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THE STABILITY OF VIGINTI UNUS FUNCTIONAL EQUATION IN VARIOUS SPACES

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Abstract

The aim of present paper is to obtain some results for the stability of viginti unus functional equation in matrix non-archimedean fuzzy normed spaces and matrix normed spaces by using the fixed point method.

1. INTRODUCTION

In 1940, an interesting talk presented by S. M. Ulam [20] triggered the study of stability problems for various functional equations. He raised a question concerning the stability of homomorphism. In the following year 1941, D. H. Hyers [8] was able to give a partial solution to Ulam's question. The result of Hyers was generalized by Aoki [4] for additive mappings. In 1978, Th. M. Rassias [15] succeeded in extending the result of Hyers theorem by weakening the condition for the Cauchy difference. In 1982 J. M. Rassias [16] solved the Ulam problem for different mappings and for many Euler-Lagrange type quadratic mappings by involving a product of different powers of norms. In 1994, a generalization of the Rassias theorem was obtained by Gavruta [7] by replacing the unbounded Cauchy difference by a general control function. A further generalization of the Hyers-Ulam stability for a large class of mapping was obtained by Isac and Th. M. Rassias [9]. They also presented some applications in non-linear analysis, especially in fixed point theory. This terminology may also be applied to the cases of other functional equations [1, 2, 6, 10, 17]. Mihet and Radu [12] investigated the stability in the settings of fuzzy, probabilistic and random normed spaces. The cocept of non-Archimedean fuzzy normed spaces has been introduced by Mirmostafafe and Moslehian [14]. Quite recently, the new results on stability of functional equations in non-Archimedean fuzzy normed spaces [3, 13, 18, 19].

In this paper, we introduce the following new functional equation

$$\begin{aligned} & f(x+11y)-21f(x+10y)+210f(x+9y)-1330f(x+8y)+5985f(x+7y) \\ & -20349f(x+6y)+54264f(x+5y)-116280f(x+4y)+203490f(x+3y) \\ & -293930f(x+2y)+352716f(x+y)-352716f(x)+293930f(x-y) \\ & -203490f(x-2y)+116280f(x-3y)-54264f(x-4y)+20349f(x-5y) \\ & -5985f(x-6y)+1330f(x-7y)-210f(x-8y)+21f(x-9y)-f(x-10y)=21!\,f(y), \end{aligned} \tag{1}$$

where $21! = 51090942170000000000$ in matrix normed spaces by using the fixed point method. The above functional equation is said to be viginti unus functional equation since the function $f(x) = cx^{21}$ is its solution.

In this paper, we study the general solution of the functional equation (1) and we also investigate the Ulam-Hyers stability of the functional equation (1) in matrix non-Archimedean fuzzy normed spaces by using fixed point approach.

2. PRELIMINARIES

In this section, we firstly restate the usual terminology, notations and conventions of the theory of non-Archimedean fuzzy normed space and we introduce the new concept of matrix non-Archimedean fuzzy normed spaces.

Definition 1. [3] Let X be a linear space over a non-Archimedean field K. A function $N : X \times R \to [0,1]$ is said to be a non-Archimedean fuzzy norm on X if for all $x, y \in X$ and all $t \in R$.

(N1) $N(x,t)=0$ for $t \le 0$;

(N2) $x=0 \Leftrightarrow N(x,t)=1$ for all $t>0$;

(N3) $N(cx,t)= N(x,\frac{t}{|c|})$ if $c \ne 0$;

(NA4) $N(x+y, \max s+t) \ge \min\{N(x,s),N(y,t)\}$;

(NA5) $\lim_{t\to\infty} N(x,t)=1$;

The pair (X,N) is called a non-Archimedean fuzzy normed space. Clearly, if $(NA4)$ holds then so is

(N4) $N(x+y,s+t) \ge \min\{N(x,s),N(y,t)\}$

A classical vector space over a complex or real field satisfying $(N1)$ and $(N5)$ is called fuzzy normed space. It is easy to see that $(NA4)$ is equivalent to the following condition

(NA4$'$) $N(x+y,t) \ge \min\{N(x,t),N(y,t)\} \quad (x,y \in X; t \in R)$.

We will use the following notations:

$M_n(X)$ is the set of all $n \times n$-matrices in X;

$e_j \in M_{1,n}(\mathbb{C})$ is that jth component is 1 and the other components are zero ;

$E_{ij} \in M_n(\mathbb{C})$ is that (i,j)-component is 1 and the other components are zero;

$E_{ij} \otimes x \in M_n(X)$ is that (i,j)-component is x and the other components are zero. For $x \in M_n(X), y \in M_k(X)$,

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Note that $(X,\{\|.\|_n\})$ is a matrix normed space if and only if $(M_n(X),\|.\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \le \|A\|\|B\|\|x\|_n$ holds for $A \in M_{k,n}(\mathbb{C})$, $B \in M_{n,k}(\mathbb{C})$ and $x=(x_{ij}) \in M_n(X)$, and that $(X,\{\|.\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X,\{\|.\|_n\})$ is a matrix normed space. A matrix normed space $(X,\{\|.\|_n\})$ is called an L^∞ -matrix normed space if $\|x\otimes y\|_{n+k} = max\{\|x\|_n,\|y\|_k\}$ holds for all $x\in M_n(X)$ and all $y\in M_k(X)$. Let E,F be vector space. For a given mapping $h:E\to F$ and a given positive integer n, define $h_n : M_n(E)\to M_n(F)$ by, $h_n([x_{ij}])=[h(x_{ij})]$ for all $[x_{ij}]\in M_n(E)$.

Definition 2. Let (X,N) be a non-Archimedean fuzzy normed space.

(M1) $(X,\{N_n\})$ is called a matrix non-Archimedean fuzzy normed space if for each positive integer n, $(M_n(X),N_n)$ is a non-archimedean fuzzy normed space and

$N_k(AxB,t)\geq N_n\left(x,\frac{t}{\|A\|.\|B\|}\right)$ for all $t>0$, $A\in M_{k,n}(\mathbb{R})$, $B\in M_{n,k}(\mathbb{R})$ and

$x=[x_{ij}]\in M_n(X)$ with $\|A\|.\|B\|\neq 0$.

(M2) $(X,\{N_n\})$ is called a complete matrix non-Archimedean fuzzy normed space if (X,N) is a non-Archimedean fuzzy Banach space and $(X,\{N_n\})$ is a matrix non-archimedean fuzzy normed space.

3. GENERAL SOLUTION OF VIGINTI UNUS FUNCTIONAL EQUATION (1)

In this section, we study the general solution of viginti unus functional equation (1). For this, let us consider $\mathcal{A}$ and $\mathcal{B}$ be real vector spaces.

Theorem 3. If a mapping $f:\mathcal{A}\to\mathcal{B}$ satisfies the functional equation (1) for all $x,y\in\mathcal{A}$, then $f(2x)=2^{21}f(x)$ for all $x\in\mathcal{A}$.

Proof. Letting $x=y=0$ in (1), one gets $f(0)=0$. Replacing $x=0$, $y=x$ and $x=x$, $y=-x$ in (1) and adding the two resulting equations, we get $f(-x)=-f(x)$. Hence, f is an odd mapping. Replacing $x=0$, $y=2x$ and $x=11x$, $y=x$ in (1) and subtracting the two resulting equations, we get

$$21f(21x)-230f(20x)+1330f(19x)-5796f(18x)+20349f(17x)$$

$$-55384f(16x)+116280f(15x)-198835f(14x)+293930f(13x)$$
$$-367080f(12x)+352716f(11x)-260015f(10x)+203490f(9x)$$
$$-178296f(8x)+54264f(7x)+66861f(6x)+5985f(5x)-91770f(4x)$$
$$+210f(3x)+(58765-21!)f(2x)-(1+21!)f(x)=0 \qquad (2)$$

for all $x \in A$. Replacing (x, y) by $(10x, x)$ in (1), we obtain that

$$f(21x)-21f(20x)+210f(19x)-1330f(18x)+5985f(17x)-20349f(16x)$$
$$+54264f(15x)-116280f(14x)+203490f(13x)-293930f(12x)-210f(2x)$$
$$+352716f(11x)-352716f(10x)+293930f(9x)-203490f(8x)+(21-21!)f(x)$$
$$+116280f(7x)-54264f(6x)+20349f(5x)-5985f(4x)+1330f(3x)=0 \qquad (3)$$

for all $x \in A$. Multiplying (3) by 21 and then subtracting (2) from the resulting equation, we get

$$211f(20x)-3080f(19x)+22134f(18x)-105336f(17x)+371945f(16x)$$
$$-1023264f(15x)+2243045f(14x)-3979360f(13x)+5805450f(12x)$$
$$-7054320f(11x)+7147021f(10x)-5969040f(9x)+4094994f(8x)$$
$$-2387616f(7x)+1206405f(6x)-421344f(5x)+33915f(4x)-27720f(3x)$$
$$+(63175-21!)f(2x)+22(21!)f(x)=0 \qquad (4)$$

for all $x \in A$. Replacing (x, y) by $(9x, x)$ in (1), we obtain that

$$f(20x)-21f(19x)+210f(18x)-1330f(17x)+5985f(16x)-20349f(15x)$$
$$+54264f(14x)-116280f(13x)+203490f(12x)-293930f(11x)$$
$$+352716f(10x)-352716f(9x)+293930f(8x)-203490f(7x)+1330f(2x)$$
$$+116280f(6x)-54264f(5x)+20349f(4x)-5985f(3x)-(209+21!)f(x)=0 \qquad (5)$$

$\forall\ x \in A$. Multiplying (5) by 211 and then subtracting (4) from the resulting equation, we get

$$1351 f(19x)-22176 f(18x)+175294 f(17x)-890890 f(16x)+3270375 f(15x)$$
$$-9206659 f(14x)+20555720 f(13x)-37130940 f(12x)+54964910 f(11x)$$
$$-67276055 f(10x)+68454036 f(9x)-57924236 f(8x)+40548774 f(7x)$$
$$-23328675 f(6x)+11028360 f(5x)-4259724 f(4x)+1235115 f(3x)$$
$$-(217455+21!) f(2x)+233(21!) f(x)=0 \quad (6)$$

for all $x \in A$. Replacing (x,y) by $(8x,x)$ in (1), we have

$$f(19x)-21 f(18x)+210 f(17x)-1330 f(16x)+5985 f(15x)-20349 f(14x)$$
$$+54264 f(13x)-116280 f(12x)+203490 f(11x)-293930 f(10x)-5984 f(2x)$$
$$+352716 f(9x)-352716 f(8x)+293930 f(7x)-203490 f(6x)+116280 f(5x)$$
$$-54264 f(4x)+20349 f(3x)+(1309-21!) f(x)=0 \quad (7)$$

$\forall\ x \in A$. Multiplying (7) by 1351, and then subtracting (6) from the resulting equation , we get

$$6195 f(18x)-108416 f(17x)+905940 f(16x)-4815360 f(15x)+18284840 f(14x)$$
$$-52754944 f(13x)+119963340 f(12x)-219950080 f(11x)+329823375 f(10x)$$
$$-408065280 f(9x)+418595080 f(8x)-356550656 f(7x)+251586315 f(6x)$$
$$-146065920 f(5x)+69050940 f(4x)-26256384 f(3x)$$
$$+(7866929-21!) f(2x)+1584(21!) f(x)=0 \quad (8)$$

for all $x \in A$. Replacing (x,y) by $(7x,x)$ in (1), it follows that

$$f(18x)-21 f(17x)+210 f(16x)-1330 f(15x)+5985 f(14x)-20349 f(13x)$$
$$+54264 f(12x)-116280 f(11x)+203490 f(10x)-293930 f(9x)$$
$$+352716 f(8x)-352716 f(7x)+293930 f(6x)-203490 f(5x)$$
$$+116280 f(4x)-54263 f(3x)+20328 f(2x)-(5775+21!) f(x)=0 \quad (9)$$

$\forall\ x \in A$. Multiplying (9) by 6195, and then subtracting (8) from the resulting equation, we get

$$21679 f(17x)-395010 f(16x)+3423990 f(15x)-18792235 f(14x)+73307111 f(13x)$$

$$-216202140 f(12x)+500404520 f(11x)-930797175 f(10x)+1412831070 f(9x)$$

$$-1766480540 f(8x)+1828524964 f(7x)-1569310035 f(6x)$$

$$+1114554630 f(5x)-651303660 f(4x)+309902901 f(3x)$$

$$-(118065031+21!) f(2x)+7779(21!) f(x)=0 \tag{10}$$

for all $x \in A$. Replacing (x, y) by $(6x, x)$ in (1), we have

$$f(17x)-21 f(16x)+210 f(15x)-1330 f(14x)+5985 f(13x)-20349 f(12x)$$

$$+54264 f(11x)-116280 f(10x)+203490 f(9x)-293930 f(8x)$$

$$+352716 f(7x)-352716 f(6x)+293930 f(5x)-203489 f(4x)$$

$$+116259 f(3x)-54054 f(2x)+(19019-21!) f(x)=0 \tag{11}$$

for all $x \in A$. Multiplying (11) by 21679, and then subtracting (10) from the resulting equation, we arrive at

$$60249 f(16x)-1128600 f(15x)+10040835 f(14x)-56441704 f(13x)$$

$$+224943831 f(12x)-675984736 f(11x)+1590036945 f(10x)$$

$$-2998628640 f(9x)+4605627930 f(8x)-5818005200 f(7x)$$

$$+6077220129 f(6x)-5257553840 f(5x)+3760134371 f(4x)+29458(21!) f(x)$$

$$-2210475960 f(3x)+(1053771635-21!) f(2x)=0 \tag{12}$$

for all $x \in A$. Replacing (x, y) by $(5x, x)$ in (1), we obtain

$$f(16x)-21 f(15x)+210 f(14x)-1330 f(13x)+5985 f(12x)-20349 f(11x)$$

$$+54264 f(10x)-116280 f(9x)+203490 f(8x)-293930 f(7x)$$

$$+352716 f(6x)-352715 f(5x)+293909 f(4x)-203280 f(3x)$$

$$+114950 f(2x)-(48279+21!) f(x)=0 \tag{13}$$

for all $x \in A$. Multiplying (13) by 60249, and then subtracting (12) from the resulting equation, we arrive at

$$136629 f(15x) - 2611455 f(14x) + 23689466 f(13x) - 135646434 f(12x)$$
$$+ 550022165 f(11x) - 1679314791 f(10x) + 4007125080 f(9x)$$
$$- 7654441080 f(8x) + 11890983370 f(7x) - 15173566160 f(6x)$$
$$+ 15993172200 f(5x) - 13947588970 f(4x) + 10036940760 f(3x)$$
$$- (5871850915 + 21!) f(2x) + 89707(21!) f(x) = 0 \quad (14)$$

for all $x \in A$. Replacing (x, y) by $(4x, x)$ in (1), we get

$$f(15x) - 21 f(14x) + 210 f(13x) - 1330 f(12x) + 5985 f(11x) - 20349 f(10x)$$
$$+ 54264 f(9x) - 116280 f(8x) + 203490 f(7x) - 293929 f(6x) + 352695 f(5x)$$
$$- 352506 f(4x) + 292600 f(3x) - 197505 f(2x) + (95931 - 21!) f(x) = 0 \quad (15)$$

for all $x \in A$. Multiplying (15) by 136629, and then subtracting (14) from the resulting equation, we obtain

$$257754 f(14x) - 5002624 f(13x) + 46070136 f(12x) - 267702400 f(11x)$$
$$+ 1100948730 f(10x) - 3406910976 f(9x) + 8232779040 f(8x) + 226336(21!) f(x)$$
$$- 15911651840 f(7x) + 24985659190 f(6x) - 32195192960 f(5x)$$
$$+ 34214953300 f(4x) - 29940704640 f(3x) + (21113059740 - 21!) f(2x) = 0 \quad (16)$$

for all $x \in A$. Replacing (x, y) by $(3x, x)$ in (1), we have

$$f(14x) - 21 f(13x) + 210 f(12x) - 1330 f(11x) + 5985 f(10x) - 20349 f(9x)$$
$$+ 54264 f(8x) - 116279 f(7x) + 203469 f(6x) - 293720 f(5x) + 351386 f(4x)$$
$$- 346731 f(3x) + 273581 f(2x) - (149226 + 21!) f(x) = 0 \quad (17)$$

for all $x \in A$. Multiplying (17) by 257754, and then subtracting (16) from the resulting equation, we obtain

$$410210 f(13x) - 8058204 f(12x) + 75110420 f(11x) - 441708960 f(10x)$$
$$+ 1838125170 f(9x) - 5753984016 f(8x) + 14059725530 f(7x)$$

$$-27459289440 f(6x)+43512311920 f(5x)-56356193740 f(4x)$$

$$+59430597530 f(3x)-(49403537340+21!) f(2x)+484090(21!) f(x)=0 \quad (18)$$

for all $x \in A$. Replacing (x, y) by $(2x, x)$ in (1), it follows that

$$f(13x)-21 f(12x)+210 f(11x)-1330 f(10x)+5985 f(9x)-20348 f(8x)$$

$$+54243 f(7x)-116070 f(6x)+202160 f(5x)-287945 f(4x)$$

$$+332367 f(3x)-298452 f(2x)+(177650-21!) f(x)=0 \quad (19)$$

for all $x \in A$. Multiplying (19) by 410210, and then subtracting (18) from the resulting equation, we obtain

$$556206 f(12x)-11033680 f(11x)+103870340 f(10x)-616981680 f(9x)$$

$$+2592969064 f(8x)-8191295500 f(7x)+20153785260 f(6x)$$

$$-39415741680 f(5x)+61761724710 f(4x)-76909669540 f(3x)$$

$$+(73024457570-21!) f(2x)+894300(21!) f(x)=0 \quad (20)$$

for all $x \in A$. Replacing (x, y) by (x, x) in (1), we get

$$f(12x)-21 f(11x)+210 f(10x)-1329 f(9x)+5964 f(8x)-20139 f(7x)$$

$$+52934 f(6x)-110295 f(5x)+183141 f(4x)-239666 f(3x)$$

$$+236436 f(2x)-(149226+21!) f(x)=0 \quad (21)$$

for all $x \in A$. Multiplying (21) by 556206, and then subtracting (20) from the resulting equation, we obtain

$$646646 f(11x)-12932920 f(10x)+122216094 f(9x)-724243520 f(8x)$$

$$+3010137130 f(7x)-9288423144 f(6x)+21930999090 f(5x)$$

$$-40102398340 f(4x)+56393997660 f(3x)-(58482664240+21!) f(2x)$$

$$+1450506(21!) f(x)=0 \quad (22)$$

for all $x \in A$. Replacing (x, y) by $(0, x)$ in (1), we obtain that

$$f(11x)-20 f(10x)+189 f(9x)-1120 f(8x)+4655 f(7x)-14364 f(6x)+33915 f(5x)$$

$$-62016f(4x)+87210f(3x)-90440f(2x)+(58786-21!)f(x)=0 \quad (23)$$

for all $x\in \mathcal{A}$. Multiplying (23) by 646646, and then subtracting (22) from the resulting equation, we can obtain that $f(2x)=2^{21}f(x)$ for all $x\in \mathcal{A}$. This completes the proof.

4. ULAM-HYERS STABILITY OF VIGINTI UNUS FUNCTIONAL EQUATION (1) IN MATRIX NON-ARCHIMEDEAN FUZZY NORMED SPACES

In this section, we will investigate the Ulam-Hyers stability for the functional equation (1) in matrix non-Archimedean fuzzy normed spaces by using the fixed point method.

Throughout this section, we assume that $\mathbb{K}$ be a non-Archimedean field, X is a vector space over $\mathbb{K}$ and $(Y,\mathcal{N}_n)$ is an complete matrix non-Archimedean fuzzy normed space over $\mathbb{K}$, and $(Z,\mathcal{N}')$ is (an Archimedean or a non-Archimedean fuzzy) normed space and minimum is denoted by T. For a mapping $f:X\to Y$, define $\mathcal{G}f:X^2\to Y$ and $\mathcal{G}f_n:M_n(X^2)\to M_n(Y)$ by,

$$\begin{aligned}\mathcal{G}f(a,b)= {} & f(a+11b)-21f(a+10b)+210f(a+9b)-1330f(a+8b)+5985f(a+7b)\\ & -20349f(a+6b)+54264f(a+5b)-116280f(a+4b)+203490f(a+3b)\\ & -293930f(a+2b)+352716f(a+b)-352716f(a)+293930f(a-b)\\ & -203490f(a-2b)+116280f(a-3b)-54264f(a-4b)+21f(a-9b)\\ & +20349f(a-5b)-5985f(a-6b)+1330f(a-7b)-210f(a-8b)\\ & -f(a-10b)-21!f(b)\end{aligned}$$

$$\begin{aligned}\mathcal{G}f_n([x_{ij}],[y_{ij}])= {} & f([x_{ij}+11y_{ij}])-21f([x_{ij}+10y_{ij}])+210f([x_{ij}+9y_{ij}])-1330f([x_{ij}+8y_{ij}])\\ & +5985f([x_{ij}+7y_{ij}])-20349f([x_{ij}+6y_{ij}])+54264f([x_{ij}+5y_{ij}])\\ & -116280f([x_{ij}+4y_{ij}])+203490f([x_{ij}+3y_{ij}])-293930f([x_{ij}+2y_{ij}])\\ & +352716f([x_{ij}+y_{ij}])-352716f([x_{ij}])+293930f([x_{ij}-y_{ij}])\\ & -203490f([x_{ij}-2y_{ij}])+116280f([x_{ij}-3y_{ij}])-54264f([x_{ij}-4y_{ij}])\\ & +20349f([x_{ij}-5y_{ij}])-5985f([x_{ij}-6y_{ij}])+1330f([x_{ij}-7y_{ij}])\end{aligned}$$

$$-210 f([x_{ij}-8y_{ij}])+21 f([x_{ij}-9y_{ij}])- f([x_{ij}-10y_{ij}])-21!\, f([y_{ij}])$$

for all $a,b \in X$ and all $x=[x_{ij}], y=[y_{ij}] \in M_n(X)$.

Theorem 4. Let $q=\pm 1$ be fixed and let $\psi : X \times X \to Z$ be a mapping such that for some $\eta \neq 2^{21}$ with $\left(\frac{\eta}{2^{21}}\right)^q < 1$

$$\mathrm{N}'(\psi(2^q a, 2^q b) \geq \mathrm{N}'\left(\psi(a,b), \eta^{-q} t\right) \tag{24}$$

for all $a,b \in X$ and $t>0$, and $\lim_{k\to\infty} \mathrm{N}(2^{-21kq}\mathrm{G}f(2^{kq}a, 2^{kq}b), t)=1$ for all $a,b \in X$ and $t>0$. Suppose an odd mapping $f: X \to Y$ with $f(0)=0$ satisfies the inequality

$$\mathrm{N}(\mathrm{G}f_n([x_{ij}],[y_{ij}]),t) \geq \sum_{i,j=1}^{n} \mathrm{N}'\left(\psi(x_{ij}, y_{ij}), t\right) \tag{25}$$

for all $x=[x_{ij}], y=[y_{ij}] \in M_n(X)$, and $t>0$. Then there exists a unique viginti unus mapping $\mathrm{V}_{\mathrm{U}} : X \to Y$ such that

$$\mathrm{N}_n(f_n([x_{ij}]) - \mathrm{V}_{\mathrm{U}n}([x_{ij}]), t) \geq \mathrm{T}\left\{\Gamma\left(x_{ij}, \frac{|\eta - 2^{21}| t}{n^2}\right) : i,j=1,2,\ldots,n\right\} \tag{26}$$

for all $x=[x_{ij}] \in M_n(X)$, and $t>0$,

where $\Gamma\left(x_{ij}, t\right) = T\left\{\mathrm{N}'\left(\psi(0, 2x_{ij}), |21!| t\right), \mathrm{N}'\left(\psi(11x_{ij}, x_{ij}), |21!| t\right), \mathrm{N}'\left(\psi(10x_{ij}, x_{ij}), \frac{|21!| t}{|21|}\right),\right.$

$$\mathrm{N}'\left(\psi(9x_{ij}, x_{ij}), \frac{|21!| t}{|211|}\right), \mathrm{N}'\left(\psi(8x_{ij}, x_{ij}), \frac{|21!| t}{|1351|}\right), \mathrm{N}'\left(\psi(7x_{ij}, x_{ij}), \frac{|21!| t}{|6195|}\right),$$

$$\mathrm{N}'\left(\psi(6x_{ij}, x_{ij}), \frac{|21!| t}{|21679|}\right), \mathrm{N}'\left(\psi(5x_{ij}, x_{ij}), \frac{|21!| t}{|60249|}\right), \mathrm{N}'\left(\psi(4x_{ij}, x_{ij}), \frac{|21!| t}{|136629|}\right),$$

$$\mathrm{N}'\left(\psi(3x_{ij}, x_{ij}), \frac{|21!| t}{|257754|}\right), \mathrm{N}'\left(\psi(2x_{ij}, x_{ij}), \frac{|21!| t}{|410210|}\right),$$

$$N'\left(\psi(x_{ij}, x_{ij}), \frac{|21!|t}{|556206|}\right), N'\left(\psi(0, x_{ij}), \frac{|21!|t}{|646646|}\right)\Bigg\}.$$

Proof. For the cases $q=1$ and $q=-1$, we consider $\eta < 2^{21}$ and $\eta > 2^{21}$, respectively. Substituting $n=1$ in (25), we obtain

$$N(Gf(a,b),t) \geq N'(\psi(a,b),t) \tag{27}$$

for all $a,b \in X$ and $t>0$. Replacing (a,b) by $(0,2a)$ in (27), we get

$$\begin{aligned} N\big(&f(22a) - 20f(20a) + 189f(18a) - 1120f(16a) + 4655f(14a) - 14364f(12a) \\ &+ 33915f(10a) - 62016f(8a) + 87210f(6a) - 90440f(4a) \\ &+ (58786 - 21!)f(2a), t\big) \geq N'(\psi(0,2a),t) \end{aligned} \tag{28}$$

for all $a \in X$ and $t>0$. Applying the same procedure of Theorem 3, we arrive at

$$\begin{aligned} N(-21!f(2a) + 2097152(21!)f(a),t) \geq T\Big\{ & N'(\psi(0,2a),t), N'(\psi(11a,a),t), \\ & N'\left(\psi(10a,a), \frac{t}{|21|}\right), N'\left(\psi(9a,a), \frac{t}{|211|}\right), N'\left(\psi(8a,a), \frac{t}{|1351|}\right), \\ & N'\left(\psi(7a,a), \frac{t}{|6195|}\right), N'\left(\psi(6a,a), \frac{t}{|21679|}\right), N'\left(\psi(5a,a), \frac{t}{|60249|}\right), \\ & N'\left(\psi(4a,a), \frac{t}{|136629|}\right), N'\left(\psi(3a,a), \frac{t}{|257754|}\right), \\ & N'\left(\psi(2a,a), \frac{t}{|410210|}\right), N'\left(\psi(a,a), \frac{t}{|556206|}\right), N'\left(\psi(0,a), \frac{t}{|646646|}\right)\Big\} \end{aligned} \tag{29}$$

for all $a \in X$ and $t>0$. It follows from (29), we can obtain

$$N(2^{21}f(a) - f(2a),t) \geq T\big\{ N'(\psi(0,2a),|21!|t), N'(\psi(11a,a),|21!|t),$$

$$N'\left(\psi(10a,a),\frac{|21!|t}{|21|}\right),N'\left(\psi(9a,a),\frac{|21!|t}{|211|}\right),N'\left(\psi(8a,a),\frac{|21!|t}{|1351|}\right),$$

$$N'\left(\psi(7a,a),\frac{|21!|t}{|6195|}\right),N'\left(\psi(6a,a),\frac{|21!|t}{|21679|}\right),$$

$$N'\left(\psi(5a,a),\frac{|21!|t}{|60249|}\right),N'\left(\psi(4a,a),\frac{|21!|t}{|136629|}\right),N'\left(\psi(3a,a),\frac{|21!|t}{|257754|}\right),$$

$$N'\left(\psi(2a,a),\frac{|21!|t}{|410210|}\right),N'\left(\psi(a,a),\frac{|21!|t}{|556206|}\right),N'\left(\psi(0,a),\frac{|21!|t}{|646646|}\right)\Bigg\}$$

(30)

Therefore,

$$N(2^{21}f(a)-f(2a),t)\geq\Gamma(a,t) \tag{31}$$

for all $a\in X$ and $t>0$. Thus

$$N\left(f(a)-\frac{1}{2^{21q}}f(2^{q}a),\frac{\eta^{\left(\frac{q-1}{2}\right)}}{|2^{21}|^{\left(\frac{1+q}{2}\right)}}t\right)\geq\Gamma(a,t) \tag{32}$$

for all $a\in X$ and $t>0$. We consider the set $M=\{f:X\to Y\}$ and introduce the generalized metric ρ on M as follows:

$$\rho(f,g)=\inf\{\mu\in R_{+}:N(f(a)-g(a),t)\geq\mu\Gamma(a,t),\forall a\in X,t>0\},$$

It is easy to check that (M,ρ) is complete generalized metric (see Lemma 3.2 in [3]).

Define the mapping $P:M\to M$ by $Pf(a)=\frac{1}{2^{21q}}f(2^{q}a)$ for all $f\in M$ and $a\in X$. Let $f,g\in M$ and ν be an arbitrary constant with $\rho(f,g)\leq\nu$. Then $N(f(a)-g(a),\nu t)\geq\Gamma(a,t)$ for all $a\in X$ and $t>0$. Therefore, using (24), we get

$$N(Pf(a)-Pg(a),\nu t)=N(f(2^{q}a)-g(2^{q}a),2^{21q}\nu t)\geq\Gamma\left(a,\frac{2^{21q}}{\eta^{q}}t\right)$$

$\forall \ a \in X$ and $t > 0$. Hence by definition $\rho(\mathsf{P}f, \mathsf{P}g) \le \left(\frac{\eta}{2^{21}}\right)^{q} \nu$. This means that P is a contractive mapping with lipschitz constant $L = \left(\frac{\eta}{2^{21}}\right)^{l} < 1$. It follows from (32) that $\rho(f, \mathsf{P}f) \le \frac{\eta^{\left(\frac{q-1}{2}\right)}}{|2^{21}|^{\left(\frac{1+q}{2}\right)}}$. Therefore according to Theorem 2.2 in [5], there exists a mapping $\mathsf{V}_{\mathsf{U}} : X \to Y$ which satisfying:

1. V_{U} is a unique fixed point of P, which is satisfied $\mathsf{V}_{\mathsf{U}}(2^{q} a) = 2^{21q} \mathsf{V}_{\mathsf{U}}(a) \ \forall \ a \in X$.
2. $\rho(\mathsf{P}^{k} f, \mathsf{V}_{\mathsf{U}}) \to 0$ as $k \to \infty$. which implies that $\lim_{k\to\infty} \frac{1}{2^{21kq}} f(2^{kq} a) = \mathsf{V}_{\mathsf{U}}(a) \ \forall \ a \in X$.
3. $\rho(f, \mathsf{V}_{\mathsf{U}}) \le \frac{1}{1-\eta} \rho(f, \mathsf{P}f)$, which implies that $\rho(f, \mathsf{V}_{\mathsf{U}}) \le \frac{1}{|2^{21} - \eta|}$.

$$\text{So,} \quad \mathsf{N}\left(f(a) - \mathsf{V}_{\mathsf{U}}(a), \frac{1}{|2^{21} - \eta|} t \right) \ge \Gamma(a,t) \tag{33}$$

for all $a \in X$ and $t > 0$. By (27),

$$\mathsf{N}(\mathsf{G}\mathsf{V}_{\mathsf{U}}(a,b), t) = \lim_{k\to\infty} \mathsf{N}(2^{-21kq} \mathsf{G}f(2^{kq} a, 2^{kq} b), t) \ge \lim_{k\to\infty} \mathsf{N}'(2^{-21kq} \psi(2^{kq} a, 2^{kq} b), t) = 1$$

Hence by (N2), $\mathsf{G}\mathsf{V}_{\mathsf{U}}(a,b)$ = 0. Thus, the function V_{U} satisfies viginti unus.

We note that $e_j \in M_{1,n}(\mathbb{R})$ means that the jth component is 1 and the others are zero, $E_{ij} \in M_n(X)$ means that (i,j)-component is 1 and the others are zero, and $E_{ij} \otimes x \in M_n(X)$ means that (i,j)-component is x and the others are zero. Since $N(E_{kl} \otimes x, t) = N(x,t)$, we have

$$N_n([x_{ij}], t) = N_n\left(\sum_{i,j=1}^{n} E_{ij} \otimes x_{ij}, t \right) \ge \mathsf{T}\{ N_n(E_{ij} \otimes x_{ij}, t_{ij}) : i, j = 1,2,\dots,n \}$$

$= T\{N(x_{ij}, t_{ij}) : i, j = 1,2,...,n\}$.

where $t = \sum_{i,j=1}^{n} t_{ij}$. So, $N_n([x_{ij}], t) \geq T\left\{N(x_{ij}, \frac{t}{n^2}) : i, j = 1,2,...,n\right\}$

By (33), we get (26).

Thus $V_U : X \to Y$ is a unique viginti unus mapping satisfying (26).

5. ULAM-HYERS STABILITY OF VIGINTI UNUS FUNCTIONAL EQUATION (1) IN MATRIX NORMED SPACES

In this section, we will investigate the Ulam-Hyers stability for the functional equation (1) in matrix normed spaces by using the fixed point method.

Throughout this section, let us consider $(X, \|.\|_n)$ be a matrix normed space, $(Y, \|.\|_n)$ be a matrix Banach space and let n be a fixed non-negative integer. For a mapping $f : X \to Y$, $Gf : X^2 \to Y$ and $Gf_n : M_n(X^2) \to M_n(Y)$ defined in section 4.

Theorem 5. Let $q = \pm 1$ be fixed and let $\psi : X^2 \to [0, \infty)$ be a function such that there exists an $\eta < 1$ with

$$\psi(a,b) \leq 2^{21q} \eta \psi(\frac{a}{2^q}, \frac{b}{2^q}) \tag{37}$$

for all $a, b \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\left\| Gf_n([x_{ij}],[y_{ij}]) \right\| \leq \sum_{i,j=1}^{n} \psi(x_{ij}, y_{ij}) \tag{38}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique viginti unus mapping $V_U : X \to Y$ such that

$$\left\| f_n([x_{ij}]) - V_{Un}([x_{ij}]) \right\|_n \leq \sum_{i,j=1}^{n} \frac{\eta^{\left(\frac{1-q}{2}\right)}}{2^{21}(1-\eta)} \overline{\psi}(x_{ij}) \tag{39}$$

for all $x = [x_{ij}] \in M_n(X)$, where

$$\overline{\psi}(x_{ij}) = \frac{1}{21!}[\psi(0, 2x_{ij}) + \psi(11x_{ij}, x_{ij}) + 21\psi(10x_{ij}, x_{ij}) + 211\psi(9x_{ij}, x_{ij}) + 1351\psi(8x_{ij}, x_{ij})$$

$$+6195\psi(7x_{ij},x_{ij})+21679\psi(6x_{ij},x_{ij})+60249\psi(5x_{ij},x_{ij})+136629\psi(4x_{ij},x_{ij})$$

$$+257754\psi(3x_{ij},x_{ij})+410210\psi(2x_{ij},x_{ij})+556206\psi(x_{ij},x_{ij})+646646\psi(0,x_{ij})]$$

Proof. Substituting $n=1$ in (38), we obtain

$$\|\mathrm{G}f(a,b)\|\le\psi(a,b) \tag{40}$$

Replacing (a,b) by $(0,2a)$ in (40), we get

$$\begin{aligned}&\|f(22a)-20f(20a)+189f(18a)-1120f(16a)+4655f(14a)-14364f(12a)\\&+33915f(10a)-62016f(8a)+87210f(6a)-90440f(4a)\\&+(58786-21!)f(2a)\|\le\psi(0,2a)\end{aligned} \tag{41}$$

for all $a\in X$. Applying the same procedure of Theorem 3, we arrive at

$$\begin{aligned}&\|-21!f(2a)+2097152(21!)f(a)\|\le\psi(0,2a)+\psi(11a,a)+21\psi(10a,a)+211\psi(9a,a)\\&+1351\psi(8a,a)+6195\psi(7a,a)+21679\psi(6a,a)+60249\psi(5a,a)+136629\psi(4a,a)\\&+257754\psi(3a,a)+410210\psi(2a,a)+556206\psi(a,a)+646646\psi(0,a)\end{aligned} \tag{42}$$

for all $a\in X$. It follows from (42), we can obtain

$$\begin{aligned}&\|2^{21}f(a)-f(2a)\|\le\frac{1}{21!}[\psi(0,2a)+\psi(11a,a)+21\psi(10a,a)+211\psi(9a,a)\\&+1351\psi(8a,a)+6195\psi(7a,a)+21679\psi(6a,a)\\&+60249\psi(5a,a)+136629\psi(4a,a)+257754\psi(3a,a)\\&+410210\psi(2a,a)+556206\psi(a,a)+646646\psi(0,a)]\end{aligned} \tag{43}$$

Therefore,

$$\|2^{21}f(a)-f(2a)\|\le\overline{\psi}(a) \tag{44}$$

for all $a\in X$. Thus

$$\left\|f(a)-\frac{1}{2^{21q}}f(2^q a)\right\|\le\frac{\eta^{\left(\frac{1-q}{2}\right)}}{2^{21}}\overline{\psi}(a) \tag{45}$$

for all $a \in X$. We consider the set $\mathsf{M} = \{f : X \to Y\}$ and introduce the generalized metric ρ on M as follows: $\rho(f,g) = \inf\{\mu \in \mathsf{R}_+ : \|f(a) - g(a)\| \le \mu\overline{\psi}(a), \forall a \in X\}$, and $\mathsf{P} : \mathsf{M} \to \mathsf{M}$ be the linear mapping defined in the proof of Theorem 4. Therefore, using (37), we get

$$\|\mathsf{P}f(a) - \mathsf{P}g(a)\| = \left\|\frac{1}{2^{21q}} f(2^q a) - \frac{1}{2^{21q}} g(2^q a)\right\| \le \eta\overline{\psi}(a) \quad \text{for all } a \in X.$$

That is P is a contractive mapping with lipschitz constant $L = \eta < 1$. It follows from (45) that $\rho(f, \mathsf{P}f) \le \frac{\eta^{\left(\frac{1-q}{2}\right)}}{2^{21}}$. So $\|f(a) - \mathsf{V}_{\mathsf{U}}(a)\| \le \frac{\eta^{\left(\frac{1-q}{2}\right)}}{2^{21}(1-\eta)}\overline{\psi}(a)$ for all $a \in X$ (46)

It follows from (37) and (38),

$$\|\mathsf{G}\mathsf{V}_{\mathsf{U}}(a,b)\| = \lim_{k\to\infty}\|\mathsf{G}f(2^{kq}a, 2^{kq}b)\| \le \lim_{k\to\infty}\frac{1}{2^{21kq}}\psi(2^{kq}a, 2^{kq}b) \le \lim_{k\to\infty}\frac{2^{kq}\eta^k}{2^{21kq}}\psi(a,b) = 0 \quad \text{for}$$

all $a, b \in X$. Hence $\mathsf{G}\mathsf{V}_{\mathsf{U}}(a,b) = 0$. Therefore, the mapping $\mathsf{V}_{\mathsf{U}} : X \to Y$ is viginti unus mapping. Utilizing Lemma 2.1 in [11] and (46), we get (39).

Thus $\mathsf{V}_{\mathsf{U}} : X \to Y$ is a unique viginti unus mapping satisfying (39).

Corollary 1. Let $q = \pm 1$ be fixed and let p, Υ be non-negative real numbers with $p \ne 21$. Let $f : X \to Y$ be a mapping such that

$$\|\mathsf{G}f_n([x_{ij}],[y_{ij}])\|_n \le \sum_{i,j=1}^{n}\Upsilon(\|x_{ij}\|^p + \|y_{ij}\|^p)\forall \mathrm{x} = [\mathrm{x_{ij}}], \mathrm{y} = [\mathrm{y_{ij}}] \in \mathrm{M_n(X)}. \tag{47}$$

Then there exists a unique viginti unus mapping $\mathsf{V}_{\mathsf{U}} : X \to Y$ such that

$$\|f_n([x_{ij}]) - \mathsf{V}_{\mathsf{U}n}([x_{ij}])\|_n \le \sum_{i,j=1}^{n}\frac{\varepsilon}{|2^{21} - 2^p|}\|x_{ij}\|^t \quad \forall \quad x = [x_{ij}] \in M_n(X),$$

where $\varepsilon = \frac{\Upsilon}{21!}[2653358 + 410211(2^p) + 257754(3^p) + 136629(4^p) + 60249(5^p)$

$$+ 21679(6^p) + 6195(7^p) + 1351(8^p) + 211(9^p) + 21(10)^p + 11^p]$$

Proof. The proof follows from Theorem 5 by taking $\psi(a,b) = \Upsilon(\|a\|^p + \|b\|^p)$ for all $a,b \in X$. Then we can choose $\eta = 2^{q(p-21)}$, and we can obtain the required result.

Corollary 2. Let $q = \pm 1$ be fixed and let p, Υ be non-negative real numbers with $p = v + w \neq 21$. Let $f : X \to Y$ be a mapping such that

$$\|\mathrm{G}f_n([x_{ij}],[y_{ij}])\|_n \leq \sum_{i,j=1}^{n} \Upsilon(\|x_{ij}\|^v . \|y_{ij}\|^w) \tag{48}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique viginti unus mapping $\mathrm{V_U} : X \to Y$ such that

$$\|f_n([x_{ij}]) - \mathrm{V}_{\mathrm{U}n}([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{\varepsilon}{|2^{21} - 2^p|} \|x_{ij}\|^t \qquad \text{for all } x = [x_{ij}] \in M_n(X),$$

where $\varepsilon = \frac{\Upsilon}{21!}\big[556206 + 410210(2^v) + 257754(3^v) + 136629(4^v) + 60249(5^v)$

$+ 21679(6^v) + 6195(7^v) + 1351(8^v) + 211(9^v) + 21(10)^v + 11^v\big]$

Proof. The proof follows from Theorem 5 by taking $\psi(a,b) = \Upsilon(\|a\|^v . \|b\|^w)$ for all $a,b \in X$. Then we can choose $\eta = 2^{q(p-21)}$, and we can obtain the required result.

Corollary 3. Let $q = \pm 1$ be fixed and let p, Υ be non-negative real numbers with $p = v + w \neq 21$. Let $f : X \to Y$ be a mapping such that

$$\|\mathrm{G}f_n([x_{ij}],[y_{ij}])\|_n \leq \sum_{i,j=1}^{n} \Upsilon(\|x_{ij}\|^v . \|y_{ij}\|^w) + (\|x_{ij}\|^{v+w} + \|y_{ij}\|^{v+w}) \tag{49}$$

$\forall$ $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique viginti unus mapping $\mathrm{V_U} : X \to Y$ such that

$$\|f_n([x_{ij}]) - \mathrm{V}_{\mathrm{U}n}([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{\varepsilon}{|2^{21} - 2^p|} \|x_{ij}\|^t \quad \forall \ x = [x_{ij}] \in M_n(X),$$

where

$$\varepsilon = \frac{\Upsilon}{21!}\Big[3209564 + 410211(2^p) + 410210(2^v) + 257754(3^p + 3^v) + 136629(4^p + 4^v) + 60249(5^p + 5^v)$$
$$+ 21679(6^p + 6^v) + 6195(7^p + 7^v) + 1351(8^p + 8^v) + 211(9^p + 9^v) + 21(10^p + 10^v) + (11^p + 11^v)\Big]$$

Proof. The proof follows from Theorem 5 by taking $\psi(a,b) = \Upsilon(\|a\|^v.\|b\|^w + (\|a\|^{v+w} + \|b\|^{v+w})$ for all $a,b \in X$. Then we can choose $\eta = 2^{q(p-21)}$, and we can obtain the required result.

6. EXAMPLES

Let

$$Df(x,y) = f(x+11y) - 21f(x+10y) + 210f(x+9y) - 1330f(x+8y) + 5985f(x+7y)$$
$$-20349f(x+6y) + 54264f(x+5y) - 116280f(x+4y) + 203490f(x+3y)$$
$$-293930f(x+2y) + 352716f(x+y) - 352716f(x) + 293930f(x-y)$$
$$-203490f(x-2y) + 116280f(x-3y) - 54264f(x-4y) + 20349f(x-5y)$$
$$-5985f(x-6y) + 1330f(x-7y) - 210f(x-8y) + 21f(x-9y) - f(x-10y) - 21!f(y).$$

Now we will provide an example to illustrate that the functional equation (1) is not stable for $p = 21$ in corollary 1.

Example 6. Let $\psi : \mathrm{R} \to \mathrm{R}$ be a function defined by

$$\psi(x) = \begin{cases} \varepsilon x^{21}, & if\,|x| < 1 \\ \varepsilon, & otherwise \end{cases} \tag{50}$$

where $\varepsilon > 0$ is a constant, and define a function $f : \mathrm{R} \to \mathrm{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{\psi(2^n x)}{2^{21n}} \tag{51}$$

for all $x \in \mathrm{R}$. Then f satisfies the inequality

$$|Df(x,y)| \le \frac{(51090942170000000000)}{2097151}(2097152)^2\varepsilon(|x|^{21} + |y|^{21}) \tag{52}$$

for all $x, y \in \mathbb{R}$. Then there do not exists a viginti unus mapping $V_U : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ such that

$$|f(x) - V_U(x)| \le \lambda |x|^{21} \tag{53}$$

for all $x \in \mathbb{R}$.

Proof. It is easy to see that f is bounded by $\dfrac{2097152\varepsilon}{2097151}$ on $\mathbb{R}$.

If $|x|^{21} + |y|^{21} = 0$, then (52) is trivial. If $|x|^{21} + |y|^{21} \ge \dfrac{1}{2^{21}}$, then L.H.S of (52) is lessthan $\dfrac{(51090942170000000000)(2097152)\varepsilon}{2097151}$. Suppose that $0 < |x|^{21} + |y|^{21} < \dfrac{1}{2^{21}}$, then there exists a non-negative integer k such that

$$\frac{1}{2^{21(k+1)}} \le |x|^{21} + |y|^{21} < \frac{1}{2^{21k}}, \tag{54}$$

so that $2^{21(k-1)}|x|^{21} < \dfrac{1}{2^{21}}, 2^{21(k-1)}|y|^{21} < \dfrac{1}{2^{21}}$, and

$$2^n(x), 2^n(y), 2^n(x+11y), 2^n(x+10y), 2^n(x+9y), 2^n(x+8y), 2^n(x+7y),$$

$$2^n(x+6y), 2^n(x+5y), 2^n(x+4y), 2^n(x+3y), 2^n(x+2y),$$

$$2^n(x+y), 2^n(x-y), 2^n(x-2y), 2^n(x-3y), 2^n(x-4y), 2^n(x-5y),$$

$$2^n(x-6y), 2^n(x-7y), 2^n(x-8y), 2^n(x-9y), 2^n(x-10y) \in (-1,1)$$

for $n = 0,1,2,\dots,k-1$. Hence

$$\psi(2^n(x+11y)) - 21\psi(2^n(x+10y)) + 210\psi(2^n(x+9y)) - 1330\psi(2^n(x+8y))$$

$$+5985\psi(2^n(x+7y)) - 20349\psi(2^n(x+6y)) + 54264\psi(2^n(x+5y))$$

$$-116280\psi(2^n(x+4y)) + 203490\psi(2^n(x+3y)) - 293930\psi(2^n(x+2y))$$

$$+352716\psi(2^n(x+y)) - 352716\psi(2^n(x)) + 293930\psi(2^n(x-y))$$

$$-203490\psi(2^n(x-2y)) + 116280\psi(2^n(x-3y)) - 54264\psi(2^n(x-4y))$$

$$+20349\psi(2^n(x-5y)) - 5985\psi(2^n(x-6y)) + 1330\psi(2^n(x-7y))$$

$$-210\psi(2^n(x-8y))+21\psi(2^n(x-9y))-\psi(2^n(x-10y))-21!\psi(2^n(y))=0$$

From the definition of f and (54), we obtain that

$$\begin{aligned}|Df(x,y)| \le \sum_{n=0}^{\infty}\frac{1}{2^{21n}}\Big|&\psi(2^n(x+11y))-21\psi(2^n(x+10y))+210\psi(2^n(x+9y))\\ &-1330\psi(2^n(x+8y))+5985\psi(2^n(x+7y))-20349\psi(2^n(x+6y))\\ &+54264\psi(2^n(x+5y))-116280\psi(2^n(x+4y))+203490\psi(2^n(x+3y))\\ &-293930\psi(2^n(x+2y))+352716\psi(2^n(x+y))-352716\psi(2^n(x))\\ &+293930\psi(2^n(x-y))-203490\psi(2^n(x-2y))+116280\psi(2^n(x-3y))\\ &-54264\psi(2^n(x-4y))+20349\psi(2^n(x-5y))-5985\psi(2^n(x-6y))\\ &+1330\psi(2^n(x-7y))-210\psi(2^n(x-8y))+21\psi(2^n(x-9y))\\ &-\psi(2^n(x-10y))-21!\psi(2^n(y))\Big|\end{aligned}$$

$$\le \sum_{n=k}^{\infty}\frac{(51090942170000000000)\varepsilon}{2^{21n}}=\frac{(2097152)(51090942170000000000)\varepsilon}{2^{21k}(2097151)}$$

$$\le \frac{(51090942170000000000)}{2097151}(2097152)^2\varepsilon(|x|^{21}+|y|^{21}).$$

Therefore f satisfies (52) for all $x,y\in\mathbb{R}$ Now, We find that the viginti unus functional equation (1) is not stable for $t=21$ in corollary 1. Suppose on the contrary that there exists a viginti unus mapping $V_U:\mathbb{R}\to\mathbb{R}$ and a constant $\lambda>0$ satisfying (53). Then there exists a constant $c\in\mathbb{R}$ such that $V_U(x)=cx^{21}$ for any $x\in\mathbb{R}$. Thus we obtain the following inequality.

$$|f(x)|\le(\lambda+|c|)|x|^{21} \tag{55}$$

Let $m\in\mathbb{N}$ with $m\varepsilon>\lambda+|c|$. If $x\in(0,\frac{1}{2^{m-1}})$, then $2^n x\in(0,1)$ for all $n=0,1,2,...,m-1$.

For this x, we get $f(x)=\sum_{n=0}^{\infty}\frac{\psi(2^n x)}{2^{21n}}\geq\sum_{n=0}^{m-1}\frac{\varepsilon(2^n x)^{21}}{2^{21n}}=m\varepsilon x^{21}>(\lambda+|c|)|x|^{21}$

which contradicts (55). Therefore the viginti unus functional equation (1) is not stable for $p=21$.

The following examples illustrates the fact that functional equation (1) is not stable for $p=v+w=21$ (when $v=\frac{21}{2}, w=\frac{21}{2}$) in corollary 2.

Example 7. Let $\psi:\mathrm{R}\to\mathrm{R}$ be a function defined by (50) and define a function $f:\mathrm{R}\to\mathrm{R}$ by (51). Then f satisfies the inequality

$$|Df(x,y)|\leq\frac{(51090942170000000000)}{2097151}(2097152)^2\varepsilon(|x|^{\frac{21}{2}}.|y|^{\frac{21}{2}}) \tag{56}$$

for all $x,y\in\mathrm{R}$. Then there do not exists a viginti unus mapping $\mathrm{V_U}:\mathrm{R}\to\mathrm{R}$ and a constant $\lambda>0$ such that (53).

Proof. The proof is analogous to the proof of Example 6.

The following examples illustrates the fact that functional equation (1) is not stable for $p=v+w=21$ (when $v=\frac{21}{2}, w=\frac{21}{2}$) in corollary 3.

Example 8. Let $\psi:\mathrm{R}\to\mathrm{R}$ be a function defined by (50) and define a function $f:\mathrm{R}\to\mathrm{R}$ by (51). Then f satisfies the inequality

$$|Df(x,y)|\leq\frac{(51090942170000000000)}{2097151}(2097152)^2\varepsilon(|x|^{\frac{21}{2}}.|y|^{\frac{21}{2}}+(|x|^{21}+|y|^{21})) \tag{57}$$

for all $x,y\in\mathrm{R}$. Then there do not exists a viginti unus mapping $\mathrm{V_U}:\mathrm{R}\to\mathrm{R}$ and a constant $\lambda>0$ such that (53).

Proof. The proof is analogous to the proof of Example 6.

CONCLUSION

In this investigation, we identified a general solution of a viginti unus functional equation (1) and established the Ulam -Hyers stability of the functional equation (1) in matrix non-Archimedean fuzzy normed spaces and matrix normed spaces by using the fixed point method. Also we illustrate the examples for non-stability.

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