On Einstein Finsler Space with Generalized $(\alpha, \beta)$ –Metrics

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Abstract

Einstein-Finsler metrics are very useful to study geometric structure of spacetime and to build applications in theory of relativity. In this paper, we consider the special $(\alpha, \beta)$-metric $L = \alpha + \frac{\beta^{m+1}}{\alpha^m}$ and we find the Riemann curvature. Then we obtained the necessary and sufficient condition for that $(\alpha, \beta)$-metric to be Einstein metric, when $\beta$ is a constant Killing form. Finally, we proved that if the metric is Einstein then it is Ricci flat.

Key Words: $(\alpha, \beta)$-metrics, Riemannian curvature, Ricci curvature, Einstein Finsler space.

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1. INTRODUCTION

A Finsler space is a manifold $M$ together with positively homogeneous metric function $L(x, y)$. In Finsler geometry, $(\alpha, \beta)$-metrics are the special class of Finsler metrics which having a major role in formulating applications in Einstein theory of relativity, Mechanics, Biology, control theory, etc., [1, 2, 5, 12]. Robles invented Einstein-Randers metrics in [3] and derived the necessary and sufficient conditions for Randers metrics to be Einstein. In [13], authors proved the Einstein Schur type lemma for $(\alpha, \beta)$-metrics.

In Finsler geometry, Einstein metrics are solutions of Einstein Field equations in general relativity. In order to characterize Einstein-Finsler $(\alpha, \beta)$-metrics, it is
necessary to compute the Riemann curvature and the Ricci curvature for \((\alpha, \beta)\)-metrics. On a Finsler manifold \(M\), Riemannian curvature \(R_y : T_xM \to T_xM\) is given by \(R_y(u) = R^i_k(y)u^k \frac{\partial}{\partial y^i}\). Then, Ricci scalar defined as \((x, y) = R^i_k\). A Finsler metric is Einstein if the Ricci scalar is of the form \(Ric = c(x)F^2(x, y)\) for some function \(c(x)\) on manifold \(M\), i.e., the Ricci scalar is a function of \(x\) alone. A manifold is called Ricci flat if Ricci tensor vanishes, which represents vacuum solution to Einstein field equations in relativity \([3, 4]\).

In \([13]\), Razaei, Razavi and Sadeghazadeh, consider the \((\alpha, \beta)\)-metrics such as generalized Kropina metric, Matsumoto metric with a constant Killing form and obtained the necessary and sufficient conditions to be Einstein metrics. In \([11]\), Rafie and Rezaei proved that the second Schur type lemma for Finsler-Matsumoto metric. Then, \([6]\) Cheng, Shen and Tian, proved \((\alpha, \beta)\)-metric is Ricci flat. In the paper \([15]\), we see the classification of projectively related Einstein Finsler metrics over compact manifold. In \([9, 10]\), authors studied Einstein criterion for Finsler special \((\alpha, \beta)\)-metrics.

In this paper we consider the special \((\alpha, \beta)\)-metric \(L = \alpha + \beta^{m+1}\), where \(\alpha\) is the Riemannian metric, \(\beta\) is a constant Killing form. Then we find the Riemannian curvature for these metrics and we obtained the necessary and sufficient condition for them to be Einstein metrics, when \(\beta\) as a constant Killing form. Finally, we proved the lemma states that the above mentioned \((\alpha, \beta)\)-metric is Einstein if and only if it is Ricci flat.

2. PRELIMINARIES

Let \((M, F)\) be an \(n\)-dimensional Finsler space. We denote the tangent space at \(x \in M\) by \(T_xM\) and the tangent bundle of \(M\) by \(TM\). Each element of \(TM\) has the form \((x, y)\), where \(x \in M\) and \(y \in T_xM\). The formal definition of Finsler space as follows;

**Definition 2.1.** A Finsler space is a triple \(F^n = (M, D, L)\), where \(M\) is an \(n\)-dimensional manifold, \(D\) is an open subset of a tangent bundle \(TM\) and \(L\) is a Finsler metric defined as function \(L : TM \to [0, 1)\) with the following properties:

(i). Regular: \(L\) is \(C^\infty\) on the entire tangent bundle \(TM\setminus\{0\}\).

(ii). Positive homogeneous: \(L(x, \lambda y) = \lambda L(x, y), \ \lambda > 0\).

(iii). Strong convexity: The \(n \times n\) Hessian matrix

\[
g_{ij} = \frac{1}{2} [L^2]_{y^iy^j}
\]

is positive definite at every point on \(TM\setminus\{0\}\), where \(TM\setminus\{0\}\) denotes the tangent
vector \( y \) is non zero in the tangent bundle \( TM \).

Matsumoto introduced the class of \((\alpha, \beta)\)-metrics \[8\]. An \((\alpha, \beta)\)-metric is a scalar function \( L \) on \( TM \) defined by \( L = \phi(s) \), where \( \phi = \phi(s) \) is a \( C^1 \) on \((-b_0, b_0)\) with certain regularity, \( \alpha \) is a Riemannian metric and \( \beta \) is a one form on \( M \).

For a Finsler metric, geodesic spray is defined by

\[
G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i},
\]

where \( G^i \) are spray coefficients is given by,

\[
G^i(x, y) = \frac{1}{4} g^{ij}(x, y) \left\{ 2 \frac{\partial g_{jl}}{\partial x^k} (x, y) - \frac{\partial g_{jk}}{\partial x^l} (x, y) \right\} y^j y^k \quad (2.1)
\]

where \((g_{ij})\) is the inverse matrix of \((g^{ij})\). For the Berwald connection the coefficients \( G^i_j, G^i_{jk} \) of spray \( G^i \) defined as,

\[
G^i_j = \frac{\partial G^i}{\partial y^j}, \text{ and } G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}.
\]

In Finsler geometry, Riemannian curvature tensor \( R_y \) is the function \( R_y = R^i_k(y) dx^k \otimes \frac{\partial}{\partial x^i} \) \( |x : T_x M \to T_x M \) is defined as,

\[
R^i_k(y) = 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^l \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}. \quad (2.2)
\]

Suppose \( \alpha = \sqrt{a_{ij} y^i y^j} \) is a Riemannian metric, then \( R^i_k = R^i_{jkl}(x)y^j y^l \), where \( R^i_{jkl}(x) \) denote the coefficients of Riemannian curvature tensor. Thus, \( R_y \) in \( (2.2) \) is called Riemannian curvature in Finsler geometry. With respect to the Riemannian curvature, Ricci scalar function for the Finsler metric defined by \( \rho = \frac{1}{L^2} R^i_i \), which is positive homogeneous function of degree 0 in \( y \). It shows that \( \rho(x, y) \) depends on the direction of the flag pole \( y \), but not its length. Then the Ricci tensor given by,

\[
Ric_{ij} = \left\{ \frac{1}{2} R^i_j \right\} y^i y^j. \quad (2.3)
\]

Suppose the Ricci tensor on a manifold becomes zero, then such manifold called as Ricci-flat \[3\].

The Ricci tensor plays major role Finsler geometry to study the Einstein criterion for Finsler spaces. A Finsler metric becomes Einstein metric if the Ricci scalar function is a function of \( x \)-alone. i.e.,

\[
Ric_{ij} = \rho(x) g_{ij}. \quad (2.4)
\]

Let \((M, L)\) be an \( n \)-dimensional Finsler space equipped with an \((\alpha, \beta)\)-metric \( L \),
where $\alpha = \sqrt{a_{ij}y^i y^j}$, $\beta = b_i y^i$. In [7] M. Matsumoto, proved that $G^i$ of $(\alpha, \beta)$-metric space are given by,

$$2G^i = \gamma^i_{00} + 2B^i,$$  \hspace{1cm} (2.5)

where,

$$B^i = (E/\alpha)y^i = (\alpha L_\beta / L_\alpha) s^i_0 - (\alpha L_{aa} / L_\alpha) C \{(y^i/\alpha) - (\alpha/\beta) b^i\}, \hspace{1cm} (2.6)$$

$$E = (\beta L_\beta / L_\alpha) C, \hspace{0.5cm} C = \alpha \beta (r_{00} L_\alpha - 2 \alpha s_0 L_\beta) / 2(\beta^2 L_\alpha + \alpha \gamma^2 L_{aa}),$$

$$b^i = a^{tr} b_r, \hspace{0.5cm} b^2 = b^r b_r, \hspace{0.5cm} \gamma^2 = b^2 \alpha^2 - \beta^2,$$

$$r_{ij} = \frac{1}{2} (b_{ij} + b_{ji}), \hspace{0.5cm} s_{ij} = \frac{1}{2} (b_{i} b_j - b_{j} b_i),$$

$$s^i_j = a^{th} s_{hj}, \hspace{0.5cm} s_i = b_i s^i_0.$$  \hspace{1cm} (2.7)

where “|” in the above formula stands for the h-covariant derivation with respect to the Riemannian connection in the space $(M, \alpha)$, and the matrix $(a^{ij})$ denotes the inverse of matrix $(a_{ij})$. The functions $\gamma^i_{jk}$ stand for the Christoffel symbols in the space $(M, \alpha)$. Now (2.6) is re-written as

$$B^i = (\bar{p} r_{00} + \bar{q} s_0) y^i + \bar{r} s^i_0 + (\bar{s}_0 r_{00} + \bar{t} s_0) b^i,$$  \hspace{1cm} (2.7)

where

$$\bar{p} = \beta (\beta L_\alpha L_\beta - \alpha LL_{aa}) / 2 BL_\alpha(\beta^2 L_\alpha + \alpha \gamma^2 L_{aa}),$$  \hspace{1cm} (2.8)$$

$$\bar{q} = -\alpha \beta L_\beta (\beta L_\alpha L_\beta - \alpha LL_{aa}) / LL_\alpha(\beta^2 L_\alpha + \alpha \gamma^2 L_{aa}),$$  \hspace{1cm} (2.9)$$

$$\bar{r} = \alpha L_\beta / L_\alpha$$  \hspace{1cm} (2.10)$$

$$\bar{s}_0 = \alpha^3 L_{aa} / 2(\beta^2 L_\alpha + \alpha \gamma^2 L_{aa}),$$  \hspace{1cm} (2.11)$$

$$\bar{t} = -\alpha^4 L_{aa} L_\beta / L_\alpha(\beta^2 L_\alpha + \alpha \gamma^2 L_{aa}).$$  \hspace{1cm} (2.12)$$

Substituting (2.7) in (2.5) and (2.2), we obtained Berwald’s formula for Riemannian curvature tensor as follows:

$$R^i_k(y) = \bar{R}^i_k + \{2B^i_{lk} - y^i(B^i_{lk})_{y^k} - (B^i_{j}y^j)_{y^k} (B^i_{j})_{y^k} + 2B^i_{j} (B^i_{j})_{y^k} \}.$$  \hspace{1cm} (2.13)$$

The 1-form $\beta$ is said to be Killing (closed) 1-form if $r_{ij} = 0$ ($s_{ij} = 0$ respectively). $\beta$ is said to be a constant Killing form if it is Killing vector and has constant length with respect to $\alpha$, equivalently $r_{ij} = 0$, $s_i = 0$. 
3. Riemannian Curvature of Finsler Space with Special (α, β)-Metrics:

In this section, we consider Finsler space with special (α, β)-metric $L = \alpha + \frac{\beta^{m+1}}{\alpha^m}$, then we derive the Riemannian curvature. For this metric partial derivatives with respect to both $\alpha$ and $\beta$ respectively given by,

$$L_\alpha = 1 - \frac{m \beta^{m+1}}{\alpha^{m+1}}, \quad L_\beta = (m + 1) \frac{\beta^m}{\alpha^m}.$$  \hfill (3.1)

Now by using values of (3.1), equation (2.10) becomes,

$$\tilde{R} = \frac{(m + 1)\alpha^2 \beta^m}{\alpha^{m+1} - m \beta^{m+1}}.$$  \hfill (3.2)

Suppose that $\_\_\_\_$ is a constant Killing form, then by substituting (3.2) in (2.7), we get

$$B^i_j = \frac{(m + 1)\alpha^2 \beta^m}{\alpha^{m+1} - m \beta^{m+1}} s_0^i.$$  \hfill (3.3)

Now, by covariant and contravariant differentiation of (3.3), we obtaine that,

$$B^i_{.j} = \frac{C_1 y_j}{\alpha^{m+1} - m \beta^{m+1}} s_0^i + \frac{C_2 b_j}{(\alpha^{m+1} - m \beta^{m+1})^2} s_0^i + \frac{(m + 1)\alpha^2 \beta^m}{(\alpha^{m+1} - m \beta^{m+1})} s_j^i,$$  \hfill (3.4)

$$B^i_{||j} = \frac{C_2 b_0 j}{(\alpha^{m+1} - m \beta^{m+1})^2} s_0^i + \frac{(m + 1)\alpha^2 \beta^m}{(\alpha^{m+1} - m \beta^{m+1})} s_0^i j,$$  \hfill (3.5)

where

$$B^i_{.j} = B^i_{|j},$$

$$C_1 = (m + 1)\beta^m - (m + 1)^2 \alpha^{m+1} \beta^m,$$

$$C_2 = m(m + 1)(\alpha^{m+1} - m \beta^{m+1})\alpha^2 \beta^{m+1} + m(m + 1)^2 \alpha^2 \beta^m,$$
From (3.4), we have
\[ B^i_{j,i} = 0, \]  
(3.6)
\[ B^i_j B^j_i = \frac{(m + 1)^2 \alpha^2 \beta^{2m}(1 - (m + 1)\alpha^{m+2})^2}{(\alpha^{m+1} - m\beta^{m+1})^2} s^i_0 s^j_0 + \frac{(m + 1)^2 \alpha^4 \beta^{2m}}{(\alpha^{m+1} - m\beta^{m+1})^2} s^{ij} s_{ij}. \]  
(3.7)
And differentiate (3.5) with respect to \( y^i \) and transecting by \( y^j \), we get
\[ y^j (B^i_{j,i}) = 0. \]  
(3.8)
Finally by substituting (3.4) to (3.8) in Berwald’s formula (2.13), we obtain,
\[ R^i_i = \overline{R^i_i} + \left\{ 2B^i_{j,i} - y^j (B^i_{j,i}) - B^i_j B^j_i + 2B^j_i (B^i_j y^j) \right\} \]
\[ = \overline{R^i_i} + \left\{ \frac{(m + 1) \alpha \beta^{m+1}}{(\alpha^{m+1} - m\beta^{m+1})^2} s^i_0 s^j_0 - \frac{(m + 1)^2 \alpha^2 \beta^{2m}(1 - (m + 1)\alpha^{m+2})^2}{(\alpha^{m+1} - m\beta^{m+1})^2} s^i_0 s^j_0 \right\}, \]  
(3.9)
where \( R^i_i \) is the Riemannian curvature of the Finsler space, thus we state the following,

**Theorem 3.1.** Let \( L = \alpha + \frac{\beta^{m+1}}{\alpha^m} \) is a Finsler space with \((\alpha, \beta)\)-metric. Suppose \( \beta \) is a constant \( k \)-illing form, then the Riemannian curvature of the Finsler space is given in the equation (3.9).

### 4. EINSTEIN CRITERION FOR FINSLER SPACE WITH SPECIAL \((\alpha, \beta)\)-METRICS:

In this section, we establish the Einstein criterion for the metric \( L = \alpha + \frac{\beta^{m+1}}{\alpha^m} \). A Finsler metric \( L = L(x, y) \) on an \( n \)-dimensional manifold \( M \) is called an Einstein metric if the Ricci scalar satisfies the following condition,
\[ Ric = (n - 1)\lambda L^2, \]  
(4.1)
where \( \lambda = \lambda(x) \) is a scalar function on \( M \). \( L \) is Ricci constant if \( \lambda \) is constant. Now, we suppose the Ricci scalar of the mentioned \((\alpha, \beta)\)-metric is the function of \( x \) alone, i.e., \( L \) is Einstein, then we have \( L^2 Ric(x) = R^i_i \), so we can derive the necessary and sufficient conditions for this to be Einstein.
Theorem 4.2. Suppose $F_n = (M, L)$ is a Finsler space with $(\alpha, \beta)$-metric $L = \alpha + \frac{\beta^{m+1}}{\alpha^m}$ and $\beta$ is constant Killing form, then $F^n$ is Einstein if and only if the Ricci scalar is of the form $\text{Rat} + aI_{\text{irrat}} = 0$, where both $\text{Rat}$ and $I_{\text{irrat}}$ are given in equation (4.4) if $m$ is odd and in (4.5) if $m$ is even, are zero.

**Proof:** By using the Riemannian curvature given in (3.9), we get the Ricci curvature as follows,

\[
\begin{align*}
\overline{\text{Ric}}_{00} &+ \frac{2(m + 1)\alpha^2 \beta^m}{\alpha^{m+1} - m\beta^{m+1}} s^i_{0j} - \frac{(m + 1)^2 \alpha^2 \beta^{2m}(1 - (m + 1)\alpha^{m+2})^2}{(\alpha^{m+1} - m\beta^{m+1})^2} s^i_0 s^i_0 \\
&- \frac{(m + 1)^2 \alpha^4 \beta^{2m}}{(\alpha^{m+1} - m\beta^{m+1})^2} - \left( \frac{\beta^{m+1}}{\alpha^m} \right)^2 \text{Ric}(x) = 0. 
\end{align*}
\]

Multiplying (4.2) by $\alpha^{2m}(\alpha^{m+1} - m\beta^{m+1})^2$ removes $y$ from the denominators and after simplification, we get:

\[
\begin{align*}
&\left[ \alpha^{4m+2} + m^2 \alpha^2 \beta^{2m+2} - 2m \alpha^{3m+1} \beta^{m+1} \right] \overline{\text{Ric}}_{00} \\
&+ [2(m + 1)\alpha^3 \beta^m - 2m(m + 1)\alpha^{2m+2} \beta^{2m+1}] s_{0i} \\
&+ [2(m + 1)^3 \alpha^{3m+4} \beta^{2m} - (m + 1)^2 \alpha^{2m+2} \beta^{2m} \\
&- (m + 1)^4 \alpha^{2m+4} \beta^{2m}] s^i_0 s^i_0 \\
&- (m + 1)^2 \alpha^{2m+2} s^i s^i_{ij} - [\alpha^{2m+4} + \alpha^{2m+2} \beta^{2m+2} + 2 \alpha^{2m+3} \beta^{m+1} \\
&+ m^2 \alpha^4 \beta^{2m+2} + m^2 \beta^{4m+4} \\
&+ 2m^2 \alpha \beta^{3m+3} - 2m \alpha^{3m+1} \beta^{m+1} - 2m \alpha^{m+1} \beta^{3m+3} - 4m \alpha^{m+2} \beta^{2m+2}] \text{Ric}(x) \\
&= 0
\end{align*}
\]

Now we have to characterize the Einstein criterion for the $(\alpha, \beta)$-metric, thus we classify both rational and irrational terms from the above equation, thus we have,

\[
\text{Rat} + aI_{\text{irrat}} = 0,
\]

where $\text{Rat}$ and $I_{\text{irrat}}$ obtained in the following cases:

**Case I:** If $m$ is odd, then we get,

\[
\begin{align*}
\text{Rat} &= (\alpha^{4m+2} + m^2 \alpha^2 \beta^{2m+2} - 2m \alpha^{m+1} \beta^{m+1}) \overline{\text{Ric}}_{00} \\
&+ 2(m + 1)[\alpha^{3m+3} \beta^m - m \alpha^{2m+2} \beta^{2m+1}] s_{0i} \\
&- [(m + 1)^2 \alpha^{2m+2} \beta^{2m} + (m + 1)^4 \alpha^{4m+8} \beta^{2m}] s^i_0 s^i_0 \\
&- (\alpha^{2m+4} + \alpha^{2m+2} \beta^{2m+2} + m^2 \alpha^4 \beta^{2m+2} + m^2 \beta^{4m+4} \\
&- 2m \alpha^{3m+1} \beta^{m+1} - 2m \alpha^{m+1} \beta^{3m+3}) \text{Ric}(x)
\end{align*}
\]
\[ \text{Irrat} = 2(m + 1)^3 \alpha^{3m+3} \beta^{2m} s_{0i}^i s_{0i} - (2 \alpha^{2m+2} \beta^{m+1} - 4m \alpha^{m+1} \beta^{m+2} + 2m^2 \beta^{3m+3}) \text{Ric}(x). \quad (4.4) \]

**Case II:** If \( m \) is even, we get,

\[
\text{Rat} = (\alpha^{4m+2} + m^2 \alpha^{2m} \beta^{2m+2}) \text{Ric}_{00} - 2m(m + 1) \alpha^{2m+2} \beta^{2m+1} s_{0i}^i s_{0i} \\
- [(m + 1)^2 \alpha^{2m+2} \beta^{m+2} + m^2 \alpha^{2m} \beta^{2m+2} + m^2 \beta^{4m+4} \\
- 4m \alpha^{m+2} \beta^{2m+2}) \text{Ric}(x) \\
\text{Irrat} = -2m \alpha^{3m} \beta^{m+1} \text{Ric}_{00} + 2m(m + 1) \alpha^{3m+3} \beta^{m} s_{0i}^i s_{0i} \\
- [2 \alpha^{2m+2} \beta^{m+1} + 2m^2 \beta^{3m+3} - 2m \alpha^{m} \beta^{3m+3}) \text{Ric}(x). \quad (4.5)\]

Clearly in both the cases \( \text{Rat} \) and \( \text{Irrat} \) are polynomials of degree \((4m + 8)\) and \((4m + 4)\) in \( y \) respectively. Let \( \text{Rat} = P(y) \) and \( \text{Irrat} = Q(y) \). We know that \( \alpha \) can never be polynomial in \( y \). Otherwise, the quadratic \( \alpha^2 = a_{ij}(x)y^i y^j \) would have been factored into linear term. It’s zero set would then consist of a hyper plane, contradicting the positive definiteness of \( a_{ij} \). Now, suppose the polynomial \( \text{Rat} \) is not zero. Then the above equation would imply that it is the product of polynomial \( \text{Irrat} \) with a non-polynomial factor \( \alpha \), this is not possible. So \( \text{Rat} \) must vanish and, since \( \alpha \) is positive at all \( y \neq 0 \), we see that \( \text{Irrat} \) also must be zero. Hence the proof.

Now consider case-I, If \( L \) is Einstein then \( \text{Rat} = 0 \), then by equation (4.4) we have

\[ 0 = \alpha^2 C_1 + C_2 \quad (4.6) \]

where \( C_1 \) and \( C_2 \) are as follows:

\[
C_1 = (\alpha^{4m} + m^2 \alpha^{2m-2} \beta^{2m+2} - 2m \alpha^{m-1} \beta^{m+1}) \text{Ric}_{00} \\
+ 2(m + 1)[\alpha^{3m+1} \beta^m - m \alpha^{2m} \beta^{2m+1}] s_{0i}^i s_{0i} \\
- [(m + 1)^2 \alpha^{2m+2} \beta^{m+2} + (m + 1)^4 \alpha^{4m+6} \beta^{2m}] s_{0i}^i s_{0i} \\
- (m + 1)^2 \alpha^{2m+2} \beta^{m+2} s_{ij}^i s_{ij} \\
- (\alpha^{2m+2} + \alpha^{2m+2} \beta^{2m+2} + 2m^2 \beta^{3m+3} - 2m \alpha^{m-1} \beta^{m+1} \\
- 2m^2 \beta^{4m+4} \text{Ric}(x)).
\]

\[
C_2 = -m^2 \beta^{4m+4} \text{Ric}(x).
\]

Thus, by (4.6) we conclude that \( \alpha^2 \) divides \( C_2 \) and so \( \beta = 0 \). Similarly, by Case-II, if \( \text{Rat} \) given in equation (4.5) is zero, then same as the case-I we arise at the result \( \beta = 0 \). Thus the Finsler metric is Riemannian for both the cases of \( m \) is odd and even.
Thus we state that

**Theorem 4.3.** Suppose Finsler metric \( L = \alpha + \frac{\beta^{m+1}}{\alpha^m} \) with constant killing form \( \beta \), is Einstein metric then it is Ricci flat.

**Example 1:** If \( m = 1 \), then the metric becomes: \( L = \alpha + \frac{\beta^2}{\alpha} \). According to equations (4.4) this metric Einstein if it satisfies,

\[
\text{Rat} + a\text{Irrat} = 0,
\]

where

\[
\text{Rat} = (\alpha^6 + \alpha^2 \beta^4 - 2\alpha^2 \beta^2)Ric_{00} + 4[\alpha^6 \beta - \alpha^4 \beta^3]s_{0|i} \\
- (4\alpha^4 \beta^2 + 16\alpha^{12} \beta^2)s_{0}^{l}s_{0|i} \\
- 4\alpha^6 \beta^2 s_{i}^{l}s_{ij} - (\alpha^6 + \alpha^4 \beta^4 + \alpha^2 \beta^4 + \beta^8 - 2\alpha^4 \beta^2 - 2\alpha^2 \beta^6)Ric(x)
\]

\[
\text{Irrat} = 16\alpha^6 \beta^2 s_{0}^{l}s_{0i} \\
- (2\alpha^4 \beta^2 - 4\alpha^2 \beta^3 + 2\beta^6)Ric(x).
\]

**Example 2:** If \( m = 2 \), then the metric becomes: \( L = \alpha + \frac{\beta^3}{\alpha^2} \). According to equations (4.5) this metric Einstein if it satisfies,

\[
\text{Rat} + a\text{Irrat} = 0,
\]

where

\[
\text{Rat} = (\alpha^{10} + 4\alpha^4 \beta^6)Ric_{00} - 12\alpha^6 \beta^5 s_{0|i} \\
- [9\alpha^6 \beta^4 + 81\alpha^{16} \beta^4 + 54\alpha^{10} \beta^4]s_{0}^{l}s_{0i} \\
- 9\alpha^8 \beta^2 s_{i}^{l}s_{ij} - (\alpha^8 + \alpha^6 \beta^6 + 4\alpha^2 \beta^6 + 4\beta^{12} - 8\alpha^4 \beta^6)Ric(x)
\]

\[
\text{Irrat} = -4\alpha^6 \beta^3 Ric_{00} + 12\alpha^9 \beta^2 s_{0|i} \\
- [2\alpha^6 \beta^3 + 8\beta^9 - 4\alpha^2 \beta^9]Ric(x).
\]

5. **CONCLUSION**

The Einstein metrics plays a major role in differential geometry and mainly connect with gravitation in general relativity. In particular, Einstein metric are solutions to Einstein field equations in general relativity containing the Ricci-flat metric. Einstein Finsler metric which represent a non Riemannian stage for the extensions of metric gravity provide an interesting source of geometric issues and the \((\alpha, \beta)\)-metric is an important class of Finsler metric appearing frequently in the study of applications in Physics.
In this paper we consider a special \((\alpha, \beta)\)-metric \(L = \alpha + \frac{\beta^{m+1}}{\alpha^m}\). For this \((\alpha, \beta)\)-metric, we obtain Riemannian curvature. Further we find the necessary and sufficient conditions for this \((\alpha, \beta)\)-metric to be Einstein metric, when \(\beta\) is a constant Killing form. Finally we prove that the above mentioned Einstein metric must be Riemannian or Ricci flat.

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