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Time series models with Pearson type III marginal

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Abstract

The Pearson type 3 distribution also called three-parameter gamma distribution is widely used in various scientific fields such as in reliability and hydrology. We focused our investigations on its applications in hydrologic frequency analysis for modelling annuals maximum flows. Two stationary first order Markov AR(1) time series models are proposed for the case where the observation or response variable comes from Pearson type 3 distribution. Both processes are constructed by means of a thinning operation. Basic properties of the proposed models are studied and fitting methods are suggested for parameters' estimation.

Keywords: Non gaussian, Time series modelling, First order autoregressive model, Estimation, Pearson type 3 model, Three-parameter gamma model.

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1 INTRODUCTION

In the classic version of the time series analysis it is usual to assume that the marginal distribution is Gaussian. However this assumption is frequently set to default. This is the case for example when it comes to analyzing sets of data in fields such as hydrology, meteorology, information theory or economics[4]. To deal with this problem, severall athors including Box & Cox[1], Lettenmaier & Burges[3] and Stedinger & Taylor [5] have suggested some kind of transformations of the original serie, with the risk of altering its structure.

In recent years, autoregressive models of very varied non-Gaussian margins have been proposed in the literature,including time series models with exponential or mixture exponential margins[2, 11, 10], weibull margins[14] as well as standard gamma margins[2, 14, 6, 7, 8, 13]. Obeysekera & Yevjevich[12] report a procedure for generating series from an autoregressive scheme of gamma margin. In their recent work, Fernandez & Salas[6] develop an first order autoregressive model with gamma margin(GAR(1)). The bias of the estimators obtained with the moment method being important, they suggest the use of a kind of correction before fitting the model. Applied to annual flow series, they claim that the GAR (1) model is more suitable than the previous to generate synthetic flow data. Tiku & al[15] propose an AR(1) model with error terms distributed according to a gamma model. They suggested the modified maximum likelihood (ML) method to estimate model parameters.

The models proposed in this paper belongs to the class of doubly stochastic autoregressive models studied by Tjostheim[16]. A general representation of this type of model is formulated as follows:

(1) $X_t = A_t X_{t-1} + \varepsilon_t,\ t \in \mathbb{Z}$

where $(\varepsilon_t)_{t\in\mathbb{Z}}$ and $(A_t)_{t\in\mathbb{Z}}$ are stochastic processes.

Previously, a detailed review of the application fields of this type of model was discussed by Vervaat[17] and later by Lewis & Lawrence[10]. General conditions for the existence of stationary solutions of the equation(1) were given by Mohsen Pourahmadi[13]. It reveals that a judicious choice of sequences $(\varepsilon_t)_{t\in\mathbb{Z}}$ and $(A_t)_{t\in\mathbb{Z}}$ could lead to stationary solutions of non-Gaussian margins. Specific cases of gamma, exponential or hyper-exponential margins were discussed. Joe Harry[8] offers autoregressive construction scheme whose margins belong in the convolution-closed infinitely divisible class. His scheme is based on a stochastic operator which he has termed the thinning operator of the marginal distribution.

The main contribution of this paper is to provide two first order autoregressive models(AR(1)) with Pearson type 3 marginal by means of the thinning operator. Innovations terms of those models are distributed according to a standard gamma or Pearson type 3 distribution. Basic Properties of these models are also proved. The paper is organized as follows: Section 2 gives details on convex and linear thinning operation in the case of Pearson type 3 distribution. Section 3 describes both models and their form of representation. Basic properties of those models are given in the next section. The later section is reserved to discuss fitting methods for the proposed models.

2 CONVEX AND LINEAR THINNING OPERATION ON PEARSON TYPE 3 FAMILY OF DISTRIBUTIONS

In this section, we aim to define different thinning operations in the case of Pearson type 3 distribution which is generally indexed by three parameters. Specifically, a random variable X is distributed according to a Pearson type 3 distribution with parameters ν, β and $\lambda_1$if its probability density function(p.d.f) is given by:

$$(2) \quad f(x,\nu,\beta,\lambda_1) = \frac{(x-\nu)^{\lambda_1}}{\beta^{\lambda_1}\Gamma(\lambda_1)}\exp(-\frac{x-\nu}{\beta})$$

where $\nu < x, \beta, \lambda_1 > 0$ design location, scale and shape parameter respectively.

When the location parameter ν is set to zero, standard gamma distribution is obtained. In the next sections, notation $X \to P_3(\nu, \beta, \lambda_1)$ will be used to denote random variable(r.v) distributed according to Pearson type 3 distribution while $P_3(0, \beta, \lambda_2)$ denote standard gamma distribution with scale parameter β and shape parameter λ_2.

Now, let us present some necessary results to derive the thinning operation on Pearson type 3 distribution.

2.1 Convolution

Lemma 2.1 :Let S be a r.v distributed according to a beta distribution with shape parameter λ_1 and scale parameter λ_2. Then the density function of the random variable $\eta S + \delta$ is given by:

$$(3) \quad f(s|\lambda_1,\lambda_2) = \frac{1}{\eta}\frac{\Gamma(\lambda_1+\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}(\frac{s-\delta}{\eta})^{\lambda_1-1}(1-\frac{s-\delta}{\eta})^{\lambda_2-1}$$

Proof : Let $F(.|\lambda_1, \lambda_2)$ be the cumulative density function(c.d.f) of the random variable $\eta S + \delta$ one has:

$$F(s|\lambda_1,\lambda_2) = F_s(\frac{s-\delta}{\eta}|\lambda_1,\lambda_2),$$

where $F_S(.|\lambda_1, \lambda_2)$denote the c.d.f of the random variable S. One derive the density function of $\eta S + \delta$ as follows.

$$f(s|\lambda_1,\lambda_2) = \frac{1}{\eta} f_s\left(\frac{s-\delta}{\eta}\right) = \frac{1}{\eta}\frac{\Gamma(\lambda_1+\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}\left(\frac{s-\delta}{\eta}\right)^{\lambda_1-1}\left(1-\frac{s-\delta}{\eta}\right)^{\lambda_2-1},$$

where the notation $f_S(.|\lambda_1, \lambda 2)$ designs the p.d.f of S.

Lemma 2.2 : Let $X \rightarrow P_3(\nu, \beta, \lambda_1)$ and $Y \rightarrow P_3(0, \beta, \lambda_2)$ and assume they are independent. Let also $Z = X + Y$ denote their sum. The following statements hold:

1. The random variable Z is distributed according to the Pearson type 3 distribution with location parameter ν, scale parameter β and shape parameter $\lambda = \lambda_1 + \lambda_2$.
2. Let $Beta(\lambda_i, \lambda_j)$ denote r.v distributed according to Beta distribution with parameters λ_i and λ_j.

   (a) The distribution of X given $Z = z$ is the same as that of the random variable $(z-\nu)Beta(\lambda_1, \lambda_2) + \nu$.

   (b) The distribution of Y given $Z = z$ is the same as that of the random variable $(z-\nu)Beta(\lambda_2, \lambda_1)$.

Proof :

1. Let $W \rightarrow P_3(\nu, \beta, \lambda)$ and $\varphi_W(t)$ its p.d.f Laplace Transformation(LT). It is straightforward to show that $\varphi_W(t) = ((\beta)/(\beta + t))^\lambda \exp\{(-\nu t)\}$. Since X and Y are independent, the LT of Z p.d.f, noted $\varphi_Z(t)$ is given by $\varphi_z(t) = \varphi_x(t)\varphi_y(t) = (\frac{\beta}{\beta+t})^{\lambda_1+\lambda_2}\exp(-\nu t)$, where $\varphi_X(.)$ and $\varphi_Y(.)$ denote LT of p.d.f of X and Y respectively. Hence, the result.
2.

   (a) Let f_X and f_Y denote the p.d.f of the random variables X , Y respectively and $f_{X,Y}$ their joint p.d.f. If $f_{X|Z}$ is the conditional density function of X given $Z = z$, one has:

$$f_{X|Z}(x|\nu,\lambda_1,\lambda_2,z) = \frac{f_{X,Y}(x, z-x)}{f_Z(z)} = \frac{\frac{(x-\nu)^{\lambda_1-1}\exp(\frac{x-\nu}{\beta})}{\beta^{\lambda_1}\Gamma(\lambda_1)}\,\frac{(z-x)^{\lambda_2-1}\exp(\frac{z-x}{\beta})}{\beta^{\lambda_2}\Gamma(\lambda_2)}}{\frac{(z-\nu)^{\lambda_1+\lambda_2-1}\exp(\frac{z-\nu}{\beta})}{\beta^{\lambda_1+\lambda_2}\Gamma(\lambda_1+\lambda_2)}}$$

   As a result

$$f_{X|Z}(x|\nu,\lambda_1,\lambda_2,z) = \frac{1}{z-\nu}\frac{\Gamma(\lambda_1+\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}\left(\frac{x-\nu}{z-\nu}\right)^{\lambda_1-1}\left(1-\frac{x-\nu}{z-\nu}\right)^{\lambda_2-1}$$

which is the p.d.f of the random variable $(z-v)S+v$ with $S \backsim Beta(\lambda_1, \lambda_2)$, from lemma(2.1).

(b) Let $f_{Y|Z}$ denote the conditional density of Y given X. Proceeding similarly as above, it follows that

$$f_{X|Z}(x|v,\lambda_1,\lambda_2,z) = \frac{1}{z-v}\frac{\Gamma(\lambda_1+\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)}(\frac{x}{z-v})^{\lambda_2-1}(1-\frac{x}{z-v})^{\lambda_1-1}$$

which is the p.d.f of the random variable $(z-v)S$ with $S \backsim Beta(\lambda_2, \lambda_1)$ (see lemma2.1).

2.2 Convex and linear thinning operation

In proposition below we show that any random variable Z distributed according to Pearson type 3 model can be decomposed in two manner. Let $Z \to P_3(v, \beta, \lambda)$ and

Proposition 2.1: Let choose λ_1, λ_2 such that $\lambda_1 + \lambda_2 = \lambda$. Let also T be a r.v.

1. If $(T|Z=z) \to (z-v)Beta(\lambda_1, \lambda_2)+v$, then the following assertions hold: T and $Z-T$ are independent random variables; $T \to P_3(v, \beta, \lambda_1)$; $Z-T \to P_3(0, \beta, \lambda_2)$
2. If $(T|Z=z) \to (z-v)Beta(\lambda_2, \lambda_1)$, then the following assertions hold: T and $Z-T$ are independent random variables; $T \to P_3(0, \beta, \lambda_2)$; $Z-T \to P_3(v, \beta,\lambda_1)$

Proof:

1. Assume that X and Y are independent with distributions $P_3(v, \beta, \lambda_1)$ and $P_3(0, \beta, \lambda_2)$ respectively, and let $W=X+Y$ denote their sum. Then, in light of lemma 2.1↑, the conditional distribution of $X|W=w$ and the marginal distribution of W are the same as the distribution of $T|Z$ and the marginal distribution of Z respectively. Consequently, the joint distribution of (X, W) is the same as the joint distribution of (T, Z), which in turn implies that the joint distribution of (X, Y) is the same as the joint distribution of $(T, Z-T)$, hence the results.
2. With the same conditions and notations as in the proof of 1. By using lemma 2.1↑, it follows that the conditional distribution of $Y|W=w$ is the same as the distribution of $T|Z$. The rest is similarly to 1.

In light of the previous results, one can define thinning operations on Pearson type 3 distributions as follows.

Definition 2.1: Let Z be a random variable distributed according to the Pearson type 3 distribution with location parameter ν, scale parameter β and shape parameter λ. Let $\alpha \in]0, 1[$;

1. The random variable $R^c(Z, \alpha)$ is called convex thinning of Z if its conditional distribution given $Z = z$ is the same as that of $(z - \nu)S + \nu$ where S is distributed according to Beta distribution with parameters $\alpha\lambda$, and $(1 - \alpha)\lambda$.
2. The random variable $R^l(Z, \alpha)$ is called linear thinning of Z if its conditional distribution given $Z = z$ is the same as that of $(z - \nu)S$ where S is distributed according to Beta distribution with parameters $(1 - \alpha)\lambda$, and $\alpha\lambda$.

From definition above, one can now formulate time series models with Pearson type 3 margin.

3 AUTOREGRESSIVE MODELS FOR TIME SERIES WITH PEARSON TYPE 3 MARGIN

The special case of stationary model with gamma margin has already been studied by many authors including [2, 14, 6, 7, 8, 13]. In this paper, we assume that the margins are distributed according to a Pearson type 3 distribution. Then, in light of proposition (*2.1*), the next formulation of the models holds:

Definition 3.1: Let $(X_t)_{t \in \mathbb{N}^*}$ denoted a time series with Pearson type 3 marginal distribution. $(X_t)_{t \in \mathbb{N}^*}$ is a first order autoregressive process if the following statements holds:

1. X_1 is a random variable distributed according to a Pearson type 3 distribution with location parameter ν, scale parameter β and shape parameter λ.
2. $(X_t)_{t \in \mathbb{N}^*, t > 1}$ satisfies one of the following recursive equations

   (4) $X_t = R_t^c(X_{t-1}, \alpha) + E_t^c$

   or

   (5) $X_t = R_t^l(X_{t-1}, \alpha) + E_t^l$

where

1. $\alpha \in]0, 1[$
2. $(R_t^c(X_{t-1}, \alpha))_{t \in \mathbb{N}^\star, t > 1}$ is a sequence of independent convex thinning operations distributed according to Pearson type 3 distribution with location parameter ν, scale parameter β and shape parameter $\alpha\lambda$;
3. $(E_t^c)_{t \in \mathbb{N}^\star, t > 1}$ is a sequence of random variables independent and identically

distributed according to gamma distribution with scale parameter β and shape parameter $(1-\alpha)\lambda$;

4. $(R_t^l(X_{t-1},\alpha))_{t\in\mathbb{N}^\star,\,t>1}$ is a sequence of independent linear thinning operations distributed according to gamma distribution with scale parameter β and shape parameter $(1-\alpha)\lambda$;
5. $(E_t^l)_{t\in\mathbb{N}^\star,\,t>1}$ is a sequence of random variables independent and identically distributed according to Pearson type 3 distribution with location parameter ν, scale parameter β and shape parameter $\alpha\lambda$;
6. The two terms on the right-hand side in each equation (4↑,5↑)are independent.

From definition2.1 and definition3.

1, one can derive others representations for model 1(4) and model 2 (6) setting above. Indeed, note that the conditional distribution of $R_t^c(X_{t-1},\alpha)$ given $X_{t-1}=x_{t-1}$ is the same as that of $(x_{t-1}-\nu)S_t+\nu$ where $(S_t)_{t\in\mathbb{N}^\star}$ are independent and identically distributed according to Beta distribution with parameters $\alpha\lambda$, and $(1-\alpha)\lambda$. Consequently, model 1 (4↑) could be represented as

(6) $$X_t = S_t(X_{t-1}-\nu)+\nu+E_t^c$$

where $S_t, X_{t-1}, E_t^c$are supposed to be independents. Similarly, the conditional distribution of $R_t^l(X_{t-1},\alpha)$ given $X_{t-1}=x_{t-1}$ is the same as that of $(x_{t-1}-\nu)S_t$ where S_t are independent and identically distributed according to Beta distribution with parameters $(1-\alpha)\lambda$ and $\alpha\lambda$. Thus, the second model given by equation(5) also could be represented as

(7) $$X_t = S_t(X_{t-1}-\nu)+\nu+E_t^l$$

with S_t, X_{t-1}, E_t^l being independents.

4 BASIC PROPERTIES OF THE MODELS

4.1 First and second order properties

Proposition 4.1 : Let $(X_t)_{t\in\mathbb{N}^\star}$ denote a time series that satisfies equation (6). The following assertions hold.

1. $\forall j\in\mathbb{N}^\star$, $E(X_{t+j}|X_t)=\alpha^j X_t+(1-\alpha^j)(\nu+\lambda\beta)$,
2. The auto-covariance function (ACF) of lag j is $\gamma(j)=\alpha^j(\lambda\beta^2)$
3. The autocorrelation function of lag j is $\rho(j)=\alpha^j$.
4. The process is stationary

5. When thinning proportion α tends to 0, an i.i.d sequence is obtained, by cons a perfectly dependent sequence is obtained if $\alpha \to 1$.

Proof :

1. In proving this formula, let us proceed by induction.

   So let $j = 1$, applying the linearity of expectation on the model representation(6) lead to

   $E(X_{t+1}|X_t = x_t) = (x_t - v)E(S_t) + v + E(E_{t+1}) \quad = \alpha x_t + (1-\alpha)(v + \lambda\beta);$

   this yields that

   (8) $E(X_{t+1}|X_t) = \alpha X_t + (1-\alpha)(v + \lambda\beta)$

   Therefore the statement holds for $j = 1$. Assuming this statement true up to $j - 1$ one has:

   $E(X_{t+j}|X_t) = E(E[((X_{t+j}|X_1, \cdots, X_{t+j-1}))]|X_t)) \quad = E(E[X_{t+j}|X_{t+j-1}]|X_t)$ (Due the markovity of the times series) $\quad = E(\alpha X_{t+j-1} + (1-\alpha)(v + \lambda\beta)|X_t)$ $= \alpha E(X_{t+j-1}|X_t) + (1-\alpha)(v + \lambda\beta) \quad = \alpha^j X_t + (1-\alpha^j)(v + \lambda\beta)$

2. It is enough to show that $Cov(X_t, X_{t+j}) = \alpha^j(\lambda\beta^2)\forall j \geq 1$. What we will do by induction.

   So, for $j = 1$ one has:

   $Cov(X_t, \quad X_{t+1}) \quad = E(Cov(X_t, \quad X_{t+1})|X_t) + Cov(E(X_t|X_t), \quad E(X_{t+1}|X_t))$
   $= E(X_t X_{t+1}|X_t) - E(X_t|X_t)E(X_{t+1}|X_t) + Cov(X_t, E(X_{t+1}|X_t))$
   $= 0 + Cov(X_t, E(X_{t+1}|X_t))$

   which in turn leads to

   $Cov(X_t, X_{t+1}) = Cov(X_t, \alpha X_t + (1-\alpha)(v + \lambda\beta)) \quad = \alpha Var(X_t) \quad = \alpha\lambda\beta^2$

   by using (8). Then, statement hold for $j = 1$.

   Now, we assume that it is true up to lag $j - 1$. Using the first result of the proposition setting above, one has:

   $Cov(X_t, \ X_{t+j}) \ = \ Cov(X_t,\mathrm{E}(X_{t+j}|X_t)) \quad = \ Cov(X_t, \alpha^j X_t + (1-\alpha^j)(v + \lambda\beta))$ $= \alpha^j Var(X_t) \quad = \ \alpha^j(\lambda\beta^2)$

3. The autocorrelation function $\rho(j)$ is given by:

   $\rho(j) = \alpha^j\lambda\beta^2/\lambda\beta^2 = \alpha^j$

   As setting above, the autocorrelation function of lag j is

   $\rho(j) = \alpha^j$

4. Since $\alpha \in]0, 1[$, the stationarity of the model hold.

5. If $\alpha = 0$, the random variable S_t is degenerate at 0. From the model(6↑) setting above, $X_t = \nu + E_t$. Since $(E_t)_{t\in\mathbb{N}^\star}$is an i.i.d sequence, the sequence $(X_t)_{t\in\mathbb{N}^\star}$is also. If $\alpha = 1$, distribution of E_t and $S_t$are degenerate at 0 and 1 respectively and then $X_t = X_{t-1}$

Proposition 4.2: Let $(X_t)_{t\in\mathrm{N}^\star}$ denote a time series generated by equation(7). The following assertions hold.

1. $\forall j \in \mathbb{N}^\star$, $E(X_{t+j}|X_t) = (1-\alpha)^j X_t + [1-(1-\alpha)^j](\nu + \lambda\beta)$,
2. The auto-covariance function (ACF) of lag j is $\gamma(j) = (1-\alpha)^j(\lambda\beta^2)$
3. The autocorrelation function of lag j is $\rho(j) = (1-\alpha)^j$.
4. The process is stationary
5. When thinning proportion α tends to 0, an i.i.d. sequence is obtained, by cons a perfectly dependent sequence is obtained if $\alpha \to 1$.

Proof : This proof can be obtained by adopting a similar approach to the previous proof.

4.2 One-step transition probability function

Proposition 4.2 : If $(X_t)_{t\in\mathrm{N}^\star}$ denote a time series generate by equation (6), then

(9) $$f_{X_t|X_{t-1}}(x_t|x_{t-1},\theta) = \int_{\min(0,\nu)}^{a_t} f_{\Sigma_t}(s_t|x_{t-1},\theta) f_{\mathrm{E}^c{}_t}(x_t - s_t|\theta)\,ds_t,$$

Where

$a_t = min(x_t, x_{t-1})$,

(10) $$f_{\mathrm{E}^c{}_t}(e_t|\theta) = \frac{1}{\beta^{(1-\alpha)\lambda}\Gamma((1-\alpha)\lambda)}(e_t)^{(1-\alpha)\lambda-1}\exp(-\frac{e_t}{\beta})1_{[0,+\infty]}(e_t)$$

and

(11) $$f_{\Sigma_t}(s_t|x_{t-1},\theta) = \frac{1}{x_{t-1}-\nu}\frac{\Gamma(\lambda)}{\Gamma(\alpha\lambda)\Gamma((1-\alpha)\lambda)}\left(\frac{s_t-\nu}{x_{t-1}-\nu}\right)^{\alpha\lambda-1}\left(1-\frac{s_t-\nu}{x_{t-1}-\nu}\right)^{(1-\alpha)\lambda-1} 1_{[\nu,x_{t-1}]}(s_t)$$

Proof : Let us consider the model given by equation(6). The conditional distribution of $X_t|X_{t-1} = x_{t-1}$ is the same as that of $S_t(x_{t-1} - \nu) + \nu + E_t^c$. Let denote by

$\Sigma_t = S_t(x_{t-1} - \nu) + \nu$. From model(6) setting above Σ_t and E_t are independents. Consequently, the distribution of the sum $\Sigma_t + E_t^c$ can be obtained by convolution. Furthermore appealing lemma(2.1) leads to the p.d.f of Σ_t given by:

$$f_{\Sigma_t}(s_t|x_{t-1},\theta) = \frac{1}{x_{t-1}-\nu}\frac{\Gamma(\lambda)}{\Gamma(\alpha\lambda)\Gamma((1-\alpha)\lambda)}\left(\frac{s_t-\nu}{x_{t-1}-\nu}\right)^{\alpha\lambda-1}\left(1-\frac{s_t-\nu}{x_{t-1}-\nu}\right)^{(1-\alpha)\lambda-1} 1_{[\nu,x_{t-1}]}(s_t)$$

Since $E_t^c \rightarrow P_3(0, \beta, (1-\alpha)\lambda)$, convolution formula leads to the one-step transition probability function. Hence, the result. If $(X_t)_{t\in \mathrm{N}^\star}$ denote a time series generate by equation (7↑), then

$$f_{X_t|X_{t-1}}(x_t|x_{t-1},\theta) = \int_0^{b_t} f_{\Sigma_t}(s_t|x_{t-1},\theta) f_{\mathrm{E}^l_t}(x_t - s_t|\theta) ds_t$$

Where

$b_t = \min(x_t, x_{t-1}) - \nu$

$$f_{\mathrm{E}^l_t}(e_t|\theta) = \frac{\Gamma(\lambda)}{\beta^{\alpha\lambda}\Gamma(\alpha\lambda)}(e_t-\nu)^{\alpha\lambda-1}\exp(-\frac{e_t-\nu}{\beta})1_{]\nu,+\infty[}(e_t)$$

$$f_{\Sigma_t}(s_t|x_{t-1},\theta) = \frac{1}{x_{t-1}-\nu}\frac{\Gamma(\lambda)}{\Gamma(\alpha\lambda)\Gamma((1-\alpha)\lambda)}\left(\frac{s_t}{x_{t-1}-\nu}\right)^{(1-\alpha)\lambda-1}\left(1-\frac{s_t}{x_{t-1}-\nu}\right)^{\alpha\lambda-1} 1_{[0,x_{t-1}-\nu]}(s_t)$$

Proof : This proof is similarly to those of proposition4.2. It is enough to denote $\Sigma_t = S_t(x_{t-1} - \nu)$with $S_t \rightarrow Beta((1-\alpha)\lambda, \alpha\lambda)$. The p.d.f of Σ_t can be obtained via the lemma(2.1). Moreover, note that innovation terms in the second model(given by equation(5,or 7)) are distributed according to Pearson type 3 distribution with location parameter ν, scale parameter β and shape parameter $\alpha\lambda$. Result is obtained by proceeding as in previous proposition.

5 NOTE ON THE MODEL FITTING PROCEDURE

In this section, we aim to suggest methods of adjustment to estimate the parameters of the proposed models. Without any restriction, the methods proposed in this section will refer to model(4), but can be easily extended to model(5).

The first method we suggest is the Conditional Least-Squares(CLS) method[14, 9]. Specifically, for a given set $x_1, x_2, ..., x_T$ that follow model(4) with unknown vector parameter $\theta = (\alpha, \nu, \lambda, \beta)$, the CLS estimate of θ is obtained by minimizing the sum of squares

$S_n(\theta) = \sum_{t=1}^{T}[x_t - E(X_t|X_{t-1} = x_{t-1})]^2,$

where $E(X_t|X_{t-1} = x_{t-1})$ is the conditional expectation of X_t given $X_{t-1} = x_{t-1}$. From Proposition4.1, it comes that

$\mathrm{E}(X_t|X_{t-1} = x_{t-1}) = \alpha x_{t-1} + (1-\alpha)(\nu + \lambda\beta)$

and thus, expression of $S_n(\theta)$ is obtained. An optimization algorithm such as the Newton Raphson method for minimization can then be used. However,the CLS estimators are generally not efficient[9].

Second method we propose the desirable maximum likelihood(ML) estimation procedure. Since process is Markovian, the ML estimate of the unknown parameter θ is obtained by solving the log-likelihood equations with the Newton-Raphson method or by numerically maximizing the log-likelihood function

$$L(\theta|(x_t)_{t=1:T}) = \log\big(f(x_1|\theta)\big) + \sum_{t=2}^{T}\log(f_{X_t|X_{t-1}}(x_t|x_{t-1},\theta))$$

where the transition density function $f_{X_t|X_{t-1}}(x_t|x_{t-1},\theta)$ is given inequation (9) and $f(x_1|\theta)$ denote Pearson type 3 p.d.f given by equation(2).

6 CONCLUDING REMARKS

In this paper, the focus is on the formulation of time series models where margins are distributed according to a Pearson type 3. The principle of models construction is based on thinning operators we have defined. Finally, two models are presented and their properties studied. One of the advantages of these two models is probably the ease of generation of the different processes. It is therefore clear that these models could serve as an alternative to generate correlated Pearson type 3 processes. We also made suggestions for estimating models parameters and detailed study is under investigation.

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