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Approximate controllability of stochastic functional differential inclusions of Sobolev- type with unbounded delay in Hilbert space

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Abstract

In this paper, we consider a class of approximate controllability of stochastic functional differential inclusions of Sobolev type with unbounded delay in Hilbert spaces. Using the semigroup theory and fixed point theorem, a set of sufficient conditions is obtained for the required result of approximate controllability of stochastic functional differential inclusions of Sobolev type with unbounded delay. Finally, an example is provided to illustrate the obtained result.

Keywords: Approximate controllability, Stochastic Sobolev-type differential inclusion, Fixed point theorem, Unbounded delay.

2010 Mathematics Subject Classification: 26A33, 34G20, 93G05.

1. INTRODUCTION

Differential inclusions have wide applications in science, engineering, economics and in optimal control theory. Many authors studied the existence, controllability and stability of differential inclusions [1-5, 14, 22, 32, 35-39]. Controllability is one of the elementary concepts in mathematical control theory, which plays a vital role in both engineering and sciences. Controllability generally means that it is possible to steer dynamical control systems from an arbitrary initial state to an arbitrary final state using the set of admissible controls. There are two basic theories of controllability can be identified as approximate controllability and exact controllability. Most of the criteria, which can be met in the literature, are formulated for finite dimensional system. But in the infinite dimensional system, many unsolved problems are still exist as for as controllability is concerned. In the case of infinite dimensional system, controllability can be distinguished as approximate and exact controllability. Approximate controllability means that the system can be governed to arbitrary small neighborhood of final state whereas exact controllability allows to govern the system to arbitrary final state. In other words the approximate controllability gives the possibility of governing the system of states which forms a dense subspace in the state space.

Recently, Mahmodev [18] et.al, studied the approximate controllability of second order neutral stochastic evolution equations using semi group methods together with Banach fixed point theorem. In [8], Henríquez studied the existence of solutions of non-autonomous second order functional differential equations with infinite delay by using Leray Schauder alternative fixed point theorem. In [37], Yan studied the approximate controllability of fractional neutral integrodifferential inclusions with state dependent delay in Hilbert spaces and in [33] Vijayakumar et.al, discuss the approximate controllability for a class of fractional neutral integrodifferential inclusions with state dependent delay using Dhage fixed point theorem. In [7], Gudenowzi investigated the approximate controllablity for a class of fractional neutral stochastic functional integrodifferential inclusions using Bohnenblust-Karlin fixed point theorem.

The stochastic differential equations have attracted great interest due to its applications in science, engineering and medical sciences. In recent years, the controllability problems of stochastic differential equations become a field of increasing interest [9, 15, 17, 19]. The existence of deterministic controllability concepts to stochastic control systems have been discussed only in limited number of publications. More precisely, there are less number of papers in the approximate controllability of non-linear stochastic systems [7, 16, 23, 24, 28 - 31, 40]. Klamka [11,12], studied stochastic relative exact and approximate controllability problem for finite dimensional linear stationary dynamical system with single time-variable point delay using open-mapping theorem. A set of necessary and sufficient conditions are established for the exact and stochastic controllability of linear system with state delays in [10]. In [21], Revathi et.al, studied the existence of stochastic functional differential equations of Sobolev type with infinite delay.

In recent years, controllability problems for various types of nonlinear dynamical systems in the infinite dimensional spaces by using different kinds of approaches have been considered in many publication [6, 15, 17, 18, 20, 23, 25 - 27, 34]. The approximate controllability problem for a nonlinear stochastic systems of Sobolev type in Hilbert space has not been investigated largely. Motivated by this consideration, in this paper, we will study the approximate controllability problem for nonlinear stochastic functional differential inclusions of Sobolev type with unbounded delay in Hilbert space which are natural generalization of controllability concepts in the theory of infinite dimensional deterministic control systems.

The paper is organized as follows: in section 2, we recall some basic definitions, notations, lemmas and some preliminary facts. In section 3, we study the approximate controllability of stochastic functional differential inclusions of Sobolev-type with unbounded delay in Hilbert spaces. An application of our theoretical results is given in Section 4.

2. PRELIMINARIES

In this section, the basic preliminaries, definitions, lemmas, notations, multivalued maps and some results which are needed to establish our main results are discussed.

Let$(H, \|\cdot\|_H)$and $(K, \|\cdot\|_k)$be two real separable Hilbert spaces and for convenience, we use the same notation$\| \cdot \|$ to denote the norms in H and K and $\langle \cdot ,\cdot \rangle$ to denote the inner product space without any confusion. Let $\mathcal{L}(K, H)$ be space of bounded linear operators from K into$H$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space satisfying that $\mathcal{F}_0$contains all P-null sets of $\mathcal{F}$. Let $\{w(t), t \geq 0\}$ represents a $Q-$ Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$with the co-variance operator Q such that $Tr(Q) < \infty$. Further, we assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in K, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, k=1,2,... and sequence of independent Wiener processes such that $\{\beta_k\}_{k\geq 1}$ such that

$$\langle w(t), e\rangle_K = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \langle e_k, e\rangle_K \beta_k(t), t \geq 0.$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}, H)$ be the space of all Hilbert- Schmidt operators from $Q^{\frac{1}{2}} K to H$with the inner product $\langle \varphi, \psi\rangle_{\mathcal{L}_2^0} = Tr[\varphi Q \psi^*]$.

In this paper, we investigate the approximate controllability of stochastic functional differential inclusions of Sobolev -type with unbounded delay in the following form

$$\frac{d[Lx(t)]}{dt} \in Ax(t) + Bu(t) + F(t, x_t) + \sum(s, x_s)dw(s)\,,\ t \in J := [0, b] \quad (2.1)$$

$$x(t) = \phi(t) \in \mathcal{B}_h\,, t \in (-\infty, 0] \quad (2.2)$$

where the state $x(\cdot)$ takes the values in the separable real Hilbert spaces H, A and $L$are linear operators on H. The histories $x_t \in (-\infty, 0] \to \mathcal{B}_h, x_t(\theta) = x(t + \theta)$ for $t \geq 0$ belongs to the phase space $\mathcal{B}_h$, which will be defined later. The initial data $\phi =$

$\{\phi(t), t \in (-\infty, 0]\}$ is an $\mathcal{F}_0$-measurable,$\mathcal{B}_h$-valued stochastic process independent of W with finite second moments. Further $F: J \times \mathcal{B}_h \rightarrow H$ and $\sum: J \times \mathcal{B}_h \rightarrow \mathcal{L}_2^0(K, H)$are appropriate mappings specified later and the control function $u(.)$ is given in $\mathcal{L}(J, U)$, a Hilbert space of admissible control functions with U as Hilbert space. B is a bounded linear operator from U into H.

The operators $A: D(A) \subset H \rightarrow H$ and $L: D(L) \subset H \rightarrow H$ satisfy the following conditions:

(A1)A and L are closed linear operators.

(A2)$D(L) \subset D(A)$ and L is bijective.

(A3)$L^{-1}: H \rightarrow D(L)$ is continuous.

Further, from **(A1)** and **(A2),** $L^{-1}$is closed and with (A3) by using the closed graph theorem, we obtain the boundedness of the linear operator $AL^{-1}: H \rightarrow H$. Further AL^{-1}generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ in H. Let us denote $max_{t\in J}\|T(t)\|^2 = M$, $\|L^{-1}\|^2 = M_L$.

Definition 2.1.(Phase space). Assume that $h: (-\infty, 0] \rightarrow (0, \infty)$ is a continuous function with $l = \int_{-\infty}^{0} h(t)dt < +\infty$ and ϕ is a $\mathcal{F}_0$- measurable functions mappings from$(-\infty, 0]$ into H.Define the phase space $\mathcal{B}_h$ by

$\mathcal{B}_h = \{\phi: (-\infty, 0] \rightarrow H,$ for any $a > 0, (E\|\phi(\theta)\|^2)^{\frac{1}{2}}$

is a bounded and measurable function on $[-a, 0]$ with $\phi(0) = 0$ and

$\int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0} \left((E\|\phi(\theta)\|^2)^{\frac{1}{2}}\right) ds\} < \infty.$

If $\mathcal{B}_h$ is endowed with the norm

$$||\phi||_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s\leq\theta\leq 0} \left((E\|\phi(\theta)\|^2)^{\frac{1}{2}}\right) ds\ , \phi \in \mathcal{B}_h,$$

then $(\mathcal{B}_h, ||\,.\,||_{\mathcal{B}_h})$ is a Banach space.

Now we consider the space of

$$\mathcal{B}'_h = \{x: x \in (-\infty, b] \rightarrow H) \text{such that } x|_J \in \mathcal{C}(J, H), x_0 = \phi \in \mathcal{B}_h\}.$$

Set $||\,.\,||_b$ be a seminorm defined by

$$||\,x||_b = ||\,\phi||_{\mathcal{B}_h} + sup_{s\in[0,b]}(E\|x(s)\|^2)^{\frac{1}{2}}, x \in \mathcal{B}'_h.$$

Lemma 2.2. Assume that $x \in \mathcal{B}'_h$, then for all $t \in J$, $x_t \in \mathcal{B}_h$. Moreover

$$l(E\|\phi(\theta)\|^2)^{\frac{1}{2}} \leq l\, sup_{s\epsilon[0,t]}(E\|x(s)\|^2)^{\frac{1}{2}} + \|\phi\|_{\mathcal{B}_h},$$

where $l = \int_{-\infty}^{0} h(s)ds < \infty$.

Definition 2.3. A multivalued map $G: H \to 2^H \setminus\{\emptyset\}$ is convex(closed) valued if $G(x)$ is convex(closed) for all $x \in H$. $G$is bounded on bounded sets if $G(B) = U_{x\in B}G(x)$ is bounded in H for any bounded set B of H i.e., $sup_{x\in B}\{\sup\{\|y\|: y \in G(x)\}\} < \infty$.

Definition 2.4. $G$is called upper semicontinuous (u.s.c for short) on H, if for each $x_0 \in H$, the set$G(x_0)$ is a nonempty closed subset of $H$and if for each open set $N$of H containing $G(x_0)$, there exits an open neighborhood $V of x_0$such that $G(V) \subseteq$ N .

Definition 2.5. The multi-valued operator G is called compact if $\overline{G(H)}$is a compact subset of H. $G$is called completely continuous if $G(B)$is relatively compact for every bounded subset $B$of H.

For more details on Multivalued maps, see the books of Deimling (1992), Hu and Papageorgiou (1997).

If the multivalued map G is completely continuous with nonempty values, then G is u.s.c., if and only if G has a closed graph, i.e., $x_n \to x_*$,$y_n \to y_*$,$y_n \in G(x_n)$ imply $y_* \in G(x_*)$. Ghas a fixed point if there is a $x \in H$ such that $x \in G(x)$. In the following, $BCC(H)$ denotes the set of all nonempty, bounded, closed and convex subset of H.

Definition 2.6. A multivalued map $G: J \to BCC(H)$is said to be measurable if, for each $x \in H$, the function $v: J \to R$, defined by$v(t) = d(x, Gx(t)) = \inf\{\|x - z\|: z \in G(t)\}$ belongs to $\mathcal{L}^1(J, R)$.

Definition 2.7. The multivalued map $\Sigma : J \times H \to BCC(H)$ is said to be $\mathcal{L}^2$ -Caratheodory if

(i) $t \to \Sigma(t, x)$ is measurable for each $x \in H$,

(ii) $x \to \Sigma(t, x)$ is upper continuous for almost all $t \in J$,

(iii) for each $r > 0$, there exists $l_r \in \mathcal{L}^1(J, \mathbb{R})$ such that

$\|\Sigma(t, x)\|^2 = \sup\{E\|\sigma\|^2: \sigma \in \Sigma(t, x)\} \leq l_r(t)$ for almost all $t \in J$ and $\|x\|^2 \leq r$.

Lemma 2.8.([13],Lasota and Opial). Let J be a compact real interval , $BCC(H)$ be the set of all nonempty, bounded, closed and convex subset of H and Σ be a multivalued map $S_{\Sigma,x} \neq \emptyset$ and let Γ be a linear combination mapping from $\mathcal{L}^2(J, H)$ to $\mathcal{C}(J, H)$ then, the operator

$$\Gamma\, o S_\Sigma: \mathcal{C} \to \mathrm{BCC}(\mathcal{C}(\mathrm{J}, \mathrm{H})), x \to (\Gamma\, o S_\Sigma)(x) = \Gamma(S_{\Sigma,x}),$$

is a closed graph operator in $\mathcal{C} \times \mathcal{C}$, where $S_{\Sigma,x}$ is known as the selected operator set from Σ, is given by

$$\sigma \in S_{\Sigma,x} = \{\sigma \in \mathcal{L}^2\big(\mathcal{L}(K,H)\big) : \sigma(t) \in \Sigma(t,x) \text{ for a.e } t \in J\}.$$

Lemma 2.9(Bohnenblust-Karlin). Let $\mathcal{D}$ be a nonempty subset of H, which is bounded , closed and convex. Suppose $G: \mathcal{D} \to 2^H \backslash \{\emptyset\}$ is u.s.c with closed, convex values and such that $G(\mathcal{D})$ is compact. Then G has a fixed point.

Definition 2.10. A continuous H- valued process x is said to be a mild solution of (2.1)-(2.2) if

(i) $x(t)$ is $\mathcal{F}_t$- adapted and $\{x_t : t \in [0,b]\}$ is $\mathcal{B}_h$-valued.

(ii) for each $t \in J$, $x(t)$ satisfies the following integral equation:

$$x(t) = L^{-1}T(t)L\phi(0) + \int_0^t L^{-1}T(t-s)f(s,x_s)ds + \int_0^t L^{-1}T(t-s)Bu(s)ds$$
$$+ \int_0^t L^{-1}T(t-s)\Sigma(s,x_s)dw(s), t \in J$$

(iii) $x(t) = \phi(t)$ on $(-\infty, 0]$ satisfying $\|\phi\|^2_{\mathcal{B}_h} < \infty$.

3. APPROXIMATE CONTROLLABILITY RESULTS

In this section, we shall formulate and prove sufficient conditions for the approximate controllability for a class of stochastic differential inclusion of Sobolev type with unbounded delay of the form (2.1)-(2.2) by using Bohnenblust-Karlin fixed point theorem. First we prove the existence of solutions for the control system and then show that under certain assumptions, the approximate controllability of the stochastic control system (2.1)-(2.2) is implied by the approximate controllability of the associated linear part.

Definition 3.1. Let $x_b(\phi,u)$be the state value of (2.1)-(2.2) at the terminal time b corresponding to the control u and the initial value ϕ. Introduce the set

$$\mathcal{R}(b,\phi) = \{x_b(\phi;u)(0) : u(.) \in \mathcal{L}(J,U)\},$$

which is called the reachable set of (2.1)-(2.2) at the time b and its closure in H is denoted by $\overline{\mathcal{R}(b,\phi)}$. The system (2.1)-(2.2) is said to be approximately controllable on J if $\overline{\mathcal{R}(b,\phi)} = H$.

Inorder to study the approximate controllability of the system (2.1)-(2.2), we consider the linear system

$$\frac{d[Lx(t)]}{dt} \epsilon Ax(t) + Bu(t)\,, t \in [0, b] \qquad (3.1)$$

$$x(0) = \phi(t) \in \mathcal{B}_h \qquad (3.2)$$

It is convienent at this point to introduce the controllability and relevant operators associated with (3.1)-(3.2),

$$\gamma_0^b = \int_0^b L^{-1}T(b-s)BB^*L^{-1}T^*(b-s)ds: H \to H,$$

$$R(\alpha, \gamma_0^b) = (\alpha I + \gamma_0^b)^{-1}: H \to H$$

where B^*denotes the adjoint of B and $T^*(t)$ is the adjoint of $T(t)$. It is straight forward that the operator γ_0^b is a linear bounded operator.

In order to establish the result, we need the following hypotheses:

(H1)$T(t), t > 0$ is compact.

(H2) The function $F: J \times \mathcal{B}_h \to H$ satisfies the following:$F(\cdot, \psi): J \to H$ is measurable for

each $\psi \in \mathcal{B}_h$ and $F(t,\cdot): \mathcal{B}_h \to H$ is continuous for a.e $t \in J$ and for $\psi \in \mathcal{B}_h$,

$F(\cdot,\cdot): J \to H$ is strongly measurable and there exists a constant $M_f > 0$ such that

$$E\|F(t, \psi)\|^2 \leq M_f(\|\psi\|_{\mathcal{B}_h}^2).$$

$(H3)$ The multivalued map $\Sigma: J \times \mathcal{B}_h \to BCC(x)$ is an $\mathcal{L}^2$caratheodory function which

satisfies the following conditions:

(i) For each $t \in J$, the function $\Sigma(\text{t},.)$ is u.s.c and for $x \in \mathcal{B}_h$, the function $\Sigma(\cdot, \psi)$ is measurable. And for each fixed $\psi \in \mathcal{B}_h$, the set

$$S_{\Sigma,\text{x}} = \{\sigma \in \mathcal{L}^2(\mathcal{L}(K, H)): \sigma(t) \in \Sigma(t, \psi) \text{ for a.e } t \in J\}$$

is nonempty.

(ii) For each positive number r there exists a positive function $l_r: J \to \mathbb{R}^+$ such that

$$sup\{E\|\sigma\|^2 : \sigma(t) \in \Sigma(t,\psi)\} \le l_r(t) \text{ a.e } t \in J. \quad \text{and}$$

$$\lim_{r\to\infty} inf \frac{\int_0^t l_r(s)ds}{r} = \Lambda < \infty$$

Lemma 3.2. For any $\bar{x}_b \in \mathcal{L}^2(\mathcal{F}_b, H)$, there exists $\bar{\phi} \in \mathcal{L}_2^{\mathcal{F}}(\Omega, \mathcal{L}^2(J, \mathcal{L}(K,H)))$ such that $\bar{x}_b = E\bar{x}_b + \int_0^b \bar{\phi}(s)dw(s)$.

Now for any $\alpha > 0$, $\bar{x}_b \in \mathcal{L}^2(\mathcal{F}_b, H)$and for $\sigma \in S_{\Sigma,\psi}$, we define the control function

$$u_\alpha(t,x) = B^*L^{-1}T^*(b-s)R(\alpha,\gamma_0^b)\Big\{E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(t)L\phi(0) - \int_0^t L^{-1}T(t-s)f(s,y_s+\hat{\phi}_s)ds + \int_0^t L^{-1}T(t-s)\,\sigma(s)dw(s)ds\Big\}(s)ds$$

Theorem 3.3. Suppose that the hypotheses **(H1)-(H3)** are satisfied, then the system (2.1)-(2.2) has a mild solution on J provided that

$$12M_L^2M^2l^2b^2M_f\left[1+5\left(\frac{M_L^2M^2M_B^2}{\alpha}\right)^2\right] + 3M^2M_L^2Tr(Q)\Lambda\left[5\left(\frac{M^2M_L^2M_B^2}{\alpha}\right)^2+1\right] < 1$$

(3.3)

and where $\|B\| = M_B$.

Proof. For any $\epsilon > 0$, we consider the operator $\Phi^\epsilon : \mathcal{B}_h{}' \to 2^{\mathcal{B}_h{}'}$defined by $\Phi^\epsilon x$ the set of $z \in \mathcal{B}_h{}'$ such that

$$z(t) = \begin{cases} \phi(t) & t \in (-\infty, 0] \\ L^{-1}T(t)L\phi(0) + \int_0^t L^{-1}T(t-s)f(s,y_s+\hat{\phi}_s)ds + \int_0^t L^{-1}T(t-s)Bu_\alpha(s,y_s+\hat{\phi}_s)ds & \\ \quad + \int_0^t L^{-1}T(t-s)\sigma(s)dw(s)\,ds, & t \in J \end{cases}$$

where $\sigma \in S_{\Sigma,x}$. We shall show that the operator Φ^{ϵ}has a fixed point, which is then a solution of (2.1)-(2.2). Clearly $x_1 = x(b) \in (\Phi^{\epsilon}x)(b)$, which means that $u_{\alpha}(t,x)$ steers system (2.1)-(2.2) from $x_0$to x_b in finite time b. For $\phi \in \mathcal{B}_h$, we define $\hat{\phi}$by

$$\hat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0] \\ L^{-1}T(t)L\phi(0) & t \in J \end{cases}$$

then $\hat{\phi} \in \mathcal{B}_h^{'}$. Let $x(t) = y(t) + \hat{\phi}(t)$, $-\infty < t \leq b$. It is easy to see that y satisfies $y_0 = 0$ and

$$y(t) = \int_0^t L^{-1}T(t-s)f\big(s, y_s + \hat{\phi}_s\big)ds + \int_0^t L^{-1}T(t-s)Bu_{\alpha}\big(s, y_s + \hat{\phi}_s\big)ds$$
$$+\int_0^t L^{-1}T(t-s)\sigma(s)dw(s)\, ds\,,\ t \in J$$

if and only if x satisfies

$$x(t) = L^{-1}T(t)L\phi(0) + \int_0^t L^{-1}T(t-s)f\big(s, y_s + \hat{\phi}_s\big)ds$$
$$+ \int_0^t L^{-1}T(b-t)BB^{*}L^{-1}T^{*}(b-s)\ R\big(\alpha, \gamma_0^b\big)$$
$$\Big\{E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(t)L\phi(0) - \int_0^t L^{-1}T(t-s)f\big(s, y_s + \hat{\phi}_s\big)ds$$
$$+ \int_0^t L^{-1}T(t-s)\sigma(s)dw(s)\, ds\,\}\,(s)ds + \int_0^t L^{-1}T(t-s)\sigma(s)dw(s)ds,\ t \in J$$

and $x(t) = \phi(t), t \in (-\infty, 0]$.

Let $\mathcal{B}_h^{''} = \{y \in \mathcal{B}_h^{'}: y_0 = 0 \in \mathcal{B}_h\}$. For any $y \in \mathcal{B}_h^{''}$, we have

$$\begin{aligned} \|y\|_b &= \|y_0\|_{\mathcal{B}_h} + sup_{s\in[0,b]}\{(E|y(s)|^2): 0 \leq s \leq b\} \\ &= sup_{s\in[0,b]}\{(E|y(s)|^2): 0 \leq s \leq b\}. \end{aligned}$$

thus $(\mathcal{B}_h^{''}, \|\cdot\|_b)$ is a Banach space. Set $\mathfrak{B}_r = \{y \in \mathcal{B}_h^{''}: \|y\|_b \leq r\}$ for some $r > 0$, then $\mathfrak{B}_r \subset \mathcal{B}_h^{''}$ is a uniformly bounded and for $y \in \mathfrak{B}_r$,from Lemma 2.2 we have

$$\begin{aligned} \big\|y_t + \hat{\phi}_t\big\|_{\mathcal{B}_h}^2 &\leq 2\left(\|y_t\|_{\mathcal{B}_h}^2 + \big\|\hat{\phi}_t\big\|_{\mathcal{B}_h}^2\right) \\ &\leq 4\left(l^2 sup_{s\in[0,t]}(E\|y(s)\|^2) + \|y_0\|_{\mathcal{B}_h}^2 + l^2 sup_{s\in[0,t]}\left(E\big\|\hat{\phi}(s)\big\|^2\right) + \big\|\hat{\phi}_0\big\|_{\mathcal{B}_h}^2\right) \\ &\leq 4l^2(r + M^2E\|\phi(0)\|^2) + 4\|\phi\|_{\mathcal{B}_h}^2 \\ &\leq r' \end{aligned}$$

Define the multivalued map $\psi: \mathcal{B}_h^{''} \to 2^{\mathcal{B}_h^{''}}$defined by ψy the set of $\bar{z} \in \mathcal{B}_h^{''}$ and there exits $\sigma \in \mathcal{L}^2(\mathcal{L}(K,H))$ such that $\sigma \in S_{\Sigma,x}$ and

$$\bar{z}(t) = \begin{cases} 0 & t \in (-\infty, 0] \\ \int_0^t L^{-1}T(t-s)f\big(s, y_s + \hat{\phi}_s\big)ds \ + \int_0^t L^{-1}T(t-s)Bu_\alpha\big(s, y_s + \hat{\phi}_s\big)ds \\ + \int_0^t L^{-1}T(t-s)\sigma(s)dw(s)ds\,, & t \in J \end{cases}$$

Obviously, the operator Φ^ϵ has a fixed point if and only if ψ has a fixed point. So our aim is to show that ψ has a fixed point. For the sake of convience, we subdivide the proof into several steps.

Step 1:$\psi$is convex for each $x \in \mathfrak{B}_r$. In fact, if φ_1, φ_2belongs to $\psi(x)$, then there exists $\sigma_1, \sigma_2 \in S_{\Sigma,x}$ such that for each $t \in J$, we have

$$\varphi(t) = \int_0^t L^{-1}T(t-s)f\big(s, y_s + \hat{\phi}_s\big)ds + \int_0^t L^{-1}T(b-t)BB^*L^{-1}T^*(b-s)R\big(\alpha, \gamma_0^b\big)$$
$$\times\left\{E\bar{x}_b \ + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(t)L\phi(0) \ - \int_0^t L^{-1}T(t-s)f\big(s, y_s + \hat{\phi}_s\big)ds\right.$$
$$\left. + \int_0^t L^{-1}T(t-s)\ \sigma(s)dw(s)\ \ ds\right\}(s)ds + \int_0^t L^{-1}T(t-s)\ \sigma(s)dw(s)\, ds$$

Let $\lambda \in [0,1]$. Then for each $t \in J$, we get

$$(\lambda\varphi_1 + (1-\lambda)\varphi_2)(t)$$
$$= \int_0^t L^{-1}T(t-s)f\big(s, y_s + \hat{\phi}_s\big)ds$$
$$+ \int_0^t L^{-1}T(b-t)BB^*L^{-1}T^*(b-s)R\big(\alpha, \gamma_0^b\big)$$
$$\times\left\{E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(t)L\phi(0) - \int_0^t L^{-1}T(t-s)f\big(s, y_s + \hat{\phi}_s\big)ds\right.$$
$$\left. + \int_0^t L^{-1}T(t-s)[\lambda\sigma_1(s) + (1-\lambda)\sigma_2(s)]dw(s)\ \ ds\right\}(s)ds$$
$$+ \int_0^t L^{-1}T(t-s)[\lambda\sigma_1(s) + (1-\lambda)\sigma_2(s)]dw(s)\, ds$$

It is easy to see that $S_{\Sigma,x}$ is convex since Σ has convex values. So$\lambda\sigma_1(s) + (1-\lambda)\sigma_2(s) \in S_{\Sigma,x}$. Thus $\lambda\varphi_1 + (1-\lambda)\varphi_2 \in \psi(x)$.

Step 2: We show that there exist some $r > 0$such that $\psi(\mathfrak{B}_r) \subset \mathfrak{B}_r$. if it is not true, then there exists $\epsilon > 0$ such that for every positive number r and $t \in J$, there exists a function $y_r \in \mathfrak{B}_r$, but $\psi \in \mathfrak{B}_r$, that is $\|(\psi(y_r))(t)\|^2 \geq r$for some $t \in J$. For such $\epsilon > 0$,elementary inequality can show that

$$r < E\|(\psi y_r)(t)\|^2$$

$$\leq 3\left\{E\left\|\int_0^t L^{-1}T(t-s)f(s,y_s+\hat{\phi}_s)ds\right\|^2 + E\left\|\int_0^t L^{-1}T(t-s)Bu_\alpha(s,y_s+\hat{\phi}_s)ds\right\|^2\right.$$

$$\left. + E\left\|\int_0^t L^{-1}T(t-s)\sigma(s)dw(s)ds\right\|^2\right\}$$

$$\leq 3\left\{E\left\|\int_0^t L^{-1}T(t-s)f(s,y_s+\hat{\phi}_s)ds\right\|^2 + E\left\|\int_0^t L^{-1}T(t-s)BB^*T^*(b-s)R(\alpha,\gamma_0^b)\right.\right.$$

$$\times\left\{E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(t)L\phi(0) - \int_0^t L^{-1}T(t-s)f(s,y_s+\hat{\phi}_s)ds\right.$$

$$\left.\left. + \int_0^t L^{-1}T(t-s)\,\sigma(s)dw(s)\ ds\right\|^2\right\} + E\left\|\int_0^t L^{-1}T(t-s)\sigma(s)dw(s)ds\right\|^2$$

$$\leq 3M_L^2M^2b^2M_f(r') + 3\left\{5\left(\frac{M_L^2M^2M_B^2}{\alpha}\right)^2 \{2E\|\bar{x}_b\|^2 + 2\int_0^t E\left\|\bar{\phi}(s)dw(s)\right\|^2\right.$$

$$\left. M_L^2M^2E\|\phi(0)\|^2 + M_L^2M^2b^2M_f(r') + M_L^2M^2TrQ\int_0^t l_r(s)ds\right\}\}$$

$$+ 3\, M_L^2M^2TrQ\ \int_0^t l_r(s)ds$$

Dividing both sides of the above inequality by r and taking $r \to \infty$ we have

$$12M_L^2M^2l^2b^2M_f\left[1+5\left(\frac{M_L^2M^2M_B^2}{\alpha}\right)^2\right] + 3M^2M_L^2Tr(Q)\Lambda\left[5\left(\frac{M^2M_L^2M_B^2}{\alpha}\right)^2+1\right] \geq 1$$

which is a contradiction to our assumption. Hence, for some positive number $r > 0$ and some $\sigma \in S_{\Sigma,x}$, $\psi(\mathfrak{B}_r) \subset \mathfrak{B}_r$.

Step 3:$\psi(\mathfrak{B}_r)$ is equicontinuous. Indeed $\epsilon > 0$ be small, $0 \leq \tau_1 \leq \tau_2 \leq b$. for each $y \in \mathfrak{B}_r$ and $\bar{z}$belongs to $\psi_1 y$, there exists $\sigma \in S_{\Sigma,x}$ such that for each $t \in J$, we have

$$E\|\bar{z}(\tau_2) - \bar{z}(\tau_1)\|^2 = 9\left\{E\left\|\int_{\tau_1}^{\tau_2} L^{-1}T(\tau_2 - s)f(s,y_s+\hat{\phi}_s)ds\right\|^2\right.$$

$$+ E\left\|\int_{\tau_1-\epsilon}^{\tau_1} L^{-1}[T(\tau_2 - s) - T(\tau_1 - s)]f(s,y_s+\hat{\phi}_s)ds\right\|^2$$

$$+ E\left\|\int_0^{\tau_1-\epsilon} L^{-1}[T(\tau_2 - s) - T(\tau_1 - s)f(s,y_s+\hat{\phi}_s)ds\right\|^2$$

$$+ E\left\|\int_{\tau_1}^{\tau_2} L^{-1}T(\tau_2 - s)Bu_\alpha^r(\eta, s)d\eta ds\right\|^2$$

$$+ E\left\|\int_{\tau_1-\epsilon}^{\tau_1} L^{-1}[T(\tau_2 - s) - T(\tau_1 - s)]Bu_\alpha^r(\eta, s)d\eta ds\right\|^2$$

$$+ E\left\|\int_0^{\tau_1-\epsilon} L^{-1}[T(\tau_2 - s) - T(\tau_1 - s)]Bu_\alpha^r(\eta, s)d\eta ds\right\|^2$$

$$+ E\left\|\int_{\tau_1}^{\tau_2} L^{-1}T(\tau_2 - s)\sigma(s)dw(s)ds\right\|^2$$

$$+ E\left\|\int_{\tau_1-\epsilon}^{\tau_1} L^{-1}[T(\tau_2 - s) - T(\tau_1 - s)]\sigma(s)dw(s)ds\right\|^2$$

$$+ E\left\|\int_0^{\tau_1-\epsilon} L^{-1}\big[T\big(\tau_2 - s\big) - T(\tau_1 - s)\big]\sigma(s)dw(s)ds\right\|^2\Bigg\}$$

$$\leq 9\Bigg\{M^2M_L^2\int_{\tau_1}^{\tau_2} E\left\|f\big(s, y_s + \hat{\phi}_s\big)\right\|^2 ds$$

$$+ M_L^2\int_{\tau_1-\epsilon}^{\tau_1}\|T(\tau_2 - s) - T(\tau_1 - s)\|^2 E\left\|f\big(s, y_s + \hat{\phi}_s\big)\right\|^2 ds$$

$$+ M_L^2\int_0^{\tau_1-\epsilon}\|T(\tau_2 - s) - T(\tau_1 - s)\|^2 E\left\|f\big(s, y_s + \hat{\phi}_s\big)\right\|^2 ds$$

$$+ M^2M_L^2\int_{\tau_1}^{\tau_2}\|Bu_\alpha^r(\eta, s)\|^2 d\eta ds$$

$$+ M_L^2\int_{\tau_1-\epsilon}^{\tau_1}\left\|T\big(\tau_2 - s\big) - T(\tau_1 - s)\right\|^2 E\|Bu_\alpha^r(\eta, s)\|^2 d\eta\, ds$$

$$+ M_L^2\int_0^{\tau_1-\epsilon}\|T(\tau_2 - s) - T(\tau_1 - s)\|^2 E\|Bu_\alpha^r(\eta, s)\|^2 d\eta\, ds$$

$$+ M^2M_L^2\int_{\tau_1}^{\tau_2} Tr(Q)E\|\sigma(s)dw(s)\|^2 ds$$

$$+ M_L^2\int_{\tau_1-\epsilon}^{\tau_1}\|T(\tau_2 - s) - T(\tau_1 - s)\|^2 Tr(Q)E\|\sigma(s)dw(s)\|^2 ds$$

$$+ M_L^2\int_0^{\tau_1-\epsilon}\|T(\tau_2 - s) - T(\tau_1 - s)\|^2 Tr(Q)E\|\sigma(s)dw(s)\|^2 ds\Bigg\}$$

Therefore for ϵ sufficiently small, we can verify that the right-hand side of the above inequality tends to zero as $\tau_2 \to \tau_1$. On the otherhand, the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Thus ψ maps $\mathfrak{B}_r$ into equicontinuous family of functions.

Step 4: The set $\prod(t) = \{\varphi(t): \varphi \in \psi(\mathfrak{B}_r)\}$ is relatively compact in H.

Let $t \in [0, b]$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $x \in \mathfrak{B}_r$, we define

$$\varphi_\epsilon(t) = \int_0^{t-\epsilon} L^{-1}T(t-s)f(s, y_s + \hat{\phi}_s)ds + \int_0^{t-\epsilon} L^{-1}T(b-t)BB^*L^{-1}T^*(b-s)R(\alpha, \gamma_0^b)$$

$$\times\left\{E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(t)L\phi(0) - \int_0^t L^{-1}T(t-s)f(s, y_s + \hat{\phi}_s)ds\right.$$

$$\left. + \int_0^t L^{-1}T(t-s)\sigma(s)dw(s)\, ds\right\}(s)ds + \int_0^{t-\epsilon} L^{-1}T(t-s)\sigma(s)dw(s)\, ds$$

$$\varphi_\epsilon(t) = T(\epsilon)\int_0^{t-\epsilon} L^{-1}T(t-s-\epsilon)f(s, y_s + \hat{\phi}_s)$$

$$+ T(\epsilon)\int_0^{t-\epsilon} L^{-1}T(t-s-\epsilon)BB^*L^{-1}T^*(b-s)R(\alpha, \gamma_0^b)$$

$$\times\left\{E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(t)L\phi(0) - \int_0^b L^{-1}T(t-s)f(s, y_s + \hat{\phi}_s)ds\right.$$

$$\left. + \int_0^b L^{-1}T(t-s)\sigma(s)dw(s)\, ds\right\}(s)ds + T(\epsilon)\int_0^{t-\epsilon} L^{-1}T(t-s-\epsilon)\sigma(s)dw(s)\, ds$$

for $\sigma \in S_{\Sigma,x}$. Since$T(t)$ is a compact operator, the set $\prod_\epsilon(t) = \{\varphi_\epsilon(t) : \varphi_\epsilon \in \psi(\mathfrak{B}_r)\}$ is relatively compact in H for each $\epsilon, 0 < \epsilon < t$. Moreover, for each $0 < \epsilon < t$, we have

$$E\|\varphi(t) - \varphi_\epsilon(t)\|$$

$$\leq \|\int_{t-\epsilon}^t L^{-1}T(t-s)f(s, y_s + \hat{\phi}_s)ds\|^2$$

$$+ E\|\int_{t-\epsilon}^t L^{-1}T(t-s)BB^*T^*(b-s)R(\alpha, \gamma_0^b)$$

$$\times\left\{E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(t)L\phi(0) - \int_0^t L^{-1}T(t-s)f(s, y_s + \hat{\phi}_s)ds\right.$$

$$\left. + \int_0^t L^{-1}T(t-s)\, \sigma(s)dw(s)\, ds\|^2\right\}(s)ds + \|\int_{t-\epsilon}^t L^{-1}T(t-s)\sigma(s)dw(s)ds\|^2$$

Therefore

$$\|\varphi(t) - \varphi_\epsilon(t)\| \to 0 \text{ as } \alpha \to 0^+$$

Hence there exists relatively compact sets arbitrarily close to the set $\prod(t) = \{\varphi(t) : \varphi \in \psi(\mathfrak{B}_r)\}$ and the set $\widetilde{\prod}(t)$ is relatively compact in H for all $t \in [0, b]$. Since it is compact at $t = 0$, hence $\prod(t)$ is relatively compact in H for all $t \in [0, b]$.

Step 5 . ψ has a closed graph. Let $y_n \to y_*$as $n \to \infty$, $\bar{z}_n \in \psi y_n$for each $y_n \in \mathfrak{B}_r$ and $\bar{z}_n \to \bar{z}_*$as $n \to \infty$. We shall show that $\bar{z}_* \in \psi y_*$. Since $\bar{z}_n \in \psi y_n$, there exists a $\sigma_n \in$

S_{Σ,y_n} such that

$$\bar{z}_n(t) = \int_0^t L^{-1}T(t-s)f\big(s,(y_n)_s+\hat{\phi}_s\big)ds + \int_0^t L^{-1}T(b-t)BB^*L^{-1}T^*(b-s)R\big(\alpha,\gamma_0^b\big) \times\Big\{E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(b)L\phi(0) - \int_0^t L^{-1}T(t-s)f\big(s,(y_n)_s+\hat{\phi}_s\big)ds + \int_0^t L^{-1}T(t-s)\sigma_n(s)dw(s)\ ds\Big\}(s)ds + \int_0^t L^{-1}T(t-s)\sigma_n(s)dw(s)\,ds, \quad t\in J$$

we must prove that there exists $\sigma_* \in S_{\Sigma,y_*}$ such that

$$\bar{z}_*(t) = \int_0^t L^{-1}T(t-s)f\big(s,(y_*)_s+\hat{\phi}_s\big)ds + \int_0^t L^{-1}T(b-t)BB^*L^{-1}T^*(b-s)R\big(\alpha,\gamma_0^b\big) \times\Big\{E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(b)L\phi(0) - \int_0^t L^{-1}T(t-s)f\big(s,(y_*)_s+\hat{\phi}_s\big)ds + \int_0^t L^{-1}T(t-s)\sigma_*(s)dw(s)\,ds\Big\}(s)ds + \int_0^t L^{-1}T(t-s)\sigma_*(s)dw(s)\,ds, \quad t\in J$$

Now, for every $t\in J$, since g is continuous and the from the definition of u^ϵ we get

$$\|(\bar{z}_n(t) - \int_0^t L^{-1}T(t-s)f\big(s,(y_n)_s+\hat{\phi}_s\big)ds + \int_0^t L^{-1}T(b-t)BB^*L^{-1}T^*(b-s)R\big(\alpha,\gamma_0^b\big) \times\Big\{E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(b)L\phi(0) - \int_0^t L^{-1}T(t-s)f\big(s,(y_n)_s+\hat{\phi}_s\big)ds + \int_0^t L^{-1}T(t-s)\sigma_n(s)dw(s)\,ds\Big\}(s)ds + \int_0^t L^{-1}T(t-s)\sigma_n(s)dw(s)\,ds\Big) - \Big(\bar{z}_*(t) - \int_0^t L^{-1}T(t-s)f\big(s,(y_*)_s+\hat{\phi}_s\big)ds + \int_0^t L^{-1}T(b-t)BB^*L^{-1}T^*(b-s)R\big(\alpha,\gamma_0^b\big)$$

$$\times \left\{ E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(b)L\phi(0) - \int_0^t L^{-1}T(t-s)f\big(s,(y_*)_s + \hat{\phi}_s\big)ds \right.$$

$$\left. + \int_0^t L^{-1}T(t-s)\sigma_*(s)dw(s)\,ds \right\}(s)ds + \int_0^t L^{-1}T(t-s)\sigma_*(s)dw(s)\,ds \Big)\|^2 \to 0 \; as \; n \to \infty.$$

Consider the linear continuous operator $\kappa\colon \mathcal{L}^1(J\times H) \to \mathcal{C}(J\times H)$,

$$(\kappa\sigma)(t) = \int_0^t L^{-1}T(t-s)\sigma(s)dw(s)\,ds - \int_0^t L^{-1}T(b-t)BB^*L^{-1}T^*(b-s)R\big(\alpha,\gamma_0^b\big)\int_0^t L^{-1}T(t-s)\sigma(s)dw(s)\,ds$$

From Lemma2.8 it follows that $\kappa o S_\Sigma$ is a closed graph operator. Also, from the definition of κ, we have that

$$\Big(\bar{z}_n(t) - \int_0^t L^{-1}T(t-s)f\big(s,(y_n)_s + \hat{\phi}_s\big)ds + \int_0^t L^{-1}T(b-t)BB^*L^{-1}T^*(b-s)R\big(\alpha,\gamma_0^b\big)$$

$$\times \left\{ E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(b)L\phi(0) - \int_0^t L^{-1}T(t-s)f\big(s,(y_n)_s + \hat{\phi}_s\big)ds \right.$$

$$\left. + \int_0^t L^{-1}T(t-s)\sigma_n(s)dw(s)\;ds \right\}(s)ds + \int_0^t L^{-1}T(t-s)\sigma_n(s)dw(s)\,ds \Big) \in \kappa(S_{\Sigma,\mathrm{y_n}})$$

Since $y_n \to y_*$for some $y_* \in S_{\Sigma,y_*}$, it follows from Lemma 2.8 that

$$\Big(\bar{z}_*(t) - \int_0^t L^{-1}T(t-s)f\big(s,(y_*)_s + \hat{\phi}_s\big)ds + \int_0^t L^{-1}T(b-t)BB^*L^{-1}T^*(b-s)R\big(\alpha,\gamma_0^b\big)$$

$$\times \left\{ E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(b)L\phi(0) - \int_0^t L^{-1}T(t-s)f\big(s,(y_*)_s + \hat{\phi}_s\big)ds \right.$$

$$\left. + \int_0^t L^{-1}T(t-s)\sigma_*(s)dw(s)\,ds \right\}(s)ds + \int_0^t L^{-1}T(t-s)\sigma_*(s)dw(s)\,ds \Big)$$

for some $\sigma_* \in S_{\Sigma,y_*}$. Therefore ψ has a closed graph.

As a consequences of **step 1** to **step 5** together with the Arzela-Ascoli theorem, we conclude that ψ is a compact multivalued map, u.s.c with convex closed values. As a consequences of Lemma 2.8, we can deduce that ψ has a fixed point x which is a mild solution of (2.1)-(2.2).

Further, in order to prove the approximate controllability result, the following additional assumption is required.

(H4) The linear inclusion (3.3)-(3.4) is approximately controllable.

(H5) $\alpha R(\alpha,\gamma_0^b) = \alpha(\alpha I + \gamma_0^b)^{-1} \to 0$ as $\alpha \to 0^+$ in the strong operator topology.

Theorem 3.4. Assume that the assumption of Theorem 3.3 hold and in addition, hypothesis (H1)-(H5) are satisfied and then the nonlinear stochastic differential inclusion (2.1)-(2.2) is approximately controllable on J.

Proof. Let $\hat{x}^\alpha(\cdot)$ be a fixed point of Φ^ϵ in $\mathfrak{B}_r$. By Theorem 3.3 any fixed point $\Phi^\epsilon$is a mild solution of (2.1)-(2.2) under the control

$$\hat{x}^\alpha(b) = x_b - R(\alpha,\gamma_0^b)$$

$$\times\left\{E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(t)L\phi(0) - \int_0^t L^{-1}T(t-s)f(s,y_s+\hat{\phi}_s)ds\right.$$

$$\left. + \int_0^t L^{-1}T(t-s)\sigma(s)dw(s)\, ds\right\}$$

Moreover by assumption on $\sigma$and Dunford-Pettis Theorem, we have that the $\{\sigma^\alpha(s)\}$ is weakly compact in $\mathcal{L}(J,H)$, so there is a subsequence, still denoted by $\{\sigma^\alpha(s)\}$, that converges weakly to $\sigma(s)$ say in $\mathcal{L}^1(J,H)$. Now we have

$$E\|\hat{x}^\alpha(b) - x_b\|^2 = 5E\|R(\alpha,\gamma_0^b) \times \left\{E\bar{x}_b + \int_0^t \bar{\phi}(s)dw(s) - L^{-1}T(t)L\phi(0)\right.$$

$$\left. - \int_0^t L^{-1}T(t-s)f(s,y_s+\hat{\phi}_s)ds - \int_0^t L^{-1}T(t-s)\sigma(s)dw(s)\, ds\right\}\|^2$$

for $0 \le s \le b$ the operator $\alpha(\alpha I + \gamma_0^b)^{-1} \to$ strongly as $\alpha \to 0^+$ and moreover$\|\alpha(\alpha I + \gamma_0^b)^{-1}\| \le 1$. It follows from Lebesgue dominated convergence theorem and the compactnesss of $T(t)$ that $E\|\hat{x}^\alpha(b) - x_b\|^2 \to 0$ as$\alpha \to 0^+$. This proves the approximate controllability of the differential inclusion (2.1)-(2.2).

4. AN APPLICATION:

Consider a control system of stochastic differential inclusion with unbounded delay of the form

$$\frac{\partial}{\partial t}[z(t,x)] \epsilon\, z_{xx}(t,x) + F(t, z(t-r), x)ds + \mu(t,x)$$

$$+G(t, y(t-r), x)dw(t), t \in [0,1], r > 0, x \in [0,1] \quad (4.1)$$

$$z(t,0) = z(t,1) = 0, 0 \le t \le 1 \quad (4.2)$$

$$z(t,x) = \phi(t,x), 0 \le x \le 1, -\infty \le t \le 0 (4.3)$$

where $w(t)$ denotes a standard cylindrical wiener process in Hdefined on a stochastic process $(\omega, \mathcal{F})$ and $H = K = \mathcal{L}^2([0,1])$. Define the operators $A: D(A) \subset H \to H$ and $L: D(L) \subset H \to H$ by $Ay = -y''$ and $Ly = y - y'$, where each domain $D(A)$ and $D(L)$ is given by $\{y \in H, y, y'$ are absolutely continuous $y'' \in H, y(0) = y(1) = 0\}$. Further A and L can be witten as $Ay = \sum_{n=1}^{\infty} n^2 \langle y, z_n \rangle z_n, y \in D(A)\,, Ly = \sum_{n=1}^{\infty}(1+n^2)\langle y, z_n\rangle z_n, y \in D(L),$ where $z_n(x) = \sqrt{\frac{2}{\pi}} \sin nx, n = 1,2,3, \ldots$is the orthogonal set of vectors of A. Also for $z \in H$, we have

$$L^{-1}z = \sum_{n=1}^{\infty} \frac{1}{1+n^2} \langle z, z_n \rangle z_n$$

and

$$AL^{-1}z = \sum_{n=1}^{\infty} \frac{n^2}{1+n^2} \langle z, z_n \rangle z_n$$

and

$$T(t)z = \sum_{n=1}^{\infty} exp \frac{n^2 t}{1+n^2} \langle z, z_n \rangle z_n$$

Further, we consider the phase space $\mathcal{B}_h$, with norm

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} g(s) \sup_{s \le \theta \le 0} (\mathrm{E}\|\phi(t)\|^2)^{1/2} \mathrm{ds}$$

where $g(s) = e^{2s}, s < 0$ and $\int_{-\infty}^{0} g(s)ds = \frac{1}{2}$. Let $z(t)(x) = z(t,x)$. Define the function $F: J \times \mathcal{B}_h \to H$ and$\Sigma: J \times \mathcal{B}_h \to \mathcal{L}_Q^0$by $F(t,z)(\cdot) = F(t, z(\cdot)), \Sigma(\mathrm{t}, \mathrm{y}(\cdot)) = \mathrm{G}(\mathrm{t}, \mathrm{y}(\cdot))$ and the bounded linear operator $Bu(t)(x) = \mu(t,x)$ respectively.

Moreover, it can be easily seen that AL^{-1} is compact and bounded with $\|L^{-1}\| \leq 1$ and AL^{-1} generates a strongly continuous semigroup $T(t), t \geq 0$ with $\|T(t)\| \leq e^{-t} \leq 1$.

Thus with the above choices (4.1)-(4.3) can be written in the abstract from of (2.1)-(2.2). Further, we can impose some suitable conditions on the above defined functions to verify the assumptions on Theorem 3.4, we can conclude that (4.1)-(4.3) is approximately controllable on $[0, b]$.

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