Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 9 (2017), pp. 5793-5803

http://www.ripublication.com

On Einstein Finsler Space with Generalized (α, β) −Metrics

Chandru K. and Narasimhamurthy S. K.

Department of P.G. Studies and Research in Mathematics, Kuvempu University, Shankaraghatta-577 451, Shivamogga, Karnataka, INDIA.

Abstract

Einstein-Finsler metrics are very useful to study geometric structure of spacetime and to build applications in theory of relativity. In this paper, we consider the special (α, β)-metric $L = \alpha + \frac{\beta^{m+1}}{\alpha^m}$ and we find the Riemann curvature. Then we obtained the necessary and sufficient condition for that (α, β)-metric to be Einstein metric, when β is a constant Killing form. Finally, we proved that if the metric is Einstein then it is Ricci flat.

Key Words: (α, β)-metrics, Riemannian curvature, Ricci curvature, Einstein Finsler space.

AMS Subject Classification (2010): 53B20, 53B40, 53C60.

1. INTRODUCTION

A Finsler space is a manifold M together with positively homogeneous metric function $L(x, y)$. In Finsler geometry, (α, β)-metrics are the special class of Finsler metrics which having a major role in formulating applications in Einstein theory of relativity, Mechanics, Biology, control theory, etc., [1, 2, 5, 12]. Robles invented Einstein-Randers metrics in [3] and derived the necessary and sufficient conditions for Randers metrics to be Einstein. In [13], authors proved the Einstein Schur type lemma for (α, β)-metrics.

In Finsler geometry, Einstein metrics are solutions of Einstein Field equations in general relativity. In order to characterize Einstein-Finsler (α, β)-metrics, it is necessary to compute the Riemann curvature and the Ricci curvature for (α,β)-metrics. On a Finsler manifold M, Riemannian curvature $R_y : T_xM \to T_xM$ is given by $R_y(u) = R^i_k\,(y)u^k \frac{\partial}{\partial x^i}$. Then, Ricci scalar defined as $(x,y) = R^i_k$. A Finsler metric is Einstein if the Ricci scalar is of the form $Ric = c(x)F^2(x,y)$ for some function $c(x)$ on manifold M, i.e., the Ricci scalar is a function of x alone. A manifold is called Ricci flat if Ricci tensor vanishes, which represents vacuum solution to Einstein field equations in relativity [3, 4].

In [13], Razaei, Razavi and Sadeghazadeh, consider the (α,β)-metrics such as generalized Kropina metric, Matsumoto metric with _ a constant Killing form and obtained the necessary and sufficient conditions to be Einstein metrics. In [11], Rafie and Rezaei proved that the second Schur type lemma for Finsler-Matsumoto metric. Then, [6] Cheng, Shen and Tian, proved (α,β)-metric is Ricci flat. In the paper [15], we see the classification of projectively related Einstein Finsler metrics over compact manifold. In [9, 10], authors studied Einstein criterion for Finsler special (α,β)-metrics.

In this paper we consider the special (α,β)-metric $L = \alpha + \frac{\beta^{m+1}}{\alpha^m}$, where α is the Riemannian metric, β is a constant Killing form. Then we find the Riemannian curvature for these metrics and we obtained the necessary and sufficient condition for them to be Einstein metrics, when β as a constant Killing form. Finally, we proved the lemma states that the above mentioned (α,β)-metric is Einstein if and only if it is Ricci flat.

2. PRELIMINARIES

Let $(M,\ F)$ be an n-dimensional Finsler space. We denote the tangent space at $x \in M$ by T_xM and the tangent bundle of M by TM. Each element of TM has the form (x,y), where $x \in M$ and $y \in T_xM$. The formal definition of Finsler space as follows;

Definition 2.1. A Finsler space is a triple $F^n = (M,D,L)$, where M is an n-dimensional manifold, D is an open subset of a tangent bundle TM and L is a Finsler metric defined as function $L : TM \to [0,1)$ with the following properties:

(i). Regular: L is C^∞ on the entire tangent bundle $TM\backslash\{0\}$.

(ii). Positive homogeneous: $L(x,\lambda y) = \lambda L(x,y),\ \lambda > 0$.

(iii). Strong convexity: The $n \times n$ Hessian matrix

$$g_{ij} = \frac{1}{2}[L^2]_{y^iy^j}$$

is positive definite at every point on $TM\backslash\{0\}$, where $TM\backslash\{0\}$ denotes the tangent vector y is non zero in the tangent bundle TM.

Matsumoto introduced the class of (α,β)-metrics [8]. An (α,β)-metric is a scalar function L on TM defined by $L = \phi(s)$, where $s = \frac{\beta}{\alpha}$. Here $\phi = \phi(s)$ is a C^1 on $(-b_0, b_0)$ with certain regularity, α is a Riemannian metric and β is a one form on M.

For a Finsler metric, geodesic spray is defined by $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y) \frac{\partial}{\partial y^i}$, where G^i are spray coefficients is given by,

$$G^i(x,y) = \frac{1}{4} g^{ij}(x,y)\left\{2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\right\} y^j y^k \tag{2.1}$$

where (g_{ij}) is the inverse matrix of (g^{ij}). For the Berwald connection the coefficients G^i_j, G^i_{jk} of spray G^i defined as,

$$G^i_j = \frac{\partial G^i}{\partial y^j}, \text{ and } G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}.$$

In Finsler geometry, Riemannian curvature tensor R_y is the function $R_y = R^i_k(y) dx^k \otimes \frac{\partial}{\partial xi} \,|x : T_xM \to T_xM$ is defined as,

$$R^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^i \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.2}$$

Suppose $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric, then $R^i_k = R^i_{jkl}(x) y^j y^l$, where $R^i_{jkl}(x)$ denote the coefficients of Riemannian curvature tensor. Thus, R_y in (2.2) is called Riemannian curvature in Finsler geometry. With respect to the Riemannian curvature, Ricci scalar function for the Finsler metric defined by $\rho = \frac{1}{L^2} R^i_i$, which is positive homogeneous function of degree 0 in y. It shows that $\rho(x,y)$ depends on the direction of the flag pole y, but not its length. Then the Ricci tensor given by,

$$Ric_{ij} = \left\{\frac{1}{2} R^i_j\right\}_{y^i y^j}. \tag{2.3}$$

Suppose the Ricci tensor on a manifold becomes zero, then such manifold called as Ricci-flat [3].

The Ricci tensor plays major role Finsler geometry to study the Einstein criterion for Finsler spaces. A Finsler metric becomes Einstein metric if the Ricci scalar function is a function of x-alone. i.e.,

$$Ric_{ij} = \rho(x) g_{ij}. \tag{2.4}$$

Let (M,L) be an n-dimensional Finsler space equipped with an (α,β)-metric L, where $\alpha = \sqrt{a_{ij}y^i y^j}$, $\beta = b_i y^i$. In [7] M. Matsumoto, proved that G^i of (α,β)-metric space are given by,

$$2G^i = \gamma^i_{00} + 2B^i, \tag{2.5}$$

where,

$$B^i = (E/\alpha)y^i = (\alpha L_\beta/L_\alpha)s^i_0 - (\alpha L_{\alpha\alpha}/L_\alpha)C\{(y^i/\alpha) - (\alpha/\beta)b^i\}, \tag{2.6}$$

$$E = (\beta L_\beta/L)C,\ C = \alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)/2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}),$$

$$b^i = a^{ir}b_r, \qquad b^2 = b^r b_r, \qquad \gamma^2 = b^2\alpha^2 - \beta^2,$$

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),$$

$$s^i_j = a^{ih}s_{hj}, \qquad s_j = b_i s^i_j.$$

where "|" in the above formula stands for the h-covariant derivation with respect to the Riemannian connection in the space (M,α), and the matrix (a^{ij}) denotes the inverse of matrix (a_{ij}). The functions γ^i_{jk} stand for the Christoffel symbols in the space (M,α). Now (2.6) is re-written as

$$B^i = (\tilde{p}r_{00} + \tilde{q}_0 s_0)y^i + \tilde{r}s^i_0 + (\tilde{s}_0 r_{00} + \tilde{t}s_0)b^i, \tag{2.7}$$

where

$$\tilde{p} = \beta(\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha})/2L(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \tag{2.8}$$

$$\tilde{q} = -\alpha\beta L_\beta(\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha})/L L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \tag{2.9}$$

$$\tilde{r} = \alpha L_\beta/L_\alpha \tag{2.10}$$

$$\tilde{s}_0 = \alpha^3 L_{\alpha\alpha}/2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \tag{2.11}$$

$$\tilde{t} = -\alpha^4 L_{\alpha\alpha}L_\beta/L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}). \tag{2.12}$$

Substituting (2.7) in (2.5) and (2.2), we obtained Berwald's formula for Riemannian curvature tensor as follows:

$$R^i_k(y) = \tilde{R}^i_k + \left\{2B^i_{|k} - y^j\left(B^i_{|k}\right)_{y^k} - \left(B^i\right)_{y^j}\left(B^j\right)_{y^k} + 2B^j\left(B^i\right)_{y^j y^k}\right\}. \tag{2.13}$$

The 1-form β is said to be Killing (closed) 1-form if $r_{ij} = 0$ ($s_{ij} = 0$ respectively). β is said to be a constant Killing form if it is Killing vector and has constant length with respect to α, equivalently $r_{ij} = 0,\ s_i = 0$.

3. RIEMANNIAN CURVATURE OF FINSLER SPACE WITH SPECIAL $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-METRICS:

In this section, we consider Finsler space with special (α, β)-metric $L = \alpha + \frac{\beta^{m+1}}{\alpha^m}$, then we derive the Riemannian curvature. For this metric partial derivatives with respect to both α and β respectively given by,

$$L_\alpha = 1 - \frac{m\beta^{m+1}}{\alpha^{m+1}}, \qquad L_\beta = (m+1)\frac{\beta^m}{\alpha^m}. \tag{3.1}$$

Now by using values of (3.1), equation (2.10) becomes,

$$\tilde{r} = \frac{(m+1)\alpha^2\beta^m}{\alpha^{m+1} - m\beta^{m+1}}. \tag{3.2}$$

Suppose that _ is a constant Killing form, then by substituting (3.2) in (2.7), we get

$$B^i = \frac{(m+1)\alpha^2\beta^m}{\alpha^{m+1} - m\beta^{m+1}} s_0^i. \tag{3.3}$$

Now, by covariant and contravariant differentiation of (3.3), we obtaine that,

$$B^i_{.j} = \frac{C_1 y_j}{(\alpha^{m+1} - m\beta^{m+1})} s_0^i + \frac{C2 b_j}{(\alpha^{m+1} - m\beta^{m+1})^2} s_0^i + \frac{(m+1)\alpha^2\beta^m}{(\alpha^{m+1} - m\beta^{m+1})} s_j^i, \tag{3.4}$$

$$B^i_{j|} = \frac{C2 b_{0|j}}{(\alpha^{m+1} - m\beta^{m+1})^2} s_0^i + \frac{(m+1)\alpha^2\beta^m}{(\alpha^{m+1} - m\beta^{m+1})} s^i_{0|j}, \tag{3.5}$$

where

$$B^i_{.j} = B^i_{y^j},$$

$$C_1 = (m+1)\beta^m - (m+1)^2\alpha^{m+1}\beta^m,$$

$$C_2 = m(m+1)(\alpha^{m+1} - m\beta^{m+1})\alpha^2\beta^{m+1} + m(m+1)^2\alpha^2\beta^{2m},$$

From (3.4), we have

$$B^i B^i_{.j.i} = 0, \tag{3.6}$$

$$B^i_{.j}\, B^j_{.i} = \frac{(m+1)^2\alpha^2\beta^{2m}(1-(m+1)\alpha^{m+2})^2}{(\alpha^{m+1} - m\beta^{m+1})^2}\, s^i_0 s_{0i} + \frac{(m+1)^2\alpha^4\beta^{2m}}{(\alpha^{m+1} - m\beta^{m+1})^2}\, s^{ij} s_{ij}\,. \tag{3.7}$$

And differentiate (3.5) with respect to y^i and transecting by y^j , we get

$$y^j\left(B^i_{|j}\right)_{.i} = 0. \tag{3.8}$$

Finally by substituting (3.4) to (3.8) in Berwald's formula (2.13), we obtain,

$$R^i_i = \overline{R^i_i} + \left\{2B^i_{|i} - y^j\left(B^i_{|j}\right)_{.i} - B^i_{.j}B^j_{.i} + 2B^j\left(B^i\right)_{y^j y^i}\right\}$$

$$= \overline{R^i_i} + \{\frac{2(m+1)\alpha^2\beta^m}{\alpha^{m+1} - m\beta^{m+1}}\, s^i_{0|j} - \frac{(m+1)^2\alpha^2\beta^{2m}(1-(m+1)\alpha^{m+2})^2}{(\alpha^{m+1} - m\beta^{m+1})^2}\, s^i_0 s_{0i} - \frac{(m+1)^2\alpha^4\beta^{2m}}{(\alpha^{m+1} - m\beta^{m+1})^2}\, s^{ij} s_{ij}\}\,, \tag{3.9}$$

where Ri is the Riemannian curvature of the Finsler space, thus we state the following,

Theorem 3.1. Let $L = \alpha + \frac{\beta^{m+1}}{\alpha^m}$ is a Finsler space with (α, β)-metric. Suppose β is a constant

killing form, then the Riemannian curvature of the Finsler space is given in the equation (3.9).

4. EINSTEIN CRITERION FOR FINSLER SPACE WITH SPECIAL $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-METRICS:

In this section, we establish the Einstein criterion for the metric $L = \alpha + \frac{\beta^{m+1}}{\alpha^m}$. A Finsler metric $L = L(x, y)$ on an n-dimensional manifold M is called an Einstein metric if the Ricci scalar satisfies the following condition,

$$Ric = (n-1)\lambda L^2, \tag{4.1}$$

where $\lambda = \lambda(x)$ is a scalar function on M. L is Ricci constant if λ is constant. Now, wesuppose the Ricci scalar of the mentioned (α, β)-metric is the function of x alone, i.e., L isEinstein, then we have $L^2 Ric(x) = R^i_i$, so we can derive the necessary and sufficient conditions for this to be Einstein.

Theorem 4.2. Suppose Fn = (M, L) is a Finsler space with (α,β)-metric $L = \alpha + \frac{\beta^{m+1}}{\alpha^m}$ and β is constant Killing form, then F^n is Einstein if and only if the Ricci scalar is of the form $Rat + \alpha Irrat = 0$, where both Rat and $Irrat$ are given in equation (4.4) if m is odd and in (4.5) if m is even, are zero.

Proof: By using the Riemannian curvature given in (3.9), we get the Ricci curvature as follows,

$$\overline{Ric_{00}} + \frac{2(m+1)\alpha^2\beta^m}{\alpha^{m+1} - m\beta^{m+1}} s^i_{0|j} - \frac{(m+1)^2\alpha^2\beta^{2m}(1-(m+1)\alpha^{m+2})^2}{(\alpha^{m+1} - m\beta^{m+1})^2} s^i_0 s_{0i}$$
$$- \frac{(m+1)^2\alpha^4\beta^{2m}}{(\alpha^{m+1} - m\beta^{m+1})^2} - \left(+\frac{\beta^{m+1}}{\alpha^m}\right)^2 Ric(x) = 0. \tag{4.2}$$

Multiplying (4.2) by $\alpha^{2m}(\alpha^{m+1} - m\beta^{m+1})^2$ removes y from the denominators and after simplification, we get:

$$\begin{aligned}
&[\alpha^{4m+2} + m^2\alpha^{2m}\beta^{2m+2} - 2m\alpha^{3m+1}\beta^{m+1}]\overline{Ric_{00}} \\
&+ [2(m+1)\alpha^{3m+3}\beta^m - 2m(m+1)\alpha^{2m+2}\beta^{2m+1}]s_{(0|i)} \\
&+[2(m+1)^3\alpha^{3m+4}\beta^{2m} - (m+1)^2\alpha^{2m+2}\beta^{2m} \\
&- (m+1)^4\alpha^{2m+8}\beta^{2m}]s^i_0 s_{0i} \\
&-(m+1)^2\alpha^{2m+2}\beta^{2m}s^{ij}s_{ij} - [\alpha^{2m+4} + \alpha^{2m+2}\beta^{2m+2} + 2\alpha^{2m+3}\beta^{m+1} \\
&+ m^2\alpha^2\beta^{2m+2} + m^2\beta^{4m+4} \\
&+2m^2\alpha\beta^{3m+3} - 2m\alpha^{3m+1}\beta^{m+1} - 2m\alpha^{m+1}\beta^{3m+3} - 4m\alpha^{m+2}\beta^{2m+2}]Ric(x) \\
&= 0
\end{aligned} \tag{4.3}$$

Now we have to characterize the Einstein criterion for the (α,β)-metric, thus we classify both rational and irrational terms from the above equation, thus we have,

$$Rat + \alpha Irrat = 0,$$

where Rat and $Irrat$ obtained in the following cases:

Case I: If m is odd, then we get,

$$\begin{aligned}
Rat = {} & (\alpha^{4m+2} + m^2\alpha^{2m}\beta^{2m+2} - 2m\alpha^{m+1}\beta^{m+1})Ric_{00} \\
& + 2(m+1)[\alpha^{3m+3}\beta^m - m\alpha^{2m+2}\beta^{2m+1}]s_{0|i} \\
& - [(m+1)^2\alpha^{2m+2}\beta^{2m} + (m+1)^4\alpha^{4m+8}\beta^{2m}]s^i_0 s_{0i} \\
& - (m+1)^2\alpha^{2m+4}\beta^{2m}s^{ij}s_{ij} \\
& - (\alpha^{2m+4} + \alpha^{2m+2}\beta^{2m+2} + m^2\alpha^2\beta^{2m+2} + m^2\beta^{4m+4} \\
& - 2m\alpha^{3m+1}\beta^{m+1} - 2m\alpha^{m+1}\beta^{3m+3})Ric(x)
\end{aligned}$$

$$Irrat = 2(m+1)^3\alpha^{3m+3}\beta^{2m}s_0^i s_{0i} - (2\alpha^{2m+2}\beta^{m+1} - 4m\alpha^{m+1}\beta^{m+2} + 2m^2\beta^{3m+3})Ric(x). \quad (4.4)$$

Case II: If m is even, we get,

$$Rat = (\alpha^{4m+2} + m^2\alpha^{2m}\beta^{2m+2})Ric_{00} - 2m(m+1)\alpha^{2m+2}\beta^{2m+1})s_{0|i} - [(m+1)^2\alpha^{2m+2}\beta^{2m} + (m+1)^4\alpha^{4m+8}\beta^{2m} + 2(m+1)^3\alpha^{3m+4}\beta^{2m}]s_0^i s_{0i} - (m+1)^2\alpha^{2m+4}\beta^{2m}s^{ij}s_{ij} - (\alpha^{2m+4} + \alpha^{2m+2}\beta^{2m+2} + m^2\alpha^2\beta^{2m+2} + m^2\beta^{4m+4} - 4m\alpha^{m+2}\beta^{2m+2})Ric(x)$$

$$Irrat = -2m\alpha^{3m}\beta^{m+1}Ric_{00} + 2m(m+1)\alpha^{3m+3}\beta^m s_{0|i} - [2\alpha^{2m+2}\beta^{m+1} + 2m^2\beta^{3m+3} - 2m\alpha^m\beta^{3m+3}]Ric(x). \quad (4.5)$$

Clearly in both the cases Rat and $Irrat$ are polynomials of degree $(4m+8)$ and $(4m+4)$ in y respectively. Let $Rat = P(y)$ and $Irrat = Q(y)$. We know that α can never be polynomial in y. Otherwise, the quadratic $\alpha^2 = a_{ij}(x)y^i y^j$ would have been factored into linear term. It's zero set would then consist of a hyper plane, contradicting the positive definiteness of a_{ij}. Now, suppose the polynomial Rat is not zero. Then the above equation would imply that it is the product of polynomial $Irrat$ with a non-polynomial factor α, this is not possible. So Rat must vanish and, since α is positive at all $y \neq 0$, we see that $Irrat$ also must be zero. Hence the proof.

Now consider case-I, If L is Einstein then $Rat = 0$, then by equation (4.4) we have

$$0 = \alpha^2 C_1 + C_2 \quad (4.6)$$

where C_1 and C_2 are as follows:

$$C_1 = (\alpha^{4m} + m^2\alpha^{2m-2}\beta^{2m+2} - 2m\alpha^{m-1}\beta^{m+1})Ric_{00} + 2(m+1)[\alpha^{3m+1}\beta^m - m\alpha^{2m}\beta^{2m+1}]s_{0|i} - [(m+1)^2\alpha^{2m}\beta^{2m} + (m+1)^4\alpha^{4m+6}\beta^{2m}]s_0^i s_{0i} - (m+1)^2\alpha^{2m+2}\beta^{2m}s^{ij}s_{ij} - (\alpha^{2m+2} + \alpha^{2m}\beta^{2m+2} + m^2\beta^{2m+2} - 2m\alpha^{3m-1}\beta^{m+1} - 2m\alpha^{m-1}\beta^{3m+3})Ric(x)$$

$$C_2 = -m^2\beta^{4m+4}Ric(x).$$

Thus, by (4.6) we conclude that α^2 divides C_2 and so $\beta = 0$. Similarly, by Case-II, if Rat given in equation (4.5) is zero, then same as the case-I we arise at the result $\beta = 0$. Thus the Finsler metric is Riemannian for both the cases of m is odd and even.

Thus we state that

Theorem 4.3. Suppose Finsler metric $L = \alpha + \frac{\beta^{m+1}}{\alpha^m}$ with constant killing form β, is Einstein metric then it is Ricci flat.

Example 1: If $m = 1$, then the metric becomes: $L = \alpha + \frac{\beta^2}{\alpha}$. According to equations (4.4) this metric Einstein if it satisfies,

$$Rat + \alpha Irrat = 0,$$

where

$$Rat = (\alpha^6 + \alpha^2\beta^4 - 2\alpha^2\beta^2)Ric_{00} + 4[\alpha^6\beta - \alpha^4\beta^3]s_{0|i} - (4\alpha^4\beta^2 + 16\alpha^{12}\beta^2)s_0^i s_{0i}$$
$$- 4\alpha^6\beta^2 s^{ij}s_{ij} - (\alpha^6 + \alpha^4\beta^4 + \alpha^2\beta^4 + \beta^8 - 2\alpha^4\beta^2 - 2\alpha^2\beta^6)Ric(x)$$
$$Irrat = 16\alpha^6\beta^2 s_0^i s_{0i} - (2\alpha^4\beta^2 - 4\alpha^2\beta^3 + 2\beta^6)Ric(x).$$

Example 2: If $m = 2$, then the metric becomes: $L = \alpha + \frac{\beta^3}{\alpha^2}$. According to equations (4.5) this metric Einstein if it satisfies,

$$Rat + \alpha Irrat = 0,$$

where

$$Rat = (\alpha^{10} + 4\alpha^4\beta^6)Ric_{00} - 12\alpha^6\beta^5)s_{0|i} - [9\alpha^6\beta^4 + 81\alpha^{16}\beta^4 + 54\alpha^{10}\beta^4]s_0^i s_{0i}$$
$$-9\alpha^8\beta^4 s^{ij}s_{ij} - (\alpha^8 + \alpha^6\beta^6 + 4\alpha^2\beta^6 + 4\beta^{12} - 8\alpha^4\beta^6)Ric(x)$$
$$Irrat = -4\alpha^6\beta^3 Ric_{00} + 12\alpha^9\beta^2 s_{0|i} - [2\alpha^6\beta^3 + 8\beta^9 - 4\alpha^2\beta^9]Ric(x). \tag{4.7}$$

5. CONCLUSION

The Einstein metrics plays a major role in differential geometry and mainly connect with gravitation in general relativity. In particular, Einstein metric are solutions to Einstein field equations in general relativity containing the Ricci-flat metric. Einstein Finsler metric which represent a non Riemannian stage for the extensions of metric gravity provide an interesting source of geometric issues and the (α,β)-metric is an important class of Finsler metric appearing frequently in the study of applications in Physics.

In this paper we consider a special (α,β)-metric $L = \alpha + \frac{\beta^{m+1}}{\alpha^m}$. For this (α,β)-metric, we obtain Riemannian curvature. Further we find the necessary and sufficient conditions for this (α,β)-metric to be Einstein metric, when β is a constant Killing form. Finally we prove that the above mentioned Einstein metric must be Riemannian or Ricci flat.

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