Generalized Form of Continuity by Using $D_\beta$-Closed Set

Purushottam Jha  
Department of Mathematics,  
Govt. P.G. College, Narayanpur, Chhattisgarh, India-494661.

Manisha Shrivastava  
Department of Mathematics,  
Govt. J.Y. Chhattisgarh College, Raipur, Chhattisgarh, India-492001.

Abstract


We establish the relationship of $D_\beta$-closed sets with some already existing generalized closed sets. We define $D_\beta$-continuous and $D_\beta$-irresolute functions and obtain their basic properties.

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1. Introduction

In general topology repeated application of interior and closure operators give rise to several different new classes of sets. Some of them are generalized form of open sets. These classes are found to have applications not only in mathematics but even in diverse fields outside the realm of mathematics ([20], [10], [24]). The most well known notion and inspiring sources are the notions of $\alpha$–open set initiated by Njasted [21] in 1965, semiopen sets by Levine (See also [11], [12]) in 1963, preopen sets by Mashour et al. [16] in 1982 and $\beta$–open (semi–preopen) sets by Abd–El–Monsef et al. [19] (by Andrijevic (See also [1], [2])). Due to this, investigation of these sets have gained momentum in recent days. By originating the concept of generalized closed (g–closed) sets, Levine [12] provided an umbrella for the researchers working in the field of generalized closed sets. Levine [12] used the closure operator and the openness of the superset in the definition of g–closed sets. Levine discussed that compactness, normality and completeness in a uniform space are inherited by g–closed subsets. He used g–closed sets to define new separation axioms, called $T_{1/2}$–spaces in which the closed sets and g–closed sets coincide.

Balchandran et al. [4] introduced the notion of generalized continuous (g–continuous) functions by using g–closed sets and obtained some of their properties. Andrijevic [1] investigated some properties of topology of $\alpha$–sets. Maheshwari et al. [13] defined and investigated the $\alpha$–irresolute mapping as a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\alpha$–irresolute if the preimage of every $\alpha$–open sets in $Y$ is $\alpha$–open in $X$. Maki et al. [15] defined and investigated the concept of generalized $\alpha$–closed sets in topological spaces as, a subset $A$ of a space is said to be generalized $\alpha$–closed set if $\alpha$–Cl$(A) \subseteq A$, whenever $A \subseteq U$, $U$ is $\alpha$–open in $X$. Noiri [22] initiated the notion of weakly $\alpha$–continuous functions in topological spaces and discussed very interesting results as, weakly $\alpha$–continuous surjection preserves connected spaces and that weakly $\alpha$–continuous functions into regular spaces are continuous.

Agashe et al. [3] introduced and studied the concept of immediate predecessor and immediate successor in the lattice of topologies and the adjacent topologies and also discussed their properties. In [23] Robert et al. originated the concept of $semi^*$–closed sets by using the closure operator $C^*$ due to Dunham [6]. They investigated many fundamental properties of $semi^*$–closed sets. This class of set lies between closed sets and semi–closed sets. They also established $semi^*$–closure of any set. Missier [18] devised and studied the new notion of sets called $\alpha^*$–open sets and $\alpha^*$–closed sets and discussed the relationship of $\alpha^*$–open sets and $\alpha^*$–closed sets with some other sets. Selvi et al. [26] defined and investigated a new class of sets called $pre^*$–closed sets by using the generalized closure operator $C^*$ due to Dunham [6].

Motivation and Contribution

Sayed et al. [25] devised a new class of generalized closed sets namely $D_{\alpha}$–closed sets in topological spaces by using the generalized closure operator $C^*$ due to Dunham [6]. They characterized the $D_{\alpha}$–closed sets and $D_{\alpha}$–open sets. The new concept and the way
of representing results motivate us to generalize this concept of $D_\alpha$–closed sets. In the present paper a new notion of generalized closed sets namely $D_\beta$–closed sets has been devised. A brief synopsis of the paper is as follows: The main objective of this paper is to introduce and study $D_\beta$–closed sets, which is the generalization of $\beta$–closed sets by using the generalized closure operator $C^*$. This class of sets are the generalization of $D_\alpha$–closed sets, $pre^*$–closed sets and $semi^*$–closed sets.

This paper is organized as follows, section 1, gives basic notions which underpin our work. In section 2, we have define $D_\beta$–closed sets and discuss their characterization and basic properties and its relationships with already existing generalized closed sets. In section 3, we define $D_\beta$–open sets. In section 4, we define $D_\beta$–continuous and $D_\beta$–irresolute functions and investigate their fundamental properties.

1.1. Preliminaries

Throughout this paper $(X, \tau)$ will always denote a topological space on which no separation axioms are assumed, unless explicitly stated. If $A$ is a subset of the space $(X, \tau)$, $Cl(A)$ and $Int(A)$ denote the closure and the interior of $A$ respectively. Here we recall the following known definitions and properties.

**Definition 1.1.** Let $(X, \tau)$ be a topological space. A subset $A$ of the space $X$ is said to be,

(i) preopen [21] if $A \subseteq Int(Cl(A))$ and preclosed if $Cl(Int(A)) \subseteq A$.

(ii) semi–open[11] if $A \subseteq Cl(Int(A))$ and semi–closed if $Cl(Cl(A)) \subseteq A$.

(iii) $\alpha$–open [21] if $A \subseteq Int(Cl(Int(A)))$ and $\alpha$–closed if $Cl(Int(Cl(A))) \subseteq A$.

(iv) $\beta$–open [19] if $A \subseteq Cl(Int(Cl(A)))$ and $\beta$–closed if $Int(Cl(Cl(A))) \subseteq A$.

(v) generalized closed (briefly g–closed)[12] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.generalized open(briefly g–open) if $X \setminus A$ is g–closed.

(vi) pre$^*$–closed set [26] if $Cl^*(Int(A)) \subseteq A$ and pre$^*$–open set if $A \subseteq Int^*(Cl(A))$.


(viii) $D_\alpha$–closed [25] if $Cl^*(Int(Cl^*(A))) \subseteq A$ and $D_\alpha$–open if $X \setminus A$ is $D_\alpha$–closed.

**Definition 1.2.** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be,

(i) $\alpha$–continuous [17](resp.$\beta$–continuous [19]) if the inverse image of each open set in $Y$ is $\alpha$–open(resp.$\beta$–open) in $X$.

(ii) $g$–continuous [4] if the inverse image of each open set in $Y$ is $g$–open in $X$.

(iii) $D_\alpha$–continuous [25] if the inverse image of each open set in $Y$ is $D_\alpha$–open in $X$. 
Theorem 2.3. Let \( A \) be a topological space. The \( \alpha \)-closure \([1]\) of a subset \( A \) of \( X \) is the intersection of all \( \alpha \)-closed sets containing \( A \) and is denoted by \( \text{Cl}_\alpha(A) \). The \( \alpha \)-interior \([1]\) of a subset \( A \) of \( X \) is the union of all \( \alpha \)-open sets contained in \( A \) and is denoted by \( \text{Int}_\alpha(A) \). The \( \beta \)-closure \([19]\) of a subset \( A \) of \( X \) is the intersection of all \( \beta \)-closed sets containing \( A \) and is denoted by \( \text{Cl}_\beta(A) \). The \( \beta \)-interior \([19]\) of a subset \( A \) of \( X \) is the union of all \( \beta \)-open sets contained in \( A \) and is denoted by \( \text{Int}_\beta(A) \). The intersection of all \( g \)-closed sets containing \( A \) is denoted by \( \text{Cl}^*_\beta(A) \) and the \( g \)-interior of \( A \) \([23]\) is the union of all \( g \)-open sets contained in \( A \) and is denoted by \( \text{Int}^*(A) \).

The family of all \( D_\beta \)-closed (resp. \( D_\alpha \)-closed, \( g \)-closed, \( \beta \)-closed) sets of \( X \) denoted by \( D_\beta \text{C}(X) \) (resp. \( D_\alpha \text{C}(X) \), \( \text{GC}(X) \), \( \beta \text{C}(X) \)). The family of all \( D_\beta \)-open (resp. \( D_\alpha \)-open, \( g \)-open, \( \beta \)-open) sets of \( X \) denoted by \( D_\beta \text{O}(X) \) (resp. \( D_\alpha \text{O}(X) \), \( \text{GO}(X) \), \( \beta \text{O}(X) \)).

\[ \beta \text{O}(X, x) = \{ U : U \in \alpha \text{O}(X, \tau) \}, \quad \alpha \text{O}(X, \tau) = \{ U : U \in \alpha \text{C}(X, \tau) \}, \quad D_\alpha \text{O}(X, x) = \{ U : U \in \alpha \text{O}(X, \tau) \}, \quad D_\alpha \text{C}(X, x) = \{ U : U \in \alpha \text{C}(X, \tau) \}. \]

Lemma 1.3. \([6]\) Let \( A \subset X \), then

(i) \( X \setminus \text{Cl}^*(A) = \text{Int}^*(X \setminus A) \).

(ii) \( X \setminus \text{Int}^*(A) = \text{Cl}^*(X \setminus A) \).

2. \( D_\beta \)-Closed Set

In this section we introduce \( D_\beta \)-closed sets and investigate some of their basic properties.

Definition 2.1. A subset \( A \) of a topological space \((X, \tau)\) is called \( D_\beta \)-closed if \( \text{Int}^*(\text{Cl}^*(\text{Int}^*(A))) \subseteq A \).

Example 2.2. Let \( X = \{a, b, c, d\} \) be any set and \( \tau = \{X, \phi, \{a, b, c\}, \{a, b\}\} \), then \((X, \tau)\) be a topological space. \( \text{C}(X) = \{\phi, X, \{d\}, \{c, d\}\} \), \( \text{GC}(X) = \{\phi, X, \{d\}, \{c, d\}, \{a, b, d\}, \{a, d\}, \{a, c, d\}, \{b, d\}, \{b, c, d\}\} \), \( \text{GO}(X) = \{X, \phi, \{a, b, c\}, \{a, b\}, \{c\}, \{b, c\}, \{a, c\}, \{b, d\}, \{a, b, c\}\} \), \( D_\alpha \text{C}(X) = \{X, \phi, \{d\}, \{c, d\}, \{a, b, d\}, \{a, d\}, \{a, c, d\}, \{b, d\}, \{b, c, d\}, \{a, c\}, \{c\}, \{b\}, \{a\}, \{b, c\}\} \) and \( D_\beta \text{C}(X) = \{X, \phi, \{d\}, \{c, d\}, \{a, b, d\}, \{a, d\}, \{a, c, d\}, \{b, d\}, \{b, c, d\}, \{a, c\}, \{c\}, \{b\}, \{a\}, \{b, c\}, \{a, b\}\} \).

Theorem 2.3. Let \((X, \tau)\) be a topological space, then

(i) Every \( \beta \)-closed subset of \((X, \tau)\) is \( D_\beta \)-closed.

(ii) Every \( g \)-closed subset of \((X, \tau)\) is \( D_\beta \)-closed.

(iii) Every \( \text{semi}^* \)-closed subset of \((X, \tau)\) is \( D_\beta \)-closed.

(iv) Every \( \text{pre}^* \)-closed subset of \((X, \tau)\) is \( D_\beta \)-closed.

(v) Every \( D_\alpha \)-closed subset of \((X, \tau)\) is \( D_\beta \)-closed.
Proof. (i) Let $A$ be any $\beta$–closed subset of the space $X$, then we have $\text{Int}(\text{Cl}(\text{Int}(A))) \subseteq A$. We know that $\text{Int}^*(A) \subseteq \text{Int}(A)$, then we have $\text{Cl}^*(\text{Int}^*(A)) \subseteq \text{Cl}^*(\text{Int}(A))$

$\text{Int}^*(\text{Cl}^*(\text{Int}^*(A))) \subseteq \text{Int}^*(\text{Cl}^*(\text{Int}(A)) \subseteq \text{Int}^*(\text{Cl}^*(\text{Int}(A))) \subseteq A$.

(ii) Let $A$ be any $g$–closed subset of the space $X$, then we have $\text{Cl}^*(A) = A$. Since $\text{Int}^*(A) \subseteq A$, then we have $\text{Cl}^*(\text{Int}^*(A)) \subseteq \text{Cl}^*(A) = A$

$\text{Int}^*(\text{Cl}^*(\text{Int}^*(A))) \subseteq \text{Int}^*(\text{Cl}^*(A)) &\subseteq A$ i.e. $A$ is $D_\beta$–closed.

(iii) Let $A$ be any $\text{semi}^*$–closed subset of the space $X$, then we have $\text{Cl}(\text{Int}^*(A)) \subseteq A$. Let $\text{Int}^*(A) \subseteq A$ and then we have $\text{Cl}^*(\text{Int}^*(A)) \subseteq \text{Cl}^*(A) \subseteq \text{Cl}(A)$. Thus we get $\text{Int}^*(\text{Cl}^*(\text{Int}^*(A))) \subseteq \text{Int}^*(\text{Cl}(A)) \subseteq A$.

([iv]) Let $A$ be any $\text{pre}^*$–closed subset of the space $X$, then we have $\text{Cl}^*(\text{Int}(A)) \subseteq A$. Let $\text{Int}^*(A) \subseteq \text{Int}(A)$, then we have $\text{Cl}^*(\text{Int}^*(A)) \subseteq \text{Cl}^*(\text{Int}(A)) \subseteq A$. This implies that $\text{Int}^*(\text{Cl}^*(\text{Int}^*(A))) \subseteq \text{Int}^*(A) \subseteq A$.

(v) It follows from (i).

\[\triangleq\]

Remark 2.4. The converse of Theorem 2.3 is not true as shown in the following example.

(i) $D_\beta$–closed set need not be $\beta$–closed. (see Example 2.5 below)

(ii) $D_\beta$–closed set need not be $g$–closed. (see Example 2.5 below)

(iii) $D_\beta$–closed set need not be $\text{semi}^*$–closed. (see Example 2.5 below)

(iv) $D_\beta$–closed set need not be $\text{pre}^*$–closed. (see Example 2.5 below)

(v) $D_\beta$–closed set need not be $D_\alpha$–closed. (see Example 2.5 below)

Example 2.5. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a, b\}, \{a, d\}, \{a\}, \{a, b, d\}\}$. Then $(X, \tau)$ be a topological space. $\text{C}(X) = \{\phi, X, \{c, d\}, \{b, c\}, \{b, c, d\}, \{c\}, \{a, b, c\}, \{a, c\}, \{a, c, d\}\}$, $\text{GC}(X) = \{\phi, X, \{c, d\}, \{b, c\}, \{b, c, d\}, \{c\}, \{a, c\}, \{a, c, d\}\}$, $\text{GO}(X) = \{X, \phi, \{a, b\}, \{a, d\}, \{a\}, \{a, b, d\}, \{b\}, \{b, d\}\}$, $\beta\text{C}(X) = \{\phi, X, \{c, d\}, \{b, c\}, \{b, c, d\}, \{c\}, \{a, b\}, \{a, c\}, \{a, c, d\}\}$, $\beta\text{O}(X) = \{\phi, X, \{a, b\}, \{a, d\}, \{a\}, \{a, b, d\}, \{b\}, \{a, c\}, \{a, c, d\}\}$, $D_\beta\text{C}(X) = \{\phi, X, \{c, d\}, \{b, c\}, \{b, c, d\}, \{c\}, \{a, c\}, \{a, c, d\}\}$, $D_\beta\text{O}(X) = \{\phi, X, \{a, b\}, \{a, d\}, \{a\}, \{a, b, d\}, \{b\}, \{a, c\}, \{a, c, d\}\}$ and $D_\alpha\text{O}(X) = \{\phi, X, \{a, b\}, \{a, d\}, \{a\}, \{a, b, d\}, \{b\}, \{a, c\}, \{a, c, d\}\}$.

$D_\beta\text{O}(X) = \{\phi, X, \{a, b\}, \{a, d\}, \{a\}, \{a, b, d\}, \{b\}, \{a, c\}, \{a, c, d\}\}$.

Let $A = \{a, b\}$ is a $D_\beta$–closed set of $X$ but $\{a, b\}$ is not a $\beta$–closed neither $g$–closed nor a $D_\alpha$–closed.
**Theorem 2.6.** Arbitrary intersection of $D_\beta$–closed sets is $D_\beta$–closed.

**Proof.** Let $\{G_\alpha : \alpha \in \Delta\}$ be a collection of $D_\beta$–closed sets in $X$. Then

\[\text{Int}^*(\text{Cl}^*(\text{Int}^*(G_\alpha))) \subseteq G_\alpha\]

for each $\alpha$. Since $\cap G_\alpha \subseteq G_\alpha$ for each $\alpha$, $\text{Int}^*(\cap G_\alpha) \subseteq \text{Int}^*(G_\alpha)$ for each $\alpha$. Therefore $\text{Int}^*(\cap G_\alpha) \subseteq \cap \text{Int}^*(G_\alpha)$, $\alpha \in \Delta$.

Hence $\text{Int}^*(\cap G_\alpha) \subseteq \cap \text{Int}^*(G_\alpha)$ for each $\alpha$. Therefore $\cap G_\alpha$ is $D_\beta$–closed. ■

**Remark 2.7.** The union of two $D_\beta$–closed sets need not be $D_\beta$–closed.

**Example 2.8.** In the Example 2.2, the sets $\{a, b\}$ and $\{b, d\}$ both are $D_\beta$–closed but their union $\{a, b\} \cup \{b, d\} = \{a, b, d\}$ is not $D_\beta$–closed.

**Remark 2.9.** The collection of $D_\beta C(X)$ does not form a topology.

**Corollary 2.10.** Let $A$ and $B$ are any two subsets of the space $X$, where $A$ is $D_\beta$–closed and $B$ is $\beta$–closed, then $A \cap B$ is $D_\beta$–closed.

**Proof.** It follows directly from the Theorems 2.3 and 2.6. ■

**Corollary 2.11.** If a subset $A$ is $D_\beta$–closed and $B$ is $g$–closed, then $A \cap B$ is $D_\beta$–closed.

**Proof.** It follows directly from Theorems 2.3 and 2.6. ■

**Definition 2.12.** Let $A$ be any subset of a space $X$. The $D_\beta$–closure of $A$ is the intersection of all $D_\beta$–closed sets in $X$ containing $A$ i.e. $\text{Cl}_{D_\beta}(A) = \bigcap \{G : A \subseteq G \text{ and } G \in D_\beta C(X)\}$. It is denoted by $\text{Cl}_{D_\beta}(A)$.

**Theorem 2.13.** Let $A$ be a subset of $X$. Then $A$ is $D_\beta$–closed set in $X$ if and only if $\text{Cl}_{D_\beta}(A) = A$.

**Proof.** Suppose $A$ is $D_\beta$–closed set in $X$. Since $\text{Cl}_{D_\beta}(A)$ is equal to the intersection of all $D_\beta$–closed sets in $X$ containing $A$. Since $A \subseteq \text{Cl}_{D_\beta}(A)$, therefore $\text{Cl}_{D_\beta}(A) = A$. Let $\text{Cl}_{D_\beta}(A) = A$. Then $A$ is $D_\beta$–closed set in $X$. ■

**Theorem 2.14.** Let $A$ and $B$ be subsets of $X$. Then the following results hold.

(i) $A \subseteq \text{Cl}_{D_\beta}(A) \subseteq \text{Cl}_\beta(A)$, $\text{Cl}_{D_\beta}(A) \subseteq \text{Cl}^*(A)$, $\text{Cl}_{D_\beta}(A) \subseteq \text{Cl}_{D_\alpha}(A)$.

(ii) $\text{Cl}_{D_\beta}(A) = \phi$ and $\text{Cl}_{D_\beta}(X) = X$.

(iii) If $A \subseteq B$, then $\text{Cl}_{D_\beta}(A) = \text{Cl}_{D_\beta}(B)$.

(iv) $\text{Cl}_{D_\beta}(\text{Cl}_{D_\beta}(A)) = \text{Cl}_{D_\beta}(A)$

(v) $\text{Cl}_{D_\beta}(A) \cup \text{Cl}_{D_\beta}(B) \subseteq \text{Cl}_{D_\beta}(A \cup B)$
(vi) \( Cl_{D_\beta}(A \cap B) \subseteq Cl_{D_\beta}(A) \cap Cl_{D_\beta}(B) \).

**Proof.**

(i) It follows directly from the Theorem 2.3.

(ii) It is trivially true.

(iii) It is trivially true.

(iv) It follows from the facts that \( Cl_{D_\beta}(A) \) is itself a \( D_\beta \)-closed set and \( D_\beta \)-closed set is a smallest closed sets containing \( A \) and from the Theorem 2.13.

(v) (v) and (vi) follows from (iii). ■

### 2.1. Interrelationship

The following diagram will describe the interrelations among \( D_\beta \)-closed set and other existing generalized–closed sets. None of these implications is reversible as shown by examples given below and well known facts.

3. \( D_\beta \)-open sets

In this section we introduce \( D_\beta \)-open sets and investigate some of their basic properties.

**Definition 3.1.** A subset \( A \) of a space \((X, \tau)\) is called an \( D_\beta \)-open if \( X \setminus A \) is \( D_\beta \)-closed. Let \( D_\beta O(X) \) denotes the collection of all \( D_\beta \)-open sets in \( X \).

**Lemma 3.2.** Let \( A \subseteq X \), then

(i) \( X \setminus Cl^*(X \setminus A) = Int^*(A) \).

(ii) \( X \setminus Int^*(X \setminus A) = Cl^*(A) \).
Proof. It is trivially true. ■

**Theorem 3.3.** A subset $A$ of a space $X$ is $D_\beta$-open if and only if $A \subseteq Cl^*(Int^*(Cl^*(A)))$.

Proof. Let $A$ be any $D_\beta$-open set. Then $X \setminus A$ is $D_\beta$-closed and $Int^*(Cl^*(Int^*(X \setminus A))) \subseteq X \setminus A$. Therefore $A \subseteq (X \setminus Int^*(Cl^*(Int^*(A)))) = Cl^*(Int^*(Cl^*(A)))$. Thus, we have $A \subseteq Cl^*(Int^*(Cl^*(A)))$. ■

**Theorem 3.4.** Let $(X, \tau)$ be a topological space. Then

(i) Every $\beta$-open subset of $(X, \tau)$ is $D_\beta$-open.

(ii) Every $g$-open subset of $(X, \tau)$ is $D_\beta$-open.

(iii) Every semi*-open subset of $(X, \tau)$ is $D_\beta$-open.

(iv) Every pre*-open subset of $(X, \tau)$ is $D_\beta$-open.

(v) Every $D_\alpha$-open subset of $(X, \tau)$ is $D_\beta$-open.

Proof. It is directly follows from the Theorem 2.3. ■

**Remark 3.5.** The converse of the above theorem is not true as seen from Example 2.2, the set $\{b, c\}$ is $D_\beta$-open but it is not $\beta$-open nor a $g$-open and nor a $D_\alpha$-open set.

**Theorem 3.6.** Arbitrary union of $D_\beta$-open set is $D_\beta$-open.

Proof. It follows from the Theorem 2.6. ■

**Remark 3.7.** The intersection of two $D_\beta$-open sets need not be $D_\beta$-open as seen from Example 2.5, in which two $D_\beta$-open sets are $A = \{a, c\}$ and $B = \{c, d\}$ but their intersection $A \cap B = \{c\}$ is not $D_\beta$-open set.

**Corollary 3.8.** If a subset $A$ is $D_\beta$-open and $B$ is $\beta$-open, then $A \cup B$ is $D_\beta$-open.

Proof. It follows from the Theorems 2.13 and 2.14. ■

**Corollary 3.9.** If a subset $A$ is $D_\beta$-open and $B$ is $g$-open, then $A \cup B$ is $D_\beta$-open.

Proof. It follows from the Theorems 2.13 and 2.14. ■

**Definition 3.10.** Let $A$ be any subset of a space $X$. The $D_\beta$-interior of $A$ is denoted by $Int_{D_\beta}(A)$, is the union of all the $D_\beta$-open sets in $X$, contained in $A$ i.e. $Int_{D_\beta}(A) = \bigcup\{U : U \subseteq A, U \in D_\beta O(X)\}$.

**Lemma 3.11.** If $A$ be any subset of $X$, then

(i) $X \setminus Cl_{D_\beta}(A) = Int_{D_\beta}(X \setminus A)$. 


(ii) \( X \setminus \text{Int}_{D_\beta}(A) = \text{Cl}_{D_\beta}(X \setminus A) \).

**Proof.** It is obvious. ■

**Theorem 3.12.** Let \( A \) be any subset of \( X \). Then \( A \) is \( D_\beta \)-open if and only if \( \text{Int}_{D_\beta}(A) = A \).

**Proof.** It follows from Theorem 2.13 and Lemma 3.11. ■

**Theorem 3.13.** Let \( A \) and \( B \) be subsets of \( X \). Then the following results hold.

1. \( \text{Int}_\beta(A) \subseteq \text{Int}_{D_\beta}(A) \subseteq A, \text{Int}^*(A) \subseteq \text{Int}_{D_\beta}(A) \).
2. \( \text{Int}_{D_\beta}(A) = X, \text{Int}_{D_\beta}(A) = \emptyset \).
3. If \( A \subseteq B \), then \( \text{Int}_{D_\beta}(A) \subseteq \text{Int}_{D_\beta}(B) \).
4. \( \text{Int}_{D_\beta}(\text{Int}_{D_\beta}(A)) = \text{Int}_{D_\beta}(A) \).
5. \( \text{Int}_{D_\beta}(A) \cup \text{Int}_{D_\beta}(B) \subseteq \text{Int}_{D_\beta}(A \cup B) \).
6. \( \text{Int}_{D_\beta}(A) \cap \text{Int}_{D_\beta}(A) \subseteq \text{Int}_{D_\beta}(A \cap B) \).

**Proof.** It is obvious. ■

**Definition 3.14.** Let \( X \) be any topological space and let \( x \in X \), then a subset \( G_x \) of \( X \) is said to be \( D_\beta \)-neighborhood of \( x \) if there exists a \( D_\beta \)-open set \( U \) in \( X \) such that \( x \in U \subseteq G_x \).

**Theorem 3.15.** Let \( x \in X \), then \( x \in \text{Cl}_{D_\beta}(A) \) if and only if \( U \cap A = \emptyset \) for every \( D_\beta \)-open set \( U \) containing \( x \).

**Proof.** Let \( x \in \text{Cl}_{D_\beta}(A) \) and on contrary assume that, there exists a \( D_\beta \)-open set \( U \) containing \( x \) such that \( U \cap A = \emptyset \). Then \( A \subseteq X \setminus U \) and \( X \setminus U \) is \( D_\beta \)-closed. Therefore \( \text{Cl}_{D_\beta}(A) \subseteq \text{Cl}_{D_\beta}(X \setminus U) = X \setminus U. \) Therefore \( x \notin \text{Cl}_{D_\beta}(A) = X \setminus U \) i.e. \( U \cap A = \phi \), which is contradiction to the assumption.

Conversely, suppose \( U \cap A \neq \phi \), on contrary, we assume that for every \( D_\beta \)-open set \( U \) containing \( x \) and \( x \notin \text{Cl}_{D_\beta}(A) \). Then there exists \( D_\beta \)-closed subset \( G \) containing \( A \) such that \( x \) not belongs to \( G \). Therefore \( x \in X \setminus G \) and \( X \setminus G \) is \( D_\beta \)-open. Since \( A \subseteq G \), \( (X \setminus G) \cap A = \phi \), which is contradiction to the assumption. Hence the result. ■

**Definition 3.16.** Let \( A \) be a subset of a space \( X \). A point \( x \in X \) is said to be a \( D_\alpha \)-limit point of \( A \) if for each \( D_\alpha \)-open set \( U \) containing \( x \), we have \( U \cap (A \setminus \{x\}) \neq \phi \). The set of all \( D_\alpha \)-limit points of \( A \) is called the \( D_\alpha \)-derived set of \( A \) and it is denoted by \( D\alpha(A) \).

**Definition 3.17.** Let \( A \) be a subset of a space \( X \). A point \( x \in X \) is said to be a \( D_\beta \)-limit point of \( A \) if for each \( D_\beta \)-open set \( U \) containing \( x \), we have \( U \cap (A \setminus \{x\}) \neq \phi \). The set of all \( D_\beta \)-limit points of \( A \) is called the \( D_\beta \)-derived set of \( A \) and is denoted by \( D\beta(A) \).
Remark 3.18. Since every open set is $D_\alpha$-open, we have $D_{D_\alpha}(A) \subseteq D(A)$ and therefore $D_{D_\beta}(A) \subseteq D(A)$ for any subset $A \subseteq X$, where $D(A)$ is the derived set of $A$. Moreover, since every closed set is $D_\alpha$-closed, we have $A \subseteq Cl_{D_\beta}(A) \subseteq Cl_{D_\alpha}(A) \subseteq Cl(A)$.

4. $D_\beta$-continuous and $D_\beta$-irresolute functions

In this section we introduce $D_\beta$-continuous functions and study some of their basic properties.

Definition 4.1. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $D_\beta$-continuous if the inverse image of each open set in $Y$ is $D_\beta$-open in $X$.

Theorem 4.2.

(i) Every $\beta$-continuous function is $D_\beta$-continuous.

(ii) Every $g$-continuous function is $D_\beta$-continuous.

(iii) Every semi*-continuous function is $D_\beta$-continuous.

(iv) Every pre*-continuous function is $D_\beta$-continuous.

(v) Every $D_\alpha$-continuous function is $D_\beta$-continuous.

Proof. It follows directly from the Theorem 2.13.

Remark 4.3.

(i) $D_\beta$-continuous function need not be $\beta$-continuous.
(see Example 4.4 below)

(ii) $D_\beta$-continuous function need not be $g$-continuous.
(see Example 4.4 below)

(iii) $D_\beta$-continuous function need not be semi*-continuous.
(see Example 4.4 below)

(iv) $D_\beta$-continuous function need not be pre*-continuous.
(see Example 4.4 below)

(v) $D_\beta$-continuous function need not be $D_\alpha$-continuous.
(see Example 4.4 below)

Example 4.4. Let $X=\{a, b, c\}$, $\tau=\{\emptyset, X, \{a\}, \{a, b\}\}$, then $(X, \tau)$ is a topological space. $C(X)=\{X, \phi, \{b, c\}, \{c\}\}$. Let $Y=\{1, 2, 3\}$, $\sigma=\{\phi, Y, \{1\}, \{1, 2\}\}$, then $(Y, \sigma)$ be another topological space. $GC(X)=\{X, \phi, \{b, c\}, \{c\}, \{a, c\}\}$,

$GO(X)=\{X, \phi, \{a, b\}, \{a\}, \{b\}\}$,
Theorem 4.5. Let 

\[ D_\alpha C(X) = \{ X, \phi, \{ b, c \}, \{ c \}, \{ a, c \}, \{ b \}, \} \]

\[ D_\beta O(X) = \{ X, \phi, \{ a, b \}, \{ b \}, \{ a, c \}, \{ b \}, \} \]

\[ \beta O(X) = \{ X, \phi, \{ a, b \}, \{ a \}, \} \]

\[ D_\beta O(X) = \{ X, \phi, \{ a, b \}, \{ a \}, \} \]

Then we get \( f( ClD_\beta (A)) \subset B \) and we get \( f(A) \subset \beta O(X) \). Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function defined by,

(i) \( f(a) = 3, f(b) = 1, f(c) = 2 \) is \( D_\beta \)-continuous, since the inverse image of each open set in \( Y \) is \( D_\beta \)-open in \( X \). But it is not \( \beta \)-continuous since the preimage of an open set \( A = \{ 1, 2 \} \) in \( Y \) is \( \{ b, c \} \), which is not \( \beta \)-open set in \( X \).

(ii) \( f(a) = 3, f(b) = 1, f(c) = 2 \) is \( D_\beta \)-continuous, since the inverse image of each open set in \( Y \) is \( D_\beta \)-open in \( X \). But it is not \( g \)-continuous, since the preimage of an open set \( A = \{ 1, 2 \} \) in \( Y \), is \( \{ b, c \} \), which is not \( g \)-open set in \( X \).

(iii) \( f(a) = 3, f(b) = 1, f(c) = 2 \) is \( D_\beta \)-continuous, since the inverse image of each open set in \( Y \) is \( D_\beta \)-open in \( X \). But it is not \( D_\alpha \)-continuous, since the preimage of an open set \( A = \{ 1, 2 \} \) in \( Y \), is \( \{ b, c \} \), which is not \( D_\alpha \)-open set in \( X \).

Theorem 4.5. Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function. Then the following are equivalent:

(i) \( f \) is \( D_\beta \)-continuous.

(ii) \( f(ClD_\beta (A)) \subset Cl(f(A)) \) for every subset \( A \) of \( X \).

(iii) The inverse image of each closed set in \( Y \) is \( D_\beta \)-closed in \( X \).

(iv) For each \( x \in X \) and each open set \( U \subset Y \) containing \( f(x) \), there exists a \( D_\beta \)-open set \( V \subset X \) containing \( x \) such that \( f(V) \subset U \).

(v) \( ClD_\beta (f^{-1}(B)) \subset f^{-1}(Cl(B)) \) for every subset \( B \) of \( Y \).

(vi) \( f^{-1}(Int(A)) \subset IntD_\beta (f^{-1}(A)) \) for every subset \( A \) of \( Y \).

Proof. (i) \( \Rightarrow \) (ii) Suppose \( f \) is \( D_\beta \)-continuous and let \( A \) be any subset of \( X \). Let \( x \in ClD_\beta (A) \), then \( f(x) \in f(ClD_\beta (A)) \). Suppose \( U \) be an open neighborhood of \( f(x) \). Then \( f^{-1}(U) \) is a \( D_\beta \)-open set of \( X \) containing \( x \) and it intersects \( A \) in the point \( y \) (other than \( x \)). Then the set \( U \) intersects \( f(A) \) in the point \( f(y) \), therefore \( f(x) \in Cl(f(A)) \) and we get \( f(ClD_\beta (A)) \subset Cl(f(A)) \).

(ii) \( \Rightarrow \) (iii) Suppose the function \( f \) is \( D_\beta \)-continuous. Let \( A \) be any closed set in \( Y \) and let \( B = f^{-1}(A) \). Since \( B \) is \( D_\beta \)-closed in \( X \). We show that \( ClD_\beta (B) = B \). For, \( f(B) = f(f^{-1}(A)) \subset A \). Suppose \( x \in ClD_\beta (B) \). Then we have \( f(x) \in f(ClD_\beta (B)) \subset Cl(f(B)) \subset Cl(A) = A \). Thus \( x \in f^{-1}(A) = B \). Therefore \( ClD_\beta (B) \subset B \). Since \( B \subset ClD_\beta (B) \). Hence we have \( B = ClD_\beta (B) \) i.e. \( B = f^{-1}(A) \) is a \( D_\beta \)-closed set in \( X \).
(iii) ⇒ (i) Since function \( f \) is \( D_\beta \)-continuous. Suppose \( U \) be any open set of \( Y \). Let \( A = Y \setminus U \) be any closed set in \( Y \). Then \( f^{-1}(A) = f^{-1}(Y) \setminus f^{-1}(U) = X \setminus f^{-1}(U) \) is a \( D_\beta \)-closed set in \( X \). Therefore \( f^{-1}(U) \) is open in \( X \) and hence \( f \) is \( D_\beta \)-continuous.

(i) ⇒ (iv) Suppose \( f \) is \( D_\beta \)-continuous function. Suppose for each \( x \in X \) and for each open set \( U \subset Y \) containing \( f(x) \), \( f^{-1}(U) \in D_\beta O(X) \). We set \( V = f^{-1}(U) \) containing \( x \), we get \( f(V) \subset U \).

(iv) ⇒ (i) Let \( U \) be an open set in \( Y \), containing \( f(x) \) for each \( x \in X \), then there exists a \( D_\beta \)-open set \( V_x \) (open neighborhood of \( x \)) containing \( x \) such that \( f(V_x) \subset U \) and then \( x \in V_x \subset f^{-1}(U) \), which shows that \( f^{-1}(U) \) is open in \( X \). Hence \( f \) is \( D_\beta \)-continuous.

(ii) ⇒ (v) Let \( B \) be any subset of \( Y \) and \( A = f^{-1}(B) \) is the subset of \( Y \). By hypothesis \( f(Cl(D_\beta(A)) \subset Cl(f(A)) \) for every subset \( A \) of \( X \), then we have \( f(Cl(D_\beta(f^{-1}(B)))) \subset Cl(f(f^{-1}(B))) \subset Cl(B) \) and therefore we get \( (Cl(D_\beta(f^{-1}(B)) \subset f^{-1}(Cl(B)) \).

(v) ⇒ (vi) Let \( F \) be any subset of \( Y \). By hypothesis \( (Cl(D_\beta(f^{-1}(Y \setminus F))) \subset f^{-1}(Cl(Y \setminus F)) \). This shows that \( (Cl(D_\beta(X \setminus f^{-1}(F))) \subset f^{-1}(Y \setminus Int(F)) \). Therefore \( X \setminus Int(D_\beta(f^{-1}(F))) \subset X \setminus f^{-1}(Int(F)) \). Hence we get \( f^{-1}(Int(F)) \subset Int(D_\beta(f^{-1}(F))) \).

Proof. It is obvious.

\[ \blacksquare \]

**Theorem 4.6.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be \( D_\beta \)-continuous and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) be continuous functions. Then their composition \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is \( D_\beta \)-continuous.

**Proof.** It is obvious.

\[ \blacksquare \]

**Remark 4.7.** Composition of two \( D_\beta \)-continuous functions need not be \( D_\beta \)-continuous.

**Example 4.8.** Let \( X = \{a, b, c\}, \tau = \{X, \phi, \{a, c\}, \{c\}\}, C(X) = \{X, \phi, \{a, b\}, \{b\}\} \), then \( (X, \tau) \) is a topological space.

\( D_\beta C(X) = \{X, \phi, \{a, b\}, \{b\}, \{a, c\}, \{c\}, \{a\}, \{b, c\}, \{a, b\}\} \).

Let \( Y = \{1, 2, 3\}, \sigma = \{Y, \phi, \{2, 3\}\} \) then \( (Y, \sigma) \) is a topological space.

\( C(Y) = \{Y, \phi, \{1\}\} \).

\( D_\beta C(Y) = \{Y, \phi, \{1, 3\}, \{1, 2\}, \{1\}, \{2\}, \{3\}\} \).

\( D_\beta O(Y) = \{Y, \phi, \{2\}, \{2, 3\}, \{3\}, \{1, 2\}, \{1, 3\}\} \).

Let \( Z = \{r, s, t\}, \eta = \{Z, \phi, \{s\}\} \) then \( (Z, \eta) \) is another topological space.

Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a function defined by \( f(a) = 2, f(b) = 3 \) and \( f(c) = 1 \) and another function \( g : (Y, \sigma) \rightarrow (Z, \eta) \) defined as \( g(1) = r, g(2) = t, g(3) = s \). Here both the functions \( f \) and \( g \) are \( D_\beta \)-continuous. Let \( A = \{s\} \) be any open set in \( Z \), but \( (g \circ f)^{-1}(s) = f^{-1}(g^{-1}(s)) = f^{-1}(3) = \{b\} \), which is not a \( D_\beta \)-open set in \( X \).

**Theorem 4.9.** If a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( D_\alpha \)-continuous and \( g : (Y, \sigma) \rightarrow (Z, \eta) \) is continuous, then \( g \circ f : (X, \tau) \rightarrow (Z, \eta) \) is also \( D_\alpha \)-continuous.

**Proof.** Let \( B \) be any open set in (\( Z, \eta) \). Since \( g \) is continuous, therefore \( g^{-1}(B) \) is open.
in $(Y, \sigma)$. Since $f$ is $D_\alpha$-continuous, we have $f^{-1}(g^{-1}(B)) = (g \circ f)^{-1}(B)$ is $D_\alpha$-open in $X$. Thus $g \circ f$ is $D_\alpha$-continuous. □

**Theorem 4.10.** If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $D_\alpha$-continuous and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is continuous, then $g \circ f : (X, \tau) \rightarrow Z, \eta$ is $D_\beta$-continuous.

**Proof.** It is obvious. □

**Definition 4.11.** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $D_\alpha$-irresolute if the preimage of each $D_\alpha$-closed($D_\alpha$-open) set in $Y$ is $D_\alpha$-closed ($D_\alpha$-open) in $X$.

**Definition 4.12.** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $D_\beta$-irresolute if the preimage of each $D_\beta$-closed($D_\beta$-open) set in $Y$ is $D_\beta$-closed ($D_\beta$-open) in $X$.

**Theorem 4.13.** If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $D_\alpha$-irresolute and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is $D_\alpha$-irresolute, then $g \circ f : (X, \tau) \rightarrow Z, \eta$ is $D_\alpha$-irresolute.

**Proof.** Let $A$ be any $D_\alpha$-closed set in the space $(Z, \eta)$. Since $g$ is $D_\alpha$-irresolute, therefore $g^{-1}(A)$ is $D_\alpha$-closed set in $Y$. Since $f$ is $D_\alpha$-irresolute, then $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$ is $D_\alpha$-closed in $X$. Hence $(g \circ f)$ is $D_\alpha$-irresolute. □

**Theorem 4.14.** If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $D_\beta$-irresolute and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is $D_\beta$-irresolute then $g \circ f : (X, \tau) \rightarrow Z, \eta$ is also $D_\beta$-irresolute.

**Proof.** Its proof is similar to the Theorem 4.13. □

**Remark 4.15.** Every $D_\alpha$-irresolute function is $D_\beta$-irresolute but the converse is not true.

**Example 4.16.** Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{b\}, \{b, c\}\}$, then $(X, \tau)$ is a topological space. $C(X) = \{\phi, X, \{a, c, d\}, \{a, d\}\}$. Let $Y = \{1, 2, 3\}$, $\sigma = \{\phi, Y, \{1, 3\}\}$, then $(Y, \sigma)$ is another topological space. $C(Y) = \{Y, \phi, \{2\}\}$.

$D_\alpha C(X) = \{X, \phi, \{a, c, d\}, \{a, d\}, \{a, b\}, \{a, b, c\}, \{d\}, \{a\}, \{a, c\}, \{b, c, d\}, \{b, d\}, \{c\}, \{a, b, d\}\}$

$D_\beta C(X) = \{X, \phi, \{a, c, d\}, \{a, d\}, \{a, b\}, \{a, b, c\}, \{d\}, \{a\}, \{a, c\}, \{b, c, d\}, \{a, b, d\}, \{c\}, \{a, b, d\}\}$

And $D_\alpha O(X) = \{X, \phi, \{b\}, \{b, c\}, \{c\}, \{a, b, d\}\}$

$D_\beta O(X) = \{X, \phi, \{b\}, \{b, c\}, \{c\}, \{a, b, d\}\}$

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by $f(a) = 1, f(c) = 3, f(b) = 2$, is $D_\beta$-irresolute. Since the preimage of every $D_\beta$-closed set in $X$ is $D_\beta$-closed in $Y$. But it is not $D_\alpha$-closed, since the preimage of the $D_\alpha$-closed set $A = \{2, 3\}$ is $\{b, c\}$, which is not $D_\alpha$-closed in $X$. 

References


