{11}}{f^2} = -\frac{R_{22}}{h^2} . \tag{4.29}$$

In the 2- dimensional space, Ricci tensor and the Curvature tensor has only one independent component. We express the Riemann curvature tensor R_{hijk} in terms of the Ricci tensor R_{ij} alone as

$$R_{hijk} = -[g_{hj}R_{ik} - g_{hk}R_{ij} - g_{ij}R_{hk} + g_{ik}R_{hj}] + \frac{R_{lm}g^{lm}}{2}(g_{hj}g_{ik} - g_{ij}g_{hk}) . \tag{4.30}$$

5. CONCLUSION:

Introduction of dyad formalism facilitates the complexity of computation. A 2-dimensional non-Riemannian space is constructed with the help of the dyad formalism. It is shown that the Einstein tensor of 2- dimensional non-Riemannian vanishes, hence the corresponding space contains no matter at all, so that there is no gravitational field either but the curvature of the space is influenced by the torsion. The results are corroborated by the method of differential forms.

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