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Fixed Point in Modified Semi-linear Uniform Spaces

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Abstract

Tallafha, A. and Alhihi, S. in [17], Defined $m-$contraction, and modefid semi-linear uniform space (X, Γ), and asked the following question. If f is an $m-$contraction from a complete modified semi-linear uniform space (X, Γ) to it self, is f has a unique fixed point. In this paper we shall answer partially the question given by Tallafha, A. and Alhihi, S. in [17] for $2-$contraction, besides we shall give an intrested properties of modefied semi-linear uniform spaces.

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1. Introduction

Fixed point results provide conditions under which maps have solutions. The theory itself is a beautiful mixture of analysis (pure and applied), topology, and geometry. Over the last 50 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. In particular, fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, engineering, game theory, and physics. Contraction functions on complete metric spaces played an

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important role in the theory of fixed point (Banach fixed point). Lipschitz condition, and contractions are usually discussed in metric and normed spaces and never been studied in other weaker spaces.

In this paper we shall study fixed point theory in a space which is a beautiful mixture of analysis and topology, a space which is weaker than metric space and stronger than topological space, called semi-linear uniform space. Semi-linear uniform space is a new type of uniform space given by Tallafha, A. and Khalil, R. in 2009 [12] . The notion of uniformity has been investigated by several mathematician as Weil [18] , [19] , and [20] . L.W. Cohen [4] , and [5] . Graves [7] . Uniform spaces are topological spaces with additional structure that is used to define uniform properties, such as completeness, uniform continuity and uniform convergence. The theory of uniform spaces was given by Burbaki in [3]. Also Wiel´s in his book [18] define uniformly continuous mapping based on uniform space. Tallafha, A. and Khalil, R. established the definition of semi-linear uniform space [12] , and they studied some cases of best approximation in such spaces, besides they defined a set valued map ρ, called metric type, on semi-linear uniform spaces. In [13], [14] and [15] . Tallafha, A. gave an example of semi-linear uniform space which is not metrizable, also he defined another set valued map called δ on $X \times X$, which is used with ρ to give more properties of semi-linear uniformspaces. Finally he showed that their is a strong relation between ρ and δ, where the relation is $\rho(x, y) = \rho(s, t)$ if and only if $\delta(x, y) = \delta(s, t)$. Also he defined Lipschitz condition, and contraction mapping on semi-linear uniform spaces, which enables one to study fixed point for such functions.

In [1] , Alhihi, S. Established more properties of semi-linear uniform space. Also in [16], Tallafha, A. and Alhihi, S. established another properties of semi-linear uniform spaces and they asked the following question. If f is a contraction from a complete semi-linear uniform space (X, Γ) to it self, is f has a unique fixed point.

In [11] Rawashdeh A. and Tallafha, A. answered the above question negatively, they gave an example of a complete semi-linear uniform space (X, Γ) and a contraction $f : (X, \Gamma) \to (X, \Gamma)$ which has infinitely many fixed points. Also they defined strong semi-linear uniform space and strong contraction, finally they gave some open questions related to the subject.

In [17] , Tallafha, A. and Alhihi, S, Defined a modified semi-linear uniform space (X, Γ), and asked the following question. **If f is a contraction from a complete modified semi-linear uniform space (X, Γ) to it self, is f has a unique fixed point.** In this paper we shall answer the question given by Tallafha, A. and Alhihi, S. Positively for 2-contraction, besides we shall give an interested properties of modified semi-linear uniform spaces.

2. Semi-linear uniform space

Uniform spaces are topological spaces with additional structure that is used to define uniform properties, such as completeness, uniform continuity and uniform convergence. For more details a bout the structure of uniform spaces, the reader can refer to [6], and [8].

Let X be a non-empty set and D_X be a collection of relations on X, such that each element V of D_X is reflexive and symmetric. D_X is called the family of all entourages of the diagonal. By a chain in $X \times X$ we mean a totally (or linearly) ordered collection of subsets of $X \times X$, ordered by set inclusion.

Definition 2.1. [12] Let Γ be a subcollection of D_X. If Γ is chain, then the pair (X, Γ) is called semi-linear uniform space if:

(i) If $V_1$and V_2 are in Γ, then $V_1 \cap V_2 \in \Gamma$.

(ii) For every $V \in \Gamma$, there exists $U \in \Gamma$ such that $U \circ U \subset V$.

(iii) $\bigcap\limits_{V \in \Gamma} V = \Delta$.

(vi) $\bigcup\limits_{V \in \Gamma} V = X \times X$.

The following is an example of a semi-linear uniform space which is metrizable.

Example 2.2. Let $V_t = \{(x, y) : y - t < x < y + t, \text{ and } -\infty < y < \infty\}$. Then $(\mathbb{R}, \Gamma)$ with $\Gamma = \{V_t : 0 < t < \infty\}$ is a semi linear uniform space.

In [13], Tallafha, gave the following example, which is a semi-linear uniform space but not metrizable.

Example 2.3. $X = \mathbb{R}^2$ and $\Gamma = \{V_\epsilon, \epsilon > 0\}$, $V_\epsilon = \{(x, y) : x^2 + y^2 < \epsilon\} \cup \{\Delta\}$.

It is known that every metric space endues a semi-linear uniform space, the following define the semi-linear uniform space which induced by a metric space (X, d).

Definition 2.4. [15]. Let (X, d) be a metric space. Define $V_\epsilon = \{(x, y) : d(x, y) < \epsilon\}$. Then (X, Γ),is a semi-linear uniform space induced by (X, d), where $\Gamma = \{V_\epsilon : \epsilon > 0\}$. This semi-linear uniform space will be denoted by (X, Γ_d).

In [12] and [13] , the authors defined the set valued map ρ and δ which played an important rule in the theory of fixed point on semi-linear uniform spaces.

Definition 2.5. [12]. Let (X, Γ) be a semi-linear uniform space. For $(x, y) \in X \times X$, let $\Gamma_{(x,y)} = \{V \in \Gamma : (x, y) \in V\}$. Then, the set valued map ρ on $X \times X$, is defined by $\rho(x, y) = \bigcap\{V : V \in \Gamma_{(x,y)}\}$.

Clearly for all $(x, y) \in X \times X$, we have $\rho(x, y) = \rho(y, x)$, and $\Delta \subseteq \rho(x, y)$.The

following definition given in [13] , the authors defined $\delta(x, x) = \phi$, now if we define $\delta(x, x) = \rho(x, x) = \Delta$, then all the results in the literature still valid and the new definition of $\delta(x, x)$ seems to be more convenient.

Definition 2.6. [13]. Let (X, Γ) be a semi-linear uniform space. For $(x, y) \in X \times X$, let $\Gamma^c_{(x,y)} = \{V \in \Gamma : (x, y) \notin V\}$. Then the set valued map δ on $X \times X$, is defined by

$$\delta(x, y) = \begin{cases} \bigcup\{V : V \in \Gamma^c_{(x,y)}\} & x \neq y \\ \Delta & x = y \end{cases}$$

The following two examples will illustrate our definitions.

Example 2.7. Referring to Example 2.2 $\delta(1, 0) = \{(x, y) : y - 1 < x < y + 1$, and $-\infty < y < \infty\}$, $\rho(2, 0) = \{\{(x, y) : y - 2 \leq x \leq y + 2$, and $-\infty < y < \infty\}$. Figure 1 shows $\delta(1, 0)$ in green color and $\rho(2, 0)$ in red color.

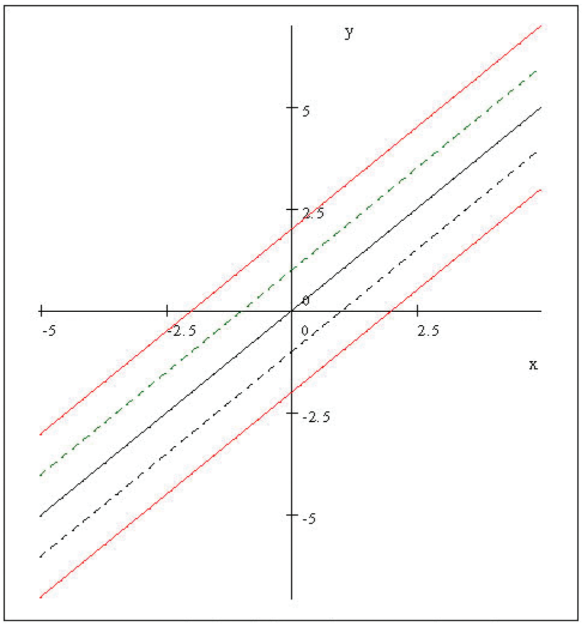

Figure 1: $\delta(1, 0)$ and $\rho(2, 0)$

Example 2.8. Referring to Example 2.3 $\delta(1, 0) = \{(x, y) : x^2 + y^2 < 1\} \cup \{\Delta\}$ and $\rho(2, 0) = \{(x, y) : x^2 + y^2 \leq 2\} \cup \{\Delta\}$. Figure 2 shows $\delta(1, 0)$ in green color and $\rho(2, 0)$ in red color.

We refer the reader to [13], for an important properties of semi-linear uniform spaces, using the set valued map ρ and δ.

In [1] Alhihi gave more properties of semi-linear uniform spaces and gives the following important definition:

Definition 2.9. [1] For $n \in \mathbb{N}$ and $A \in \Lambda$, where $\Lambda = \delta \cup \rho \cup \Gamma$. Define $\frac{1}{n}A = \bigcup_{U \in \Gamma}\{U : nU \subseteq A\}$.

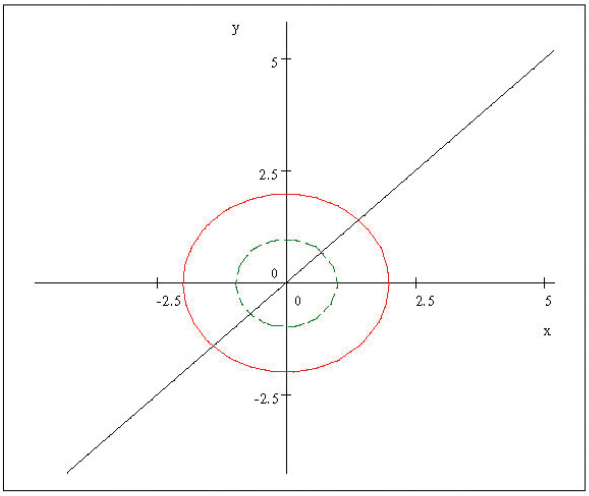

Figure 2: $\delta(1, 0)$ and $\rho(2, 0)$

Clearly $\frac{1}{n}\Delta = \Delta$ and $\frac{1}{n}A \in D_X$ for all $A \in \Lambda$.

To illustrate our definitions, if we take $t = 1$ in Example 2.2, then $V_1, 2V_1$ and $\frac{1}{2}V$ are sketched in figure (c) by red, green and blue color respectively, where $\{\Delta\}$ is sketched by black color which common in all of them.Also if we take $\epsilon = 2$ in Example 2.3, then $V_1, 2V_1$ and $\frac{1}{2}V$ are sketched in figure (d) by red, green and blue color respectively. where $\{\Delta\}$ is sketched by black color which common in all of them.

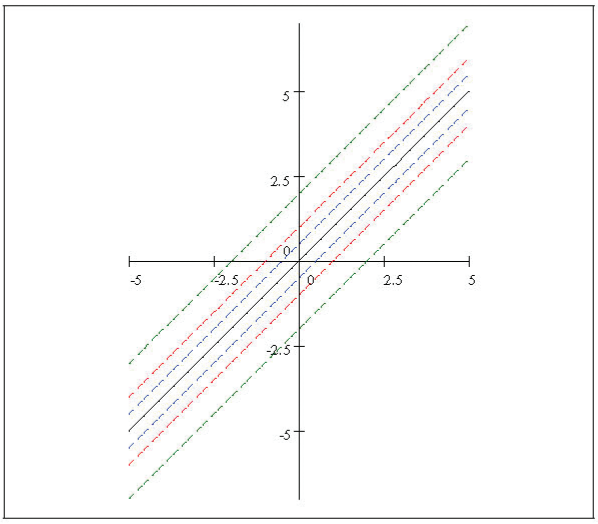

Figure 3: $V, 2V$, and $\frac{1}{2}V$

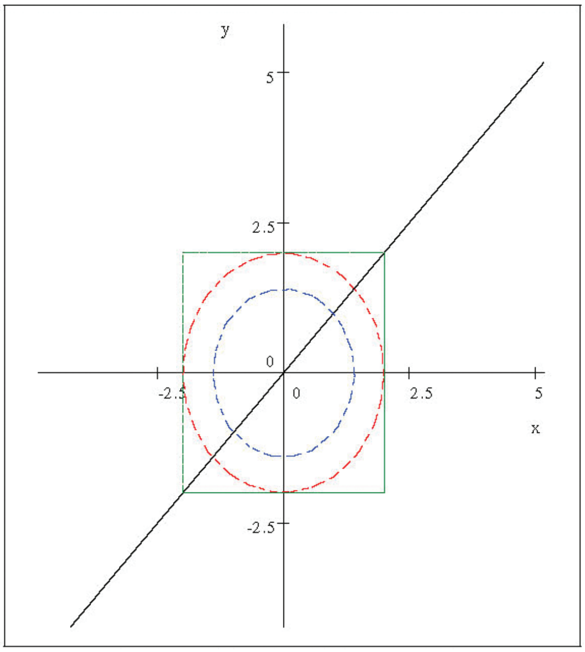

Figure 4: $V_1, 2V_1$ and $\frac{1}{2}V_1$

Using Definition 2.9, Alhihi gave the following important theorem.

Theorem 2.10. [1] Let $A \in \Lambda$, and σ a sub collection of Λ. For $n \in \mathbb{N}$, we have

(i) $n\left(\frac{1}{n}A\right) \subseteq A$

(ii) If $B \in \Gamma$satisfies $nB \subseteq A$, then $B \subseteq \frac{1}{n}A$.

(iii) $\frac{1}{n+1}A \subseteq \frac{1}{n}A$.

(iv) $\frac{1}{n}A \subseteq A$

(v) $\frac{1}{n}\bigcap_{A\in\sigma} A = \bigcap_{A\in\sigma}\frac{1}{n}A$.

(vi) $\bigcup_{A\in\sigma}\frac{1}{n}A \subseteq \frac{1}{n}\bigcup_{A\in\sigma} A$.

(vii) $\frac{1}{n}\,\delta(x,y) = \left\{ \bigcup_{V\in\Gamma^c_{(x,y)}} \frac{1}{n}V \qquad if\ x \neq y\phi \qquad if\ x = y. \right\}$

(viii) $\frac{1}{n}\rho(x,y) = \frac{1}{n} \bigcap_{V \in \Gamma_{(x,y)}} V \subseteq \bigcap_{V \in \Gamma_{(x,y)}} \frac{1}{n} V.$

(ix) Let x, y be any distinct points in semi-linear uniform spaces (X, Γ) . Then, $n\left(\frac{1}{n}\delta(x,y)\right) \subseteq \delta(x,y) \subseteq \frac{1}{n}(n\delta(x,y))$.

In [2], S. Alhihi and M. Fayyad showed that every semi-linear uniform space induced a Tychonoff space (X, T_Γ), where T_Γ induced by local base $B_x = \{B(x,U) : U \in \Gamma\}$, where $B(x,U) = \{y : (x,y) \in U\}$. For more topological properties of T_Γ we refer the reader to [2].

The following is an equivalent definition of continuity and uniform continuity in semi-linear uniform spaces, for more details see [12] .

Definition 2.11. [12] Let $f : (X, \Gamma_X) \to (Y, \Gamma_Y)$, then

1. f is continuous at $x_\circ$ if for all $U \in \Gamma_Y$, there exists $V \in \Gamma_X$, such that if $(x, x_\circ) \in V$, then $(f(x), f(x_\circ)) \in U$.
2. f is uniformly continuous if for all $U \in \Gamma_Y$, there exists $V \in \Gamma_X$, such that if $(x,y) \in V$, then $(f(x), f(y)) \in U$.

The following definitions are an equivalent definitions to convergent, Cauchy and completeness in semi-linear uniform spaces.

Definition 2.12. [12] Let (X, Γ) be a semi-linear uniform space and (x_n) be a sequence in X. then,

1. (x_n) converges to x in X and denoted by $x_n \to x$, if for every $V \in \Gamma$ there exists k such that $(x_n, x) \in V$ for every $n \geq k$.
2. (x_n) is called Cauchy if for every $V \in \Gamma$ there exists k such that $(x_n, x_m) \in V$ for every $n, m \geq k$.

Definition 2.13. [12] Let (X, Γ) be a semi-linear uniform space. Then (X, Γ) is called complete, if every Cauchy sequence is convergent.

3. Contraction mapping

Banach fixed-point theorem (also known as the contraction mapping theorem or contraction mapping principle) is an important tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points. The theorem is named after Stefan Banach (1892–1945), and was first stated by him in 1922. Contractions plays an important rule in the theory of fixed point. Tell 2013 to define a contraction we need a metric space. In 211, Tallafha gave an example of a semi-linear uniform, which is not metrizable, also in 2013 he defined contraction function on semi-linear uniform spaces, which is the first time we had a contraction mapping on a space which is weaker than metric space. So the theory of fixed point can be discussed in semi-linear uniform spaces.

Definition 3.1. [15] Let $f : (X, \Gamma) \longrightarrow (X, \Gamma)$, f satisfied Lipschitz condition if there exist $m, n \in N$ such that $m\delta(f(x), f(y)) \subseteq n\delta(x, y)$ $\forall x, y \in X$. Moreover if $m > n$, then we call f a contraction.

The well known definition of contraction says if $f : (X, d) \longrightarrow (X, d)$, then f is a contraction if there exists a positive real number $r < 1$ such that $d(f(x), f(y)) \leq rd(x, y)$. Clearly this is equivalent to exist $r \in \mathbb{Q}^+, r < 1$, such that $d(f(x), f(y)) \leq rd(x, y)$.Using this idea Alhihi, S. gave a definition of other type of Lipschize condition and contraction called $r-$Lipschitz condition and r-contraction.

Definition 3.2. [1] Let $\Lambda = \delta \cup \rho \cup \Gamma$. For $A \in \Lambda$ and $r = \frac{n}{m}, n, m \in \mathbb{N}$ and the greatest common devisor of n, m is 1. Define $rA = n\left(\frac{1}{m}A\right)$.

In the above definition Alhihi, S. assumed that $r = \frac{n}{m}, n, m \in \mathbb{N}$, are relatively prime. In [17] , Tallafha, A. and Alhihi, S. used the same idea to define rA for $r \in \mathbb{Q}^+$ and $A \in \delta \cup \rho \cup \Gamma$. If $r = \frac{n}{m} \in \mathbb{Q}^+$, let $k = g.c.d\,(n, m)$, the greatest common devisor of n and m. then $r = \frac{n}{m} = \frac{kn_1}{km_1}$, n_1, m_1 are relatively prime.

Definition 3.3. [17] Let $\Lambda = \delta \cup \rho \cup \Gamma$. For $A \in \Lambda$ and $r = \frac{n}{m}, n, m \in \mathbb{N}$ define $rA = n_1\left(\frac{1}{m_1}A\right)$, where n_1, m_1 are relatively prime and $\frac{n}{m} = \frac{n_1}{m_1}$.

Now we shall give the definition of contraction from a semi-linear uniform space (X, Γ) to it self.

Definition 3.4. [1] Let $f : (X, \Gamma) \longrightarrow (X, \Gamma)$, then f satisfied r-Lipschitz condition if there exist $r \in \mathbb{Q}^+$ such that $\delta(f(x), f(y)) \subseteq r\delta(x, y)$. Moreover if $r < 1$, then f is called r-contraction.

In [9] , Khalil, R. and in [10] , Menger, K. defined M-space and convex metric spaces, respectively as follows.

Definition 3.5. [9] A metric space (X, d) is M-space, if for all $(x, y) \in X \times X$, and $\lambda = d\,(x, y)$, if $\alpha \in [0, \lambda]$, there exists a unique $z_\alpha \in X$ such $B\,[x, \alpha] \cap B\,[y, \lambda - \alpha] = \{z_\alpha\}$.

Definition 3.6. [10] Let (X, d) be a metric space. For $x \in X, r > 0$, let $B[x, r] = \{t : d(x, t) \leq r\}$. A metric space (X, d) is convex, if for all $x, y \in X$, $B[x, r_1] \cap B[y, r_2] \neq \phi$ whenever $r_1 + r_2 \geq d(x, y)$.

Rawashdeh, A. and Tallafha, A. in [11] shoed that M-space and convex metric space are equivalent except uniqueness, also they gave the following nice results.

Lemma 3.7. [11] Let (X, Γ_d) be a semi-linear uniform space induced by the metric space (X, d). Then

(1) $\rho(x, y) = \{(s, t) \in X \times X : d(s, t) \leq d(x, y)\}$.

(2) $\delta(x, y) = \{(s, t) \in X \times X : d(s, t) < d(x, y)\}$.

Theorem 3.8. [11] Let (X, Γ) be a semi-linear uniform space induced by unbounded convex metric space (X, d). Then $f : (X, d) \longrightarrow (X, d)$ is a contraction if and only if $f : (X, \Gamma) \longrightarrow (X, \Gamma)$ is a contraction.

Lemma 3.9. [11] Let (X, Γ_d) be a semi-linear uniform space induced by a convex metric space (X, d). Then $V_{\varepsilon_1} \circ V_{\varepsilon_2} = V_{\varepsilon_1 + \varepsilon_2}$.

Rawashdeh, A and Tallafha, A. in [11], proved that if (X, Γ_d) be a semi-linear uniform space induced by a convex metric space (X, d), then $n\rho(x, y) = \{(s, t) \in X \times X : d(s, t) \leq nd(x, y)\}$. Now we shall prove a similar result for the set valued function $\delta(x, y)$.

Lemma 3.10. Let (X, Γ_d) be a semi-linear uniform space induced by a metric space (X, d), then $n\delta(x, y) \subseteq \{(s, t) \in X \times X : d(s, t) < nd(x, y)\}$.

Proof. Let $(s, t) \in n\delta(x, y)$, then there exist $x_1, x_2, \ldots, x_{n-1}$ such that $(s, x_1), (x_1, x_2),$
$\ldots (x_{n-1}, t) \in \delta(x, y)$, therefore $d(s, t) \leq d(s, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, t) < nd(x, y)$. ■

To prove the converse of the previous Lemma, we need a convex metric space. So we have the following.

Theorem 3.11. Let (X, Γ_d) be a semi-linear uniform space induced by a convex metric space (X, d), then $n\delta(x, y) = \{(s, t) \in X \times X : d(s, t) < nd(x, y)\}$.

Proof. Let (X, Γ_d) be a semi-linear uniform space induced by a convex metric space (X, d). By Lemma 3.10, we want to show $\{(s, t) \in X \times X : d(s, t) < nd(x, y)\} \subseteq n\delta(x, y)$.

By Lemma 3.9, we have $n\delta(x, y) = \bigcup\limits_{u_\epsilon \in \Gamma^c(x,y)} nU_\epsilon = \bigcup\limits_{u_\epsilon \in \Gamma^c(x,y)} U_{n\epsilon}$.

Let $(s, t) \in X \times X$ be such that $d(s, t) < nd(x, y)$, so it is enough to show $(s, t) \in \bigcup\limits_{u_\epsilon \in \Gamma^c(x,y)} U_{n\epsilon}$.

Since $d(s,t) < nd(x,y)$, let r be such that $\frac{d(s,t)}{n} < r < d(x,y)$. So $(x,y) \notin U_r$ and $(s,t) \in U_{nr} \subseteq \bigcup_{u_\epsilon \in \Gamma^c(x,y)} U_{n\epsilon}$. ■

Theorem 3.12. Let (X, Γ_d) be a semi-linear uniform space induced by a convex metric space (X,d), then $\frac{1}{m}\delta(x,y) = \{(s,t) \in X \times X : md(s,t) < d(x,y)\}$.

Proof. Let $(s,t) \in X \times X$ such that $md(s,t) < d(x,y)$, so $(s,t) \in V_{\frac{d(x,y)}{m}}$.We want to show $mV_{\frac{d(x,y)}{m}} \subseteq \delta(x,y)$.

let $(q,r) \in mV_{\frac{d(x,y)}{m}} = V_{m(\frac{d(x,y)}{m})} = V_{d(x,y)}$, so we have $d(q,r) < d(x,y)$. Hence $(q,r) \in \delta(x,y)$. Now, let $(s,t) \in \frac{1}{m}\delta(x,y) = \bigcup_{U_\epsilon \in \Gamma^c(x,y)} \frac{1}{m}U_\epsilon$. Then there exist $0 < \epsilon \leq d(x,y)$ such that $(s,t) \in \frac{1}{m}U_\epsilon$, hence there exist $U \in \Gamma$, such that $(s,t) \in U$ and $mU \subseteq U_\epsilon \subseteq \delta(x,y)$, so there exist $\beta > 0$ such that $(s,t) \in U_\beta$ and $mU_\beta = U_{m\beta} \subseteq U_\epsilon \subseteq \delta(x,y)$. Therefore $md(s,t) < \epsilon \leq d(x,y)$. ■

The above Theorem show that $\frac{1}{m}\delta(x,y) = \{(s,t) \in X \times X : md(s,t) < d(x,y)\} = U_{\frac{d(x,y)}{m}}$. So $\frac{n}{m}\delta(x,y) = n\left(\frac{1}{m}\delta(x,y)\right) = nU_{\frac{d(x,y)}{m}} = U_{\frac{nd(x,y)}{m}} = \{(s,t) \in X \times X : d(s,t) < \frac{n}{m}d(x,y)\}$. Therefore we complete the prove of the following Corollary.

Corollary 3.13. Let (X, Γ_d) be a semi-linear uniform space induced by a convex metric space (X,d), then $\frac{n}{m}\delta(x,y) = \{(s,t) \in X \times X : d(s,t) < \frac{n}{m}d(x,y)\}$.

Theorem 3.14. Let (X, Γ) be a semi-linear uniform space induced by convex metric space (X,d). Then $f : (X,d) \longrightarrow (X,d)$ is a contraction if and only if $f : (X,\Gamma) \longrightarrow (X,\Gamma)$ is r-contraction.

Proof. Let (X,Γ) be a semi-linear uniform space induced by convex metric space (X,d), and $f : (X,d) \longrightarrow (X,d)$ be a contraction. Then there exists $0 < r < 1$ such that $d(f(x), f(y)) \leq rd(x,y)$ $\forall x, y \in X$, hence, there exists a relatively prime $m, n \in N$ such that $n < m$ and $r \leq \frac{n}{m} = r' < 1$, which implies $d(f(x), f(y)) \leq \frac{n}{m}d(x,y)$ for all $x, y \in X$. Thus if $d(s,t) < d(f(x), f(y))$, then $d(s,t) < \frac{n}{m}d(x,y)$, which means $\delta(f(x), f(y) \subseteq r'\delta(x,y)$ and $0 < r' < 1$. Which mean $f : (X,\Gamma) \longrightarrow (X,\Gamma)$ is r'-contraction.

Now suppose $f : (X,\Gamma) \longrightarrow (X,\Gamma)$ is a $r$contraction, then there exists a relatively prime $m, n \in \mathbb{N}$ such that $m > n$ and $\delta(f(x), f(y) \subseteq \frac{n}{m}\delta(x,y)$ $\forall x, y \in X$. By Lemma

3.7 and Corollary 3.13, we have,

$$\{(s,t) \in X \times X : d(s,t) < d(f(x), f(y))\} = \delta(f(x), f(y) \subseteq \frac{n}{m}\delta(x,y)$$

$$= \{(s,t) \in X \times X : d(s,t) < \frac{n}{m}d(x,y)\},$$

therefore $d(f(x), f(y)) \leq \frac{n}{m}d(x,y)$. ■

4. Main result

In [1], Alhihi, S. defined r-Lipschitz condition on semi-linear uniform space, where $f : (X,\Gamma) \longrightarrow (X,\Gamma)$ is r-Lipschitz, if there exist $r \in \mathbb{Q}^+$ such that $\delta(f(x), f(y)) \subseteq r\delta(x,y)$. Moreover if $r < 1$, then f is called r-contraction. Also in [1], Alhihi, S. asked the following question. If f is r-contraction from a complete semi-linear uniform space (X,Γ) to it self, is f has a unique fixed point. In [11], Rawashdeh A. and Tallafha, A. answered the above question negatively, they gave an example of a complete semi-linear uniform space (X,Γ) and a contraction $f : (X,\Gamma) \to (X,\Gamma)$ which has infinitely many fixed points. Also they defined strong semi-linear uniform space and strong contraction.

In [17] , Tallafha, A. and Alhihi, S. defined modified semi-linear uniform spaces, and asked the following question.

Question. Let (X,Γ) be a complete modified semi-linear uniform space. And $f : (X,\Gamma) \longrightarrow (X,\Gamma)$ be a m-contraction. Is f has unique fixed point.

In this section we shall answer this question positively for 2-contraction.

Remark 4.1. Let (X,Γ) be a semi-linear uniform space and $x, y \in X$. If $\delta(x,y) \subseteq \frac{1}{2}\delta(x,y)$, then $2\delta(x,y) = \delta(x,y)$.

Let x_n be a sequence in X, for $y \in X$ the sets $\rho(x_n, y)$ and $\delta(x_n, y)$ are sequences of subsets of $X \times X$. Now we shall define what we mean by the convergent of $\rho(x_n, y)$ and $\delta(x_n, y)$.

Definition 4.2. Let (X,Γ) be a semi-linear uniform space, (x_n) a sequence in X and $x, y \in X$. Then we say $\rho(x_n, y)$ converge to $\rho(x,y)$ and denote by $\rho(x_n, y) \to \rho(x,y)$ if for all $U \in \Gamma_{(x,y)}$, there exist $k \in \mathbb{N}$ such that $(x_n, y) \in U \circ U$ for all $n \geq k$.

Definition 4.3. Let (X,Γ) be a semi-linear uniform space and (x_n) be a sequence in X and $x, y \in X$.Then we say $\delta(x_n, y)$ converge to $\delta(x,y)$ and denote by $\delta(x_n, y) \to \delta(x,y)$ if for all $U \in \Gamma_{(x,y)}$, there exist $k \in \mathbb{N}$ such that $(x_n, y) \in U \circ U$ for all $n \geq k$.

Definition 4.4. Let (X,Γ) be a semi-linear uniform space and (x_n) be a sequence in X and $x, y \in X$.Then we say $\delta(x_n, x_m)$ converge to Δ and denote by $\delta(x_n, x_m) \to \Delta$ if for all $U \in \Gamma$, there exist $k \in \mathbb{N}$ such that $(x_n, x_m) \in U \circ U$ for all $n, m \geq k$.

From definition 4.2 and definition 4.3 we can conclude the following Proposition:

Proposition 4.5. Let (X, Γ) be a semi-linear uniform space and (x_n) be a sequence in X and $x, y \in X$. Then $\delta(x_n, y) \to \delta(x, y)$ if and only if $\rho(x_n, y) \to \rho(x, y)$.

Remark 4.6. In the above definitions, the limit is not unique. Consider Example 2.3, $\delta\left(1 - \frac{1}{n}, 1\right) \to \delta(x, y)$, and $\rho\left(1 - \frac{1}{n}, 1\right) \to \rho(x, y)$, for all $x, y \in \mathbb{R}$ such that $x^2 + y^2 \geq 2$.

The following lemma give us a nice connection between the convergent of $\delta(x_n, y)$ and the convergent of x_n.

Lemma 4.7. Let (X, Γ) be a semi-linear uniform spaces and x_n be a sequence in X. If $x_n \to x$, then $\delta(x_n, y) \to \delta(x, y)$.

Proof. Let $U \in \Gamma_{(x,y)}$ and x_n be a sequence in X. If $x_n \to x$, then there exist $k \in \mathbb{N}$ such that $(x_n, x) \in U$ for all $n \geq k$. Hence $(x_n, y) \in U \circ U$ for all $n \geq k$. ■

Using definition 4.4, we can show following.

Lemma 4.8. Let (X, Γ) be a semi-linear uniform spaces and x_n be any sequence in X. If $\delta(x_n, x_m) \to \Delta$, then x_n is Cauchy.

In [11], Rawashdeh, A. and Tallafha, A. answered the following question negatively, if f is a contraction from a complete semi-linear uniform space (X, Γ) to it self, is f has a unique fixed point.

In [17] , Tallafha and Alhihi defined Modified semi-linear uniform spaces and m-contraction.

Definition 4.9. [17] A semi-linear uniform space (X, Γ) is called a modified semi-linear uniform space, if Γ satisfies the following additional conditions.

1. For all $V \in \Gamma$,we have $\bigcup_{n=1}^{\infty} nV = X \times X$.

2. $\delta(x, y) \subseteq \delta(x, z) \circ \delta(z, y)$ for all x, y and $z \in X$.

Definition 4.10. [17] Let (X, Γ) be a semi-linear uniform space. $f : (X, \Gamma) \longrightarrow (X, \Gamma)$ is called m-contraction if there exists $m \in \{2, 3, 4, \ldots\}$ such that $m\delta(f(x), f(y)) \subseteq \delta(x, y)$, for all $x, y \in X$.

Clearly from definition 4.9, a modified semi-linear uniform space (X, Γ) satisfies $U \subsetneqq 2U$ for all $U \in \Gamma$. Hence we have the following.

Proposition 4.11. Let (X, Γ) be a modified semi-linear uniform space. Then $\delta(x, y) \subsetneqq 2\delta(x, y)$ for all $x, y \in X, x \neq y$.

Proof. Let (X, Γ) be a strong semi-linear uniform space. Suppose not, i.e., there exists $x, y \in X$ with $x \neq y$, and $\delta(x, y) = 2\delta(x, y)$. Since $x \neq y$, then $\Gamma^c_{(x,y)} \neq \phi$. Let $V \in \Gamma^c_{(x,y)}$, by induction. $V \subseteq \delta(x, y)$ and if $KV \subseteq \delta(x, y)$ then $(K + 1)V = KV \circ V \subseteq \delta(x, y) \circ \delta(x, y) = 2\delta(x, y) = \delta(x, y)$ then for each $n \in \mathbb{N}$, we have $nV \subseteq \delta(x, y)$, which mean $\bigcup\limits_{n=1} nV \subseteq \delta(x, y)$. Which is a contradiction. ■

The following Proposition and lemma will be needed in the end of this section.

Proposition 4.12. Let (X, Γ) be a semi-linear uniform space. For $n \in \mathbb{N}$, we have $\left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{2}\delta(x, y)\right) \subseteq \left(\frac{1}{2}\right)^{n}\delta(x, y)$ for all $x, y \in X$.

Proof. Let $(s, t) \in \left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{2}\delta(x, y)\right)$, then there exist $U \in \Gamma$ such that $(s, t) \in U$ and $2^{n-1}U \subseteq \frac{1}{2}\delta(x, y)$, if $2^nU \subseteq \delta(x, y)$ we complete the proof. Let $(k, l) \in X \times X$ and $(k, l) \in 2^nU$, then $(k, l) \in 2(2^{n-1}U)$ so there exist $z \in X$ such that $(k, z) \in 2^{n-1}U$ and $(z, l) \in 2^{n-1}U$, but as $2^{n-1}U \subseteq \frac{1}{2}\delta(x, y)$, we have $(k, z), (z, l) \in \frac{1}{2}\delta(x, y)$, so there is V_1 and V_2 such that $(k, z) \in V_1 \subseteq 2V_1 \subseteq \delta(x, y)$ and $(z, l) \in V_2 \subseteq 2V_2 \subseteq \delta(x, y)$, by chain we have $(k, l) \in 2V \subseteq \delta(x, y)$ where $V \in \{V_1, V_2\}$ which means $(k, l) \in \delta(x, y)$. ■

Lemma 4.13. Let (X, Γ) be a modified semi-linear uniform space. Then for all $V \in \Gamma$ and $x_\circ, x_1 \in X$, there exist $N_\circ \in \mathbb{N}$ such that $\frac{1}{2^{N_\circ}}\delta(x_\circ, x_1) \subseteq V$.

Proof. Let $V \in \Gamma$. As $\bigcup\limits_{n=1}^{\infty} nV = X \times X$, there exist $k \in \mathbb{N}$ such that $(x_\circ, x_1) \in kV$. For k there is $N_\circ \in \mathbb{N}$ such that $k \leq 2^{N_\circ}$. Now as $(x_\circ, x_1) \in kV$ there exist $t_1, ...t_{k-1}$ such that $(x_\circ, t_1) \in V$, ... and $(t_{k-1}, x_1) \in V$, so $\delta(x_\circ, t_1) \subsetneq V$..... and $\delta(t_{k-1}, x_1) \subsetneq V$.

Thus $\delta(x_\circ, t_1) \circ ....\circ \delta(t_{k-1}, x_1) \subseteq kV$, but as $\delta(x_\circ, x_1) \subseteq \delta(x_\circ, t_1) \circ ....\circ \delta(t_{k-1}, x_1)$ and as $(x_\circ, x_1) \in kV$ so we have $\delta(x_\circ, x_1) \subsetneq kV \subseteq 2^{N_\circ} V$. Thus $\delta(x_\circ, x_1) \subsetneq 2^{N_\circ} V$. Now we want to show $\bigcup\limits_{U\in\Gamma}\{U : 2^{N_\circ} U \subseteq \delta(x_\circ, x_1)\} \subseteq V$, suppose not, then there is $U \in \Gamma$ with $2^{N_\circ} U \subseteq \delta(x_\circ, x_1)$ and $U \nsubseteq V$, therefore by chain we have $V \subset U$, so $2^{N_\circ} V \subseteq 2^{N_\circ} U \subseteq \delta(x_\circ, x_1)$, but we showed that $\delta(x_\circ, x_1) \subsetneq 2^{N_\circ} V$, so we have $\delta(x_\circ, x_1) \subsetneq 2^{N_\circ}V \subseteq \delta(x_\circ, x_1)$ which is contradiction. Thus $\bigcup\limits_{U\in\Gamma}\{U : 2^{N_\circ} U \subseteq \delta(x_\circ, x_1)\} \subseteq V$. Hence $\frac{1}{2^{N_\circ}}\delta(x_\circ, x_1) \subseteq V$. ■

In [17], Tallafha, A. And Alhihi, S. proved that, if (X, Γ_d) be a semi-linear uniform space induced by a convex metric space (X, d), then $\delta(x, y) \subseteq \delta(x, z) \circ \delta(z, y)$ for all x, y and $z \in X$.

In the following Proposition we shall use lemma 3.10 to give a new prove for, $\delta(x, y) \subseteq \delta(x, z) \circ \delta(z, y)$ for all x, y and $z \in X$.

Proposition 4.14. Let (X, Γ_d) be a semi-linear uniform space induced by a convex metric space (X, d), then $\delta(x, y) \subseteq \delta(x, z) \circ \delta(z, y)$ for all x, y and $z \in X$.

Proof. Let (X, Γ_d) be a semi-linear uniform space induced by a convex metric space (X, d). For $x, y \in X$ we want to show $\delta(x, y) \subseteq \delta(x, z) \circ \delta(z, y)$ for all $z \in X$. Let $(s, t) \in \delta(x, y)$, then for $z \in X$ we have, $d(s, t) < d(x, y) \leq d(x, z) + d(z, y)$, so we have $(s, t) \in V_{d(x,z)+d(z,y)} \in \Gamma_d$. But by lemma 3.9, we have $(s, t) \in V_{d(x,z)+d(z,y)} = V_{d(x,z)} \circ V_{d(z,y)}$, which means there is $w \in X$ such that $(s, w) \in V_{d(x,z)}$ and $(w, t) \in V_{d(z,y)}$. Thus $d(s, w) < d(x, z)$ and $d(w, t) < d(z, y)$, hence $(s, w) \in \delta(x, z)$ and $(w, t) \in \delta(z, y)$. ■

The following Theorems is an important consequences of convex metric spaces, for proof, we refer the reader to [17] .

Theorem 4.15. [17] Let (X, Γ) be a semi-linear uniform space induced by a convex metric space (X, d), then (X, Γ) a modified semi-linear uniform space.

Theorem 4.16. [17] Let (X, Γ) be semi-linear uniform space, then $\frac{1}{m}\delta(x, y) \circ \frac{1}{n}\delta(x, y) = \left(\frac{1}{m} + \frac{1}{n}\right)\delta(x, y)$, for all $x, y \in X$.

The following Proposition show that $m-$contraction is $\frac{1}{m}$-contraction.

Proposition 4.17. [17] Let (X, Γ) be a complete modified semi-linear uniform space. If $f : (X, \Gamma) \longrightarrow (X, \Gamma)$ is m-contraction, then f is $\frac{1}{m}$-contraction.

Now we shall answer gave a partial answer to the question given by Tallafha and Alhihi for $\frac{1}{2}-$ contraction a complete modified semi-linear uniform space,

Theorem 4.18. Let (X, Γ) be a complete modified semi-linear uniform space. If $f : (X, \Gamma) \longrightarrow (X, \Gamma)$ is 2-contraction, then f has a unique fixed point.

Proof. Fix $x_\circ \in X$, set $x_1 = f(x_\circ)$ and $x_{n+1} = f(x_n) = f^{n+1}(x_\circ)$. Firstly, by using Proposition 4.12 we see for $n \in \mathbb{N}$,

$$\delta(x_n, x_{n+1}) = \delta(f(x_{n-1}), f(x_n)) \subseteq \frac{1}{2}(\delta(x_{n-1}, x_n))$$

$$\subseteq \frac{1}{2}(\frac{1}{2}\delta(x_{n-2}, x_{n-1}))..... \subseteq \frac{1}{2^n}\delta(x_\circ, x_1).$$

Since (X, Γ) is a modified semi-linear uniform space, it follows that for $m \geq n$,

$$\delta(x_n, x_m) \subseteq \delta(x_n, x_{n+1}) \circ \delta(x_{n+1}, x_{n+2}) \circ .... \circ \delta(x_{m-1}, x_m)$$

$$\subseteq \left(\frac{1}{2}\right)^n \delta(x_\circ, x_1) \circ \left(\frac{1}{2}\right)^{n+1} \delta(x_\circ, x_1) \circ .... \circ \left(\frac{1}{2}\right)^{m-1} \delta(x_\circ, x_1).$$

By Theorem 4.16,

$$\subseteq \left(\frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}}\right) \delta(x_\circ, x_1)$$

$$= \frac{1}{2^n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}}\right) \delta(x_\circ, x_1).$$

Since $\frac{1}{2^n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}}\right) \leq \frac{1}{2^{n-1}}$ and by definition of $\frac{n}{m}\delta(x_\circ, x_1)$, we have $\delta(x_n, x_m) \subseteq \frac{1}{2^{n-1}}\delta(x_\circ, x_1)$.

By Lemma 4.13 we have, $\delta(x_n, x_m) \to \Delta$ as $n \to \infty$ and by Lemma 4.8, the sequence is Cauchy, hence there must be a point $x \in X$ such that the sequence converge to x. The fact that the limit x is a fixed point of f follows from the following: $\delta(x, f(x)) \subseteq \delta(x, x_m) \circ \delta(x_m, f(x)) \subseteq \delta(x, x_m) \circ \frac{1}{2}\delta(x, x_{m-1}) \subseteq \delta(x, x_m) \circ \delta(x, x_{m-1})$ As $x_n \to x$ by Lemma 4.7 for large m we have $\delta(x, f(x)) = \Delta$ which means $x = f(x)$. To show that the fixed point is unique, suppose there is another fixed point $y \in X$. Since f is $\frac{1}{2}$-contraction, we have $\delta(x, y) = \delta(f(x), f(y)) \subseteq \frac{1}{2}\delta(x, y)$ which means that $2\delta(x, y) = \delta(x, y)$, from Proposition **??** we have a contradiction. Hence x is a unique fixed point of f. ■

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